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FUNCTIONAL EQUATIONS FOR FRACTAL INTERPOLANTS

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This paper is dedicated to Professor R. Ž. Djordjević

Abstract. For a given data set $Y = \{(x_i, y_i)\}_{i=0}^n$, a fractal interpolating function $\varphi : [x_0, x_n] \to \mathbf{R}$ can be defined by a hyperbolic iterated function system $\Sigma_{Y,\mathbf{d}} = \{\mathbf{R}^2; \{w_i\}_{i=1}^n\}$ where w_i are affine contractions in $\mathbf{R}^2$ depending on an n-dimensional vector $\mathbf{d}$. Typically, $w_i : (x, y) \mapsto \big(u_i(x), v_i(x, y)\big)$ where $u_i(\cdot)$ and $v_i(x, \cdot)$ are contractions, and $v_i(\cdot\,, y)$ is a Lipschitz mapping.

The system $\Sigma_{Y,\mathbf{d}}$ has the unique attractor $\Phi_{Y,\mathbf{d}}$ which is the graph of a continuous function φ interpolating Y and satisfying the Read-Bajraktarević functional equation

$$\varphi(x) = v_i\big(u_i^{-1}(x),\, \varphi(u_i^{-1}(x))\big)\,, \quad x \in [x_{i-1},\, x_i]\,, \quad i = 1, \ldots, n\,.$$

Using this equation, it is shown that φ has only a limited affine invariant property. Correspondingly, the general form of affine transformations ω such that $\omega(\Phi_{Y,\mathbf{d}}) = \Phi_{\omega(Y),\mathbf{d}}$ is specified. An application of the functional equation and some examples are also given.

1. Introduction

Let an *interpolatory data set* $Y = \{(x_i, y_i)\}_{i=0}^n$ $(n \geq 2)$ be given, that is a set of points from $\mathbf{R}^2$ such that either $\Delta x_i = x_{i+1} - x_i > 0\ \forall i$, or $\Delta x_i < 0\ \forall i$. Also let a *scaling vector* $\mathbf{d} = [d_1 \ldots d_n]^T$ be given, namely a vector from $\mathbf{R}^n$ whose components are intended as *vertical scaling factors*.

With the pair $(Y, \mathbf{d})$ one can associate the iterated function system (IFS for short) $\Sigma_{Y,\mathbf{d}} = \{\mathbf{R}^2; \{w_i\}_{i=1}^n\}$, in which $w_1, \ldots, w_n$ are the affine transformations of $\mathbf{R}^2$ defined by

(1) $$w_i : (x, y) \mapsto (a_i x + e_i,\ c_i x + d_i y + f_i)\,,$$

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with the coefficients

$$
\begin{aligned}
a_i &= \frac{\Delta x_{i-1}}{x_n - x_0}\,, \quad c_i = \frac{\Delta y_{i-1}}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}\,, \\
e_i &= x_i - a_i x_n, \quad f_i = y_i - c_i x_n - d_i y_n\,.
\end{aligned} \tag{2}
$$

Let $\mathcal{H}(\mathbf{R}^2)$ denote the set of nonempty compact subsets of $\mathbf{R}^2$ and let h_θ be the Hausdorff metric on $\mathcal{H}(\mathbf{R}^2)$ generated by the norm $\|\cdot\|_\theta$ defined as $\|(x,y)\|_\theta = |x| + \theta|y|$, $0 < \theta < \min_i\{(1-|a_i|)/(1+|c_i|)\}$. The space $(\mathcal{H}(\mathbf{R}^2), h_\theta)$ is a complete metric space. With $\Sigma = \Sigma_{Y,\mathbf{d}}$ is canonically associated the *Hutchinson* operator W_Σ acting on $(\mathcal{H}(\mathbf{R}^2), h_\theta)$ and defined by

$$
W_\Sigma(\cdot) = \cup_1^n w_i(\cdot)\,. \tag{3}
$$

If $\|\mathbf{d}\| = \max_i\{|d_i|\} < 1$, then $\Sigma_{Y,\mathbf{d}}$ is hyperbolic, and W_Σ is a contraction in $(\mathcal{H}(\mathbf{R}^2), h_\theta)$. The unique fixed point of W_Σ, namely the nonempty, closed, bounded set $\Phi_{Y,\mathbf{d}} \subset \mathbf{R}^2$ such that

$$
W_\Sigma(\Phi_{Y,\mathbf{d}}) = \Phi_{Y,\mathbf{d}}
$$

is the unique *attractor* of $\Sigma_{Y,\mathbf{d}}$. Under such conditions the following theorem holds [1].

