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ON THE LOG–QUADRATIC FUNCTIONAL EQUATION

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Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. Our attention was drawn on log–quadratic functional equation mentioned by Hiroshi Haruki and Themistocles M. Rassias in [4]:

$$f\colon \mathbb{C} \to \mathbb{C}\ , \quad f(x+y)f(x-y) = f(x)^2 f(y)^2\ , \quad x, y \in \mathbb{C}\ .$$

They stated the following result: *The only entire solutions of this equation are given by $f(z) \equiv 0$, $f(z) = e^{az^2}$ and $f(z) = -e^{az^2}$, where a is an arbitrary complex constant.* In our paper, using some *elementary methods* we consider the log–quadratic functional equation for functions $f\colon \mathbb{R} \to \mathbb{C}$, supposing only that f is continuous. The expected solutions are evidently $f(x) \equiv 0$, $f(x) = e^{(\alpha+i\beta)x^2}$ and $f(x) = -e^{(\alpha+i\beta)x^2}$, but we will determine these solutions independently of the above mentioned result.

1. We start with the solution of

$$f\colon \mathbb{C} \to \mathbb{C}\ , \quad f(x+y)f(x-y) = f(x)^2 f(y)^2\ , \quad x, y \in \mathbb{C}\ , \tag{1}$$

for real functions of a real variable, $f\colon \mathbb{R} \to \mathbb{R}$, under the assumption that f is continuous:

$$f\colon \mathbb{R} \to \mathbb{R}\ , \quad f(x+y)f(x-y) = f(x)^2 f(y)^2\ , \ x, y \in \mathbb{R}\ . \tag{1'}$$

With $x = y = 0$ we obtain $f(0) = 0$, $f(0) = 1$, $f(0) = -1$. The case $f(0) = 0$ leads to $f(x) \equiv 0$ (putting $y = 0$ in (1′)). In the case $f(0) = 1$ we

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shall first show that $f(x) > 0$, $x \in \mathbb{R}$. Suppose that there exists x_0 such that $f(x_0) = 0$, with $x = y = x_0/2$, we obtain $f(x_0/2) = 0$, and repeating the procedure we reach to $f(x_0/2^n) = 0$. By continuity we have a contradiction. Hence $f(0) = 1$, it follows $f(x) > 0$ for each $x \in \mathbb{R}$. Now we can apply the log and then we obtain the quadratic functional equation

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + 2\varphi(y)$$

for $\varphi(x) := \log f(x)$, whose continuous (nontrivial) solution $\varphi(x) = ax^2$ (a is an arbitrary real constant) is known. But we will determine the continuous solution of $(1')$ in the case $f(0) = 1$ directly from this equation without using the log-process. With $x = y = t$ we have $f(2t) = f(t)^4$. With $x = 2t$, $y = t$ we have $f(3t)f(t) = f(2t)^2 f(t)^2$, $f(3t) = f(2t)^2 f(t)$, because $f(t) > 0$. Therefore, $f(3t) = f(t)^9$ and so on, we obtain $f(nx) = f(x)^{n^2}$. With $x = 1/n$, we have $f(1/n) = f(1)^{1/n^2}$, because $f(1) > 0$. Finally, $f(m/n) = f(1)^{(m/n)^2}$ and we obtained the values of unknown function on a dense set (see [1]). By continuity, $f(x) = b^{x^2}$, $b = f(1) > 0$, $f(x) = e^{ax^2}$, where a is an arbitrary real constant. The case $f(0) = -1$ can be reduced to the previous case with $g(x) = -f(x)$ also satisfying $(1')$.