Theorem 1. *The iterated function system $\Sigma_{Y,\mathbf{d}}$ corresponding to an arbitrary interpolatory data set Y and to a scaling vector $\mathbf{d}$ such that $\|\mathbf{d}\| = \max_i\{|d_i|\} < 1$ is hyperbolic, and its attractor is the graph of a continuous function $\varphi : [x_0, x_n] \to \mathbf{R}$ that interpolates Y.*

Because of this result, we refer to $\Sigma_{Y,\mathbf{d}}$ with $\|\mathbf{d}\| < 1$, as a fractal interpolatory scheme, and call the function $\varphi = \varphi_{Y,\mathbf{d}}$, whose graph is the attractor of $\Sigma_{Y,\mathbf{d}}$, a *fractal interpolating function.* Also, we use the acronym IFS with the meaning of iterated function system.

The Functional Equation

For $i = 1, \ldots, n$, denote the x and y components of $w_i(x,y)$ in (1) by

$$
u_i(x) = a_i x + e_i\,, \quad v_i(x,y) = c_i x + d_i y + f_i\,, \tag{4}
$$

respectively.

Consider the functional equation

$$\varphi(x) = v_i(u_i^{-1}(x), \varphi(u_i^{-1}(x))), \quad x \in [x_{i-1}, x_i], \ i = 1, \ldots, n. \tag{5}$$

If $u_i(x)$ and $v_i(x, y)$ are given by (4) with $|d_i| < 1$, then $u_i(x)$ is a bijection $[x_0, x_n] \to \mathbf{R}$, $v_i(x, \cdot\,) \in \mathrm{Lip}^{(<1)}(\mathbf{R})$, and $v_i(\cdot\,, y) \in \mathrm{Lip}(\mathbf{R})$, so that (5) is a functional equation of Read-Bajraktarević type [7]. It can be written as

$$\varphi(x) = c_i \frac{x - e_i}{a_i} + d_i \, \varphi\Big(\frac{x - e_i}{a_i}\Big) + f_i\,, \quad x \in [x_{i-1}, x_i]\,, \ i = 1, \ldots, n\,. \tag{6}$$

Also, introducing the operator $T : \varphi \mapsto T\varphi$ defined piecewise by

$$(T\varphi)(x) = (T_i\varphi)(x) = c_i \frac{x - e_i}{a_i} + d_i \, \varphi\Big(\frac{x - e_i}{a_i}\Big) + f_i\,, \quad x \in [x_{i-1}, x_i]\,, \tag{7}$$

where $i = 1, \ldots, n$, equation (6) can be put in the more compact form

$$\varphi(x) = (T\varphi)(x)\,, \quad x \in [x_0, x_n]\,. \tag{8}$$

By Theorem 2, below, fractal interpolating functions are characterized as solutions of equation (8). The following lemma states a preliminary result, namely that bounded solutions of (8) are continuous.

Lemma 1. *Let $\varphi : [x_0, x_n] \to \mathbf{R}$ be a bounded function that satisfies equation* (8)*. Then φ is continuous.*

Proof. Suppose that φ be discontinuous at the point $t_0 \in [x_{i_1-1}, x_{i_1}]$, $i_1 \in \{1, \ldots, n\}$, with the jump $L_0 = |\varphi(t_0 - 0) - \varphi(t_0 + 0)| > 0$. Consider the point $t_1 = u_{i_1}^{-1}(t_0)$ and let $i_2 \in \{1, \ldots, n\}$ be the index such that $t_1 = u_{i_1}^{-1}(t_0) \in [x_{i_2-1}, x_{i_2}]$. Being, by (8), $\varphi = T_{i_1}\varphi$ on the interval $[x_{i_1-1}, x_{i_1}]$, and being $|d_{i_1}| \le \|\mathbf{d}\| < 1$, we have

$$0 < L_0 = |(T_{i_1}\varphi)(t_0 - 0) - (T_{i_1}\varphi)(t_0 + 0)| = |d_{i_1}|\,|\varphi(t_1 - 0) - \varphi(t_1 + 0)| \le \|\mathbf{d}\| L_1$$

so that t_1 is also a discontinuity point with a corresponding positive jump L_1. The discontinuity at t_1 is, in turn, the "image" of a discontinuity at $t_2 = u_{i_2}^{-1}(t_1)$, with a corresponding jump L_2, and $L_1 \le \|\mathbf{d}\|\, L_2$. Continuing this process, after k steps, a point is reached where φ has a jump L_k such that

$$\|\mathbf{d}\|^k L_k \ge \cdots \ge \|\mathbf{d}\|^2 L_2 \ge \|\mathbf{d}\|\, L_1 \ge L_0\,, \quad k \ge 1$$

Therefore $L_k \ge L_0\, \|\mathbf{d}\|^{-k} \to \infty$ if $k \to \infty$, which leads to the conclusion that φ is unbounded. Thus, φ must be a continuous function. $\square$

Theorem 2. *Let $\mathcal{B}$ denote the set of bounded functions $[x_0, x_n] \to \mathbf{R}$. Let $\varphi \in \mathcal{B}$. A necessary and sufficient condition for φ to be a fractal interpolating function of the data set Y is that φ satisfy the Read-Bajraktarević functional equation* (8) *with T given by* (7), *where a_i, c_i, e_i, f_i are given by* (2) *and $\|\mathbf{d}\| < 1$.*

Proof. (i) Let φ be a fractal interpolant for Y associated with the vertical scaling vector $\mathbf{d} = [d_1 \dots d_n]^T$, $\|\mathbf{d}\| < 1$. Then, by Theorem 1, its graph $\Phi_Y \subset \mathbf{R}^2$ is the fixed point of the operator W_Σ given by (3). Therefore, for any $P = (x, \varphi(x)) \in \Phi_Y$ there exist $i \in \{1, \dots, n\}$ and $P' = (x', \varphi(x')) \in \Phi_Y$ such that $x \in [x_{i-1}, x_i]$, and $P = w_i(P')$, which means

$$(9) \qquad x = a_i x' + e_i\ , \qquad \varphi(x) = c_i x' + d_i \varphi(x') + f_i, \qquad x \in [x_{i-1}, x_i].$$

Being $a_i \neq 0$ since $\Delta x_i \neq 0$ for each i, (9) yields $x' = (x - e_i)/a_i$ and

$$\varphi(x) = c_i \frac{x - e_i}{a_i} + d_i\, \varphi\Big(\frac{x - e_i}{a_i}\Big) + f_i,$$

which is (6).

(ii) Let φ satisfy (8). Let $(\mathcal{B}, d)$ be the metric space obtained by endowing $\mathcal{B}$ with the max-norm $d(\psi, \varphi) = \max\{|\psi(x) - \varphi(x)|, x \in [x_0, x_n]\}$. The operator T defined by (7) is a contraction in $(\mathcal{B}, d)$ since

$$|(T\varphi)(x) - (T\psi)(x)| = |d_i|\, |\varphi(u_i^{-1}(x)) - \psi(u_i^{-1}(x))| \le |d_i|\, d(\varphi, \psi),$$

and therefore $d(T\psi, T\varphi) \le \|\mathbf{d}\| d(\psi, \varphi)$.

Since T is a contraction in $(\mathcal{B}, d)$, it has a unique fixed point and, according to (8), this is φ. By Lemma 1, φ is also continuous. Let Φ' be the graph of φ. As a consequence of (6) we have

$$(T\varphi)(u_i(x)) = v_i(x, \varphi(x))\,,\ x \in [x_0, x_n]\ ,\ i = 1\,, \dots, n\,,$$

which implies that, for $x \in [x_0, x_n]$,

$$w_i(x, \varphi(x)) = (u_i(x), v_i(x, \varphi(x))) = (u_i(x), (T\varphi)(u_i(x))) = (u_i(x), \varphi(u_i(x)))$$

which, in connection with the continuity of φ, leads to

$$\Phi' = \bigcup_{i=1}^{n} w_i(\Phi')\,.$$

This means that Φ' is a fixed point of the Hutchinson operator (3). Since it is also a nonempty compact set of $\mathbf{R}^2$, it must be the unique attractor of the IFS $\Sigma_{Y,\mathbf{d}}$, which, by Theorem 1, is the graph of the interpolating fractal function. □

Remark 1. Notice that the interpolation property of φ can also be derived directly from (8), by the following argument. From (6) it can be seen immediately that $\varphi(x_0) = y_0$ and $\varphi(x_n) = y_n$. Furthermore, being the fixed point of T, φ satisfies $\varphi(x_i) = (T\varphi)(x_i) = c_i u_i^{-1}(x_i) + d_i\varphi(u_i^{-1}(x_i)) + f_i = c_i x_n + d_i\varphi(x_n) + f_i = c_i x_n + d_i y_n + f_i = c_i x_n + d_i y_n + y_i - c_i x_n - d_i y_n = y_i$ for $i = 1, \ldots, n$. Therefore φ interpolates the set of data Y.