2. Now we consider the functional equation for a continuous function $f\colon \mathbb{R} \to \mathbb{C}$,

$$f(x+y)f(x-y) = f(x)^2 f(y)^2\,, \quad x, y \in \mathbb{R}\,. \tag{1''}$$

With $f(x) = u(x) + i\,v(x)$ we obtain the system of equations

$$\left\{\begin{aligned} & u(x+y)u(x-y) - v(x+y)v(x-y) \\ & \quad = \big(u(x)^2 - v(x)^2\big)\big(u(y)^2 - v(y)^2\big) - 4u(x)v(x)u(y)v(y)\,, \\ & u(x+y)v(x-y) + u(x-y)v(x+y) \\ & \quad = 2u(x)v(x)\big(u(y)^2 - v(y)^2\big) + 2u(y)v(y)\big(u(x)^2 - v(x)^2\big)\,, \end{aligned}\right. \tag{2}$$

where $u, v\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. With $x = y = 0$ we obtain

$$\left\{\begin{aligned} u(0)^2 - v(0)^2 &= \big(u(0)^2 - v(0)^2\big)^2 - 4u(0)^2 v(0)^2\,, \\ 2u(0)v(0) &= 4u(0)v(0)\big(u(0)^2 - v(0)^2\big)\,. \end{aligned}\right. \tag{3}$$

The real solutions of (3) are

$$u(0) = 0\,,\ v(0) = 0\,; \quad u(0) = 1\,,\ v(0) = 0\,; \quad u(0) = -1\,,\ v(0) = 0\,.$$

The first solution of (3) leads to the trivial solution of (2), i.e. $u(x) \equiv 0$, $v(x) \equiv 0$, and therefore $f(x) \equiv 0$ for $(1'')$. By squaring and addition in (2) we find

$$\begin{aligned}[u(x+y)^2 + v(x+y)^2]\,[u(x-y)^2 + v(x-y)^2] \\ = [u(x)^2 + v(x)^2]^2\,[u(y)^2 + v(y)^2]^2\,.\end{aligned}$$

Therefore, with $g(x) = u(x)^2 + v(x)^2$ we obtain the functional equation $(1')$, $g(0) = 1$, in the both cases $u(0) = 1$, $u(0) = -1$.

Consequently, $u(x)^2 + v(x)^2 = e^{2\alpha x^2}$,

$$\left(\frac{u(x)}{e^{\alpha x^2}}\right)^2 + \left(\frac{v(x)}{e^{\alpha x^2}}\right)^2 = 1\,.$$

We put now $u(x) = e^{\alpha x^2}\cos\omega(x)$, $v(x) = e^{\alpha x^2}\sin\omega(x)$ and substitute them in the first equation of (2). In this way we get

$$\omega(x+y) + \omega(x-y) = 2k\pi \pm \big(2\omega(x) + 2\omega(y)\big)\,.$$

In different cases we have:

“ + ”: $\omega(x) = \beta x^2 - k\pi\,, \quad k \in \mathbb{Z}$,

“ − ”: $\omega(x) = k\pi/3\,, \quad k \in \mathbb{Z}$.

“ + ”: It leads to $u(x) = e^{\alpha x^2}\cos(\beta x^2 - k\pi)\,,\ v(x) = e^{\alpha x^2}\sin(\beta x^2 - k\pi)\,,$

$$f(x) = \pm e^{(\alpha + i\,\beta)x^2}\,. \tag{4}$$

“ − ”: It leads to $k = 3q$, $q \in \mathbb{Z}$, and therefore $f(x) = \pm e^{\alpha x^2}$. That is a particular case of (4) with $\beta = 0$.

Remark 1. From $(1'')$ we get

$$\big|f(x+y)\big|\,\big|f(x-y)\big| = \big|f(x)\big|^2\,\big|f(y)\big|^2\,, \tag{5}$$

which represents the equation $(1')$ for nonnegative functions. The solutions of (5) are $\big|f(x)\big| \equiv 0$, $\big|f(x)\big| = e^{ax^2}$, where a is an arbitrary real constant.

With $f(x) = \big|f(x)\big|e^{i\,\arg f(x)}$ we obtain

$$\begin{aligned}\arg f(x+y) + \arg f(x-y) = 2\arg f(x) + 2\arg f(y) + 2k\pi\,, \\ \arg f(x) = bx^2 - k\pi\,, \quad f(x) = \pm e^{(a+i\,b)x^2}\,.\end{aligned}$$

In this way we avoid the second case of the functional equation for $\omega(x)$ (“ $-$ ”).