Affine Transformation of The Interpolatory Scheme

Lemma 2. *Let ω be a regular affine mapping $\mathbf{R}^2 \to \mathbf{R}^2$ given by*

$$\omega : (x, y) \mapsto (px + qy + g,\ rx + sy + h), \quad p, q, r, s, g, h \in \mathbf{R} \tag{10}$$

and let F denote the graph of an arbitrary function $f : I \to \mathbf{R}$ ($I \subset \mathbf{R}$). The set $\omega(F)$ is the graph of a function if and only if $q = 0$.

Proof. Let $\alpha(x) = px + qf(x) + g$ and $\beta(x) = rx + sf(x) + h$, so that the image under ω of the point $(x,\ f(x)) \in F$ has coordinates $(\alpha(x),\ \beta(x))$.

(i) Suppose $q = 0$. Since ω is regular, it must be $p \neq 0$, therefore $\alpha(x)$ is invertible and $\beta(x) = \beta(\alpha^{-1} \circ \alpha(x)) = \beta \circ \alpha^{-1}(\alpha(x))$ is obviously a function, mapping $\alpha(I) \to \mathbf{R}$.

(ii) Let now $\hat{F} = \omega(F)$ be the graph of a function $\hat{f}$. Let $\xi, \eta \in I$ and $\xi \neq \eta$. Since ω is a regular affine mapping, it maps $P = (\xi,\ f(\xi)) \in F$ and $Q = (\eta,\ f(\eta)) \in F$, $P \neq Q$, into two different points $\omega(P) = (\alpha(\xi),\ \beta(\xi)) \in \hat{F}$ and $\omega(Q) = (\alpha(\eta),\ \beta(\eta)) \in \hat{F}$. Two cases are possible: either $\beta(\xi) = \beta(\eta)$, and then, since $\omega(P) \neq \omega(Q)$, it must be $\alpha(\xi) \neq \alpha(\eta)$; or $\beta(\xi) \neq \beta(\eta)$, and in this case again it must be $\alpha(\xi) \neq \alpha(\eta)$ by the fact that $\hat{f}$ is a function. Therefore, the implication $\xi \neq \eta \Rightarrow \alpha(\xi) \neq \alpha(\eta)$ holds in any case. Now, since $\alpha(\xi) - \alpha(\eta) = p(\xi - \eta) + q[f(\xi) - f(\eta)]$ for any f, in order that the inequality $\alpha(\xi) - \alpha(\eta) \neq 0$ be valid whenever $\xi - \eta \neq 0$, it must be $q = 0$. Otherwise, a function f can be found such that $[f(\xi) - f(\eta)]/(\xi - \eta) = -p/q$ and then $\alpha(\xi) - \alpha(\eta) = 0$. □

Lemma 2 suggests that affine transformation of a fractal interpolating function φ can only be performed by means of a mapping ω with $q = 0$, i.e.,

$$\omega : (x, y) \mapsto (\alpha(x),\ \beta(x, y)) = (px + g,\ rx + sy + h)\,. \tag{11}$$

In fact, ω maps the data set Y into the data set $\hat{Y} = \omega(Y) = \{(\hat{x}_i, \hat{y}_i)\}_{i=0}^n$, with $\hat{x}_i = \alpha(x_i)$, $\hat{y}_i = \beta(x_i, y_i)$. Since $\operatorname{sign}(\Delta\hat{x}_i) = \operatorname{sign}(p)\operatorname{sign}(\Delta x_i)$, $\forall i$, coefficients $\hat{a}_i$, $\hat{c}_i$, $\hat{e}_i$, $\hat{f}_i$ can be defined according to (2) for the new data set $\hat{Y}$, and the new fractal interpolating scheme is well defined. So is also the function $\hat{\varphi}$ having graph $\hat{\Phi}$, the attractor of $\Sigma_{\hat{Y},\mathbf{d}}$. Note that the new coefficients are related to the old ones by