Therefore, the continuous nontrivial solution of (1) for the complex functions of real variable (equation (1″)) is $f(x) = \pm e^{Ax^2}$, A is an arbitrary complex constant, $A = a + i\,b$, $a, b \in \mathbb{R}$. This solution was expected in view of the above cited result for (1), but here we have obtained this result independently using elementary methods and functional equation techniques.

3. A new situation occurs in the case $f\colon \mathbb{R} \to \mathcal{A}$, where $\mathcal{A}$ is a finite–dimensional real algebra with other elements satisfying the equation

$$f(0)^2 = f(0)^4 \tag{6}$$

besides the zero and the unity. In this case we have besides the trivial solution (corresponding to the solution $f(0) = \mathbf{0}$ of the equation (6)) and a “regular” solution (corresponding to the root $f(0) = \mathbf{1}$ of (6)), also other solutions corresponding to the other roots of (6), named “singular solutions” (the norm of these solutions are equal to the zero). For example, let $\mathcal{A}_3$ be an algebra of the real square matrices of the form

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

(a subalgebra of the complete algebra of square matrices of third order $\mathcal{M}_3(\mathbb{R})$). This algebra is isomorphic with the algebra of “hypercomplex” numbers $a + \theta b + \theta^2 c$, $\theta^3 = 1$, used by Lagrange in the solution of the algebraic equation of third degree (see [6]).

Now, we consider the log–quadratic functional equation for $f\colon \mathbb{R} \to \mathcal{A}_3$, $f(x) = a(x) + \theta b(x) + \theta^2 c(x)$, ($a, b, c\colon \mathbb{R} \to \mathbb{R}$ are continuous),

$$f(x+y)f(x-y) = f(x)^2 f(y)^2\ , \quad f\colon \mathbb{R} \to \mathcal{A}_3\,. \tag{1‴}$$

As in the previous cases we have

$$f(0)^2 = f(0)^4\,, \tag{7}$$

but now we have seven distinct solutions of this equation

$$j_1 = \mathbf{0}\ ,\ j_{2,3} = \pm\mathbf{1}\ ,\ j_{4,5} = \pm\frac{1}{3}(1+\theta+\theta^2)\ ,\ j_{6,7} = \pm\Big(\frac{2}{3} - \frac{1}{3}\theta - \frac{1}{3}\theta^2\Big)\,.$$

Multiply $(1''')$ with $1+\theta+\theta^2$. It gives

$$\begin{aligned}&\big(a(x+y)+b(x+y)+c(x+y)\big)\cdot\big(a(x-y)+b(x-y)+c(x-y)\big)\\&=\big(a(x)+b(x)+c(x)\big)^2\cdot\big(a(y)+b(y)+c(y)\big)^2,\end{aligned}$$

i.e. (6):

$$a(x)+b(x)+c(x)=\begin{cases}0\,,\\ \pm e^{\gamma x^2}\,.\end{cases}$$

Denoting $\mathcal{A}_3^0=\big\{a+\theta b-(a+b)\theta^2\ \big|\ a,b\in\mathbb{R}\big\}$, we can establish an isomorphism between $\mathcal{A}_3^0$ and $\mathbb{C}$ considered it as a real linear two–dimensional algebra by

$$a+\theta b-(a+b)\theta^2\to\frac{3}{2}a+i\,\frac{\sqrt{3}}{2}(a+2b)\,.$$

In view of this isomorphism the equation $(1''')$ in the case $f(0)=\pm\frac{1}{3}(2-\theta-\theta^2)$ reduces to

$$\begin{aligned}&\Big[\frac{3}{2}a(x+y)+i\,\frac{\sqrt{3}}{2}\big(a(x+y)+2b(x+y)\big)\Big]\\&\cdot\Big[\frac{3}{2}a(x-y)+i\,\frac{\sqrt{3}}{2}\big(a(x-y)+2b(x-y)\big)\Big]\\&=\Big[\frac{3}{2}a(x)+i\,\frac{\sqrt{3}}{2}\big(a(x)+2b(x)\big)\Big]^2\Big[\frac{3}{2}a(y)+i\,\frac{\sqrt{3}}{2}\big(a(y)+2b(y)\big)\Big]^2.\end{aligned}$$