$$\begin{aligned} &\hat{a}_i = a_i\,, \quad \hat{c}_i = (r/p)(a_i - d_i) + (s/p)c_i\,, \quad \hat{e}_i = pe_i + g(1 - a_i)\,, \\ &\hat{f}_i = sf_i + re_i + h(1 - d_i) + (rg/p)(d_i - a_i) + (sg/p)c_i, \end{aligned} \tag{12}$$

and by Theorem 2, $\hat{\varphi}$ satisfies a functional equation having the form of (6) with the coefficients (12), namely

$$\hat{\varphi}(x) = \hat{c}_i \frac{x - \hat{e}_i}{\hat{a}_i} + d_i\,\hat{\varphi}\Big(\frac{x - \hat{e}_i}{\hat{a}_i}\Big) + \hat{f}_i\,, \quad x \in [\hat{x}_{i-1}, \hat{x}_i]. \tag{13}$$

The following theorem establishes affine invariance of the fractal interpolatory scheme $\Sigma_{Y,\mathbf{d}}$ under a regular transformation of the type of ω.

Theorem 3. *Let φ be the fractal interpolant of Y associated with $\mathbf{d}$ and let $\Phi = \Phi_{Y,\mathbf{d}}$ be the graph of φ. Let ω be the regular affine mapping given by (11), and let $\hat{Y} = \omega(Y)$. If $\hat{\varphi}$ is the interpolant of $\hat{Y}$ associated with the same $\mathbf{d}$, and $\hat{\Phi} = \Phi_{\omega(Y),\mathbf{d}}$ is its graph, then*

$$\omega(\Phi) \;=\; \hat{\Phi}. \tag{14}$$

Proof. Denoting the interval $[x_0, x_n]$ by I, consider the function $\varphi : I \to \mathbf{R}$, and its graph Φ. By Lemma 2, the set $\omega(\Phi)$ is also the graph of a function, say $f\ :\ \alpha(I) \to \mathbf{R}$, so that any point $P \in \omega(\Phi)$ has coordinates $P = (x, f(x))\,,\ x \in \alpha(I)$. On the other hand, this point is the image under ω of some point $Q = (\alpha^{-1}(x), \varphi(\alpha^{-1}(x))) \in \Phi$, therefore its ordinate must also satisfy

$$f(x) = \beta(\alpha^{-1}(x),\ \varphi(\alpha^{-1}(x))), \quad x \in \alpha(I). \tag{15}$$

So, equation (15) characterizes functions whose graph is the image under ω of the graph of φ. Therefore (14) is equivalent to

$$\hat{\varphi}(x) = \beta(\alpha^{-1}(x),\ \varphi(\alpha^{-1}(x))), \quad x \in \alpha(I). \tag{16}$$

Plugging this equation into (13) yields the functional equation for φ

$$\begin{aligned}\beta(\alpha^{-1}(x),\varphi(\alpha^{-1}(x))) &= \hat{c}_i\frac{x-\hat{e}_i}{\hat{a}_i}\\ &\quad + d_i\beta\Big(\alpha^{-1}(\frac{x-\hat{e}_i}{\hat{a}_i}),\varphi(\alpha^{-1}(\frac{x-\hat{e}_i}{\hat{a}_i}))\Big)+\hat{f}_i,\end{aligned} \tag{17}$$

that is also equivalent to (14).

We will show that (17) reduces to (6). This will lead to the conclusion that (14) holds if and only if φ is a solution of (6), namely (by Theorem 2) the fractal interpolating function for $(Y,\mathbf{d})$. So, our assertion will be proved.

Since any affine transformation is a composition of a translation and a linear transformation, it is sufficient to prove the equivalence for these two special cases of ω, separately:

a) The affine transformation given by (11) is a translation if $p=s=1$, $r=0$, and $g,h\neq 0$.