Therefore

$$\frac{3}{2}a(x)+i\,\frac{\sqrt{3}}{2}\big(a(x)+2b(x)\big)=\pm e^{(\alpha+i\,\beta)x^2}\,,$$

$$\text{(8)}\qquad\begin{cases}a(x)&=\pm\frac{2}{3}e^{\alpha x^2}\cos\beta x^2\,,\\ b(x)&=\mp\frac{1}{3}e^{\alpha x^2}\cos\beta x^2\pm\frac{1}{\sqrt{3}}e^{\alpha x^2}\sin\beta x^2\,,\\ c(x)&=-\big(a(x)+b(x)\big)\,.\end{cases}$$

This is a solution corresponding to $f(0)=\pm\frac{1}{3}(2-\theta-\theta^2)$.

We denote $\mathcal{A}_3^1=\big\{a+\theta b+\theta^2(\pm e^{\gamma x_1^2}-a-b)\ \big|\ a,b\in\mathbb{R}\big\}$, where γ is an arbitrary real constant, and x_1 is fixed if a and b are given.

The mapping

$$a+\theta b+\theta^2(\pm e^{\gamma x_1^2}-a-b)\ \rightarrow\ \frac{3}{2}a\mp\frac{1}{2}e^{\gamma x_1^2}+i\,\frac{\sqrt{3}}{2}(a+2b\mp e^{\gamma x_1^2})$$

represents an isomorphism between $\mathcal{A}_3^1$ and $\mathbb{C}$. Based on this isomorphism we obtain

$$\frac{3}{2}a(x)\mp\frac{1}{2}e^{\gamma x^2}+i\,\frac{\sqrt{3}}{2}\big(a(x)+2b(x)\mp e^{\gamma x^2}\big)=\pm e^{(\alpha+i\,\beta)x^2}\,.$$

Therefore

$$\text{(9)}\qquad \begin{cases} a(x) &= \pm\frac{1}{3}e^{\gamma x^2}\pm\frac{2}{3}e^{\alpha x^2}\cos\beta x^2\,,\\ b(x) &= \pm\frac{1}{3}e^{\gamma x^2}\mp\frac{1}{3}e^{\alpha x^2}\cos\beta x^2\pm\frac{1}{\sqrt{3}}e^{\alpha x^2}\sin\beta x^2\,,\\ c(x) &= \pm e^{\gamma x^2}-\big(a(x)+b(x)\big)\,.\end{cases}$$

This is a solution of $(1''')$ corresponding to $f(0)=\pm\mathbf{1}$. In the case $f(0)=\pm\frac{1}{3}(1+\theta+\theta^2)$ from the relation $f(x)^2=f(x)^2f(0)^2$ (obtained with $y=0$) it gives

$$\begin{cases} a(x)^2+2b(x)c(x) &= \frac{1}{3}\big(a(x)+b(x)+c(x)\big)^2\,,\\ c(x)^2+2a(x)b(x) &= \frac{1}{3}\big(a(x)+b(x)+c(x)\big)^2\,,\\ b(x)^2+2a(x)c(x) &= \frac{1}{3}\big(a(x)+b(x)+c(x)\big)^2\,,\end{cases}$$

consequently $a(x)=b(x)=c(x)$. Then we have $f(x)=a(x)(1+\theta+\theta^2)$ and

$$3a(x+y)a(x-y)=27a(x)^2a(y)^2\,.$$

Hence

$$\text{(10)}\qquad a(x)=b(x)=c(x)=\pm\frac{1}{3}e^{\gamma x^2}\,.$$

Thus, this is the solution corresponding to $f(0)=\pm\frac{1}{3}(1+\theta+\theta^2)$.