In this case $\alpha(x)=x+g\,,\quad \beta(x,y)=y+h\,$, and, by (12),

$$\hat{a}_i=a_i\,,\quad \hat{c}_i=c_i\,,\quad \hat{e}_i=e_i+g(1-a_i)\,,\quad \hat{f}_i=f_i+h(1-d_i)-c_ig\,.$$

Therefore, equation (17) takes the form

$$\begin{aligned}\varphi(x-g)+h &= c_i\Big(\frac{x-e_i-g(1-a_i)}{a_i}\Big)+d_i\varphi\Big(\frac{x-e_i-g(1-a_i)}{a_i}-g\Big)\\ &\quad + d_ih+f_i+h-c_ig-d_ih\,,\end{aligned}$$

which, by easy computations, becomes

$$\varphi(x-g)=c_i\Big(\frac{(x-g)-e_i}{a_i}\Big)+d_i\varphi\Big(\frac{(x-g)-e_i}{a_i}\Big)+f_i\,,$$

which, in turn, by the position $t=x-g$ yields

$$\varphi(t)=c_i\Big(\frac{t-e_i}{a_i}\Big)+d_i\varphi\Big(\frac{t-e_i}{a_i}\Big)+f_i\,,$$

that is (6).

b) Transformation ω in (11) is a linear map if $g=h=0$ and $ps\neq 0$.

In this case $\alpha(x) = p\,x$, $\beta(x,y) = r\,x + s\,y$, and

$$\hat{a}_i = a_i\,,\quad \hat{c}_i = (r/p)(a_i - d_i) + (s/p)c_i\,,\quad \hat{e}_i = pe_i\,,\quad \hat{f}_i = sf_i + re_i\,,$$

so (17) becomes

$$rx/p + s\varphi(x/p) = sc_i\Big(\frac{x/p - e_i}{a_i}\Big) + sd_i\varphi\Big(\frac{x/p - e_i}{a_i}\Big) + r(a_i - d_i)\Big(\frac{x - pe_i}{pa_i}\Big) + rd_i\Big(\frac{x - pe_i}{pa_i}\Big) + re_i + sf_i\,,$$

which gives

$$s\varphi(x/p) = sc_i\Big(\frac{x/p - e_i}{a_i}\Big) + sd_i\varphi\Big(\frac{x/p - e_i}{a_i}\Big) + ra_i\Big(\frac{x - pe_i}{pa_i}\Big) + re_i + sf_i - (rx)/p\,,$$

and finally

$$\varphi(x/p) = c_i\Big(\frac{x/p - e_i}{a_i}\Big) + d_i\varphi\Big(\frac{x/p - e_i}{a_i}\Big) + f_i\,,$$

which again yields (6) by the variable transformation $t = x/p$. $\square$

Remark 2. For another proof of (14), see [6]. For monotonicity preserving property of φ, see [4].

Application and Examples

It was suggested in [2, Chap. 6] that the integral of the interpolating fractal function φ satisfying equation (8) can be calculated by

$$I = \int_{x_0}^{x_n} \varphi(x)dx = \int_{x_0}^{x_n} (T\varphi)(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (T_i\varphi)(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \Big[c_i\frac{x - e_i}{a_i} + d_i\varphi\Big(\frac{x - e_i}{a_i}\Big) + f_i\Big]dx,$$

which, by the substitution $x = a_i t + e_i$ becomes

$$I = \sum_{i=1}^{n} \int_{x_0}^{x_n} \big(c_i t + d_i\varphi(t) + f_i\big)a_i dt = \alpha I + \beta, \tag{18}$$

where

$$\alpha = \sum_{i=1}^{n} a_i d_i, \quad \beta = \sum_{i=1}^{n} a_i \int_{x_0}^{x_n} (c_i t + f_i) dt. \tag{19}$$

And it follows from (18) that

$$I = \frac{\beta}{1-\alpha}. \tag{20}$$

Also in [2, p. 221], Barnsley suggests that β equals $\int_{x_0}^{x_n} \varphi_0(x)dx$ where

$$\varphi_0(x) = y_i + \frac{\Delta y_i}{\Delta x_i}(x - x_i)\,, \quad x_i \le x \le x_{i+1}\,, \quad i = 0, 1, \ldots, n-1\,, \tag{21}$$

is the piecewise linear interpolant to the data Y. But, by (19),

$$\begin{aligned}\beta = \sum_{i=1}^{n} a_i \int_{x_0}^{x_n} (c_i t + f_i)dt &= \sum_{i=1}^{n} a_i \left(c_i \frac{x_n^2 - x_0^2}{2} + f_i(x_n - x_0) \right) \\ &= \sum_{i=1}^{n} \frac{\Delta x_{i-1}}{2} \big[c_i(x_n + x_0) + 2 f_i \big],\end{aligned}$$

which, after replacing a_i, c_i and f_i by their expression (2) gives

$$\beta = \sum_{i=1}^{n} \frac{\Delta x_{i-1}}{2} \big[y_{i-1} + y_i - d_i(y_0 + y_n) \big],$$

or

$$\beta = \sum_{i=1}^{n} \frac{y_{i-1} + y_i}{2} \Delta x_{i-1} - \frac{y_0 + y_n}{2} \sum_{i=1}^{n} d_i \Delta x_{i-1}\,. \tag{22}$$

Note that the first term on the right hand side of (22) is, by itself, the value of $\int_{x_0}^{x_n} \varphi_0(x)dx$. Therefore $\beta = \int_{x_0}^{x_n} \varphi_0(x)dx$ is valid if and only if

$$y_0 + y_n = 0\,, \tag{23}$$

or

$$d_i = 0\,, \quad i = 1, 2, \ldots, n\,. \tag{24}$$

Example 1. The *Cantor function* $x \mapsto f(x)$ is the interpolating fractal function that is defined by the data $Y = \{(0,0), (1/3,1/2), (2/3,1/2), (1,1)\}$ and the scaling vector $\mathbf{d} = [1/2\ 0\ 1/2]^T$.

Thus, by (19), $\alpha = 1/3$ and by (22) $\beta = 1/3$, which gives, by (20), $I = \int_0^1 f(x)dx = 1/2$, which is the known result [3]. Here, neither condition (23) nor (24) is satisfied, and in fact β differs from $\int_0^1 \varphi_0(x)dx = 1/2$, where φ_0 is the piecewise linear interpolant to Y.

Direct computation, based on (18) and on the functional equation for Cantor function

$$f(x) = \begin{cases} \dfrac{1}{2} f(3x)\,, & 0 \le x \le \dfrac{1}{3}\,, \\ \dfrac{1}{2}\,, & \dfrac{1}{3} \le x \le \dfrac{2}{3}\,, \\ \dfrac{1}{2} f(3x-2) + \dfrac{1}{2}\,, & \dfrac{2}{3} \le x \le 1\,. \end{cases}$$

also gives

$$I = \int_0^1 \varphi_0(x)dx = \int_0^1 \frac{1}{2} f(t)\frac{dt}{3} + \int_0^1 \frac{1}{2}\frac{dt}{3} + \int_0^1 \left(\frac{1}{2}f(t) + \frac{1}{2}\right)\frac{dt}{3},$$

i.e., $I = \dfrac{1}{3}I + \dfrac{1}{3}$, which yields $I = 1/2$.

Example 2. Consider the functional equation of type (6)

$$f(x) = \begin{cases} \lambda f(2x) + x\,, & 0 \le x \le 1/2\,, \\ \mu f(2x-1) - x + 1\,, & 1/2 \le x \le 1\,, \end{cases} \tag{25}$$

where $|\lambda| < 1$, $|\mu| < 1$.

Comparing (25) with (6) immediately gives $d_1 = \lambda$, $d_2 = \mu$ and $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$. Letting $x = x_i$, $i = 0,1,2$ in (25) results in the following linear system:

$$\begin{aligned} f(0) &= \lambda f(0), \\ f(1/2) &= \lambda f(1) + 1/2, \\ f(1/2) &= \mu f(0) + 1/2, \\ f(1) &= \mu f(1), \end{aligned}$$

wherefrom it follows $f(0) = f(1) = 0$ and $f(1/2) = 1/2$. Thus, the interpolating data set is $Y = \{(0,0),\ (1/2,1/2),\ (1,0)\}$, and therefore, $a_1 = a_2 = 1/2$ which gives (by (19)) $\alpha = (\lambda + \mu)/2$ and (by (22)) $\beta = 1/4$.

Thus,

$$I = \frac{1}{2(2-\lambda-\mu)}.$$

Note that the condition (23) is valid and, accordingly, $\beta = \int_0^1 f_0(x)dx$, where f_0 is a "tent function" which interpolates the data Y.

It is interesting to note that for $\lambda = \mu = 1/4$ the interpolant given by (25) is in fact a smooth function, namely the quadratic polynomial $f(x) = 2\,x\,(1-x)$. For other smooth fractal objects and their applications, see [5].

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