4. Following Hille–Phillips book [5] the continuous solutions of

$$f\colon\mathbb{R}\rightarrow\mathcal{A}\,,\quad f(x+y)=f(x)f(y)\,,$$

are given by

$$f(x) = j + \sum_{n=1}^{+\infty} \frac{x^n}{n!} A^n \,, \tag{11}$$

where $\mathcal{A}$ is Banach algebra, j is an idempotent of $\mathcal{A}$ and A is a constant from this algebra such that $Aj = jA = A$.

We will show that the solutions of $(1''')$ can be represented in the form given by (11), putting x^2 instead of x.

For the solution (10) we have $A = \delta(1+\theta+\theta^2)$, where δ is a real constant, $(1+\theta+\theta^2)^n = 3^{n-1}(1+\theta+\theta^2)$. Substituting in (11) we get

$$\begin{aligned} f(x) &= \frac{1}{3}(1+\theta+\theta^2) + \Big[\sum_{n=1}^{\infty} \frac{(x^2)^n}{n!} 3^{n-1}(1+\theta+\theta^2)\delta^n\Big] \\ &= \frac{1}{3}(1+\theta+\theta^2)\Big[1 + \sum_{n=1}^{\infty} \frac{(3\delta)^n x^{2n}}{n!}\Big] = \frac{1}{3}(1+\theta+\theta^2)e^{3\delta x^2} \,. \end{aligned}$$

With $3\delta = \gamma$ we obtain (10). (The function $-f(x)$ also verifies the equation $(1''')$.

In what concerns the solution (8) we have $A = \delta + \theta\varepsilon - \theta^2(\delta+\varepsilon)$. Hence

$$\begin{aligned} f(x) &= \frac{2}{3} - \frac{1}{3}\theta - \frac{1}{3}\theta^2 + \big(\delta + \varepsilon\theta - \theta^2(\delta+\varepsilon)\big)x^2 \\ &\quad + \frac{1}{2}\big(\delta + \varepsilon\theta - (\delta+\varepsilon)\theta^2\big)^2 x^4 + \frac{1}{6}\big(\delta + \varepsilon\theta - (\delta+\varepsilon)\theta^2\big)^3 x^6 + \cdots \,, \end{aligned}$$

i.e.

$$\begin{aligned} f(x) = \Big[&\frac{2}{3} + \delta x^2 + \frac{1}{2}(\delta^2 - 2\delta\varepsilon - 2\varepsilon^2)x^4 \\ &+ \frac{1}{6}\big(\delta^3 + \varepsilon^3 - 6\delta\varepsilon(\delta+\varepsilon) - (\delta+\varepsilon)^3\big)x^6 + \cdots\Big] \\ &+ \theta\Big[-\frac{1}{3} + \varepsilon x^2 + \frac{1}{2}(\delta^2 + 4\delta\varepsilon + \varepsilon^2)x^4 + \cdots\Big] \\ &- \theta^2\Big[\frac{1}{3} + (\delta+\varepsilon)x^2 + \frac{1}{2}(2\delta^2 + 2\delta\varepsilon - \varepsilon^2)x^4 + \cdots\Big] \,. \end{aligned}$$

Putting $\delta = \frac{2}{3}\alpha$, $\varepsilon + \frac{1}{2}\delta = \frac{\beta}{\sqrt{3}}$, $\varepsilon = -\frac{1}{3}\alpha + \frac{\beta}{\sqrt{3}}$, we have

$$\begin{aligned}
f(x) &= \frac{2}{3} + \frac{2}{3}\alpha x^2 + \frac{1}{3}(\alpha^2 - \beta^2)x^4 - \frac{3}{2}(-\frac{2}{27}\alpha^3 + \frac{2}{9}\beta^2\alpha)x^6 + \cdots \\
&\quad + \theta\big[-\frac{1}{3} + \varepsilon x^2 + \frac{1}{2}(\delta^2 + 4\delta\varepsilon + \varepsilon^2)x^4 + \frac{1}{2}(\delta^3 - \varepsilon^3 + 3\delta^2\varepsilon)x^6 + \cdots\big] \\
&\quad - \theta^2\big[\frac{1}{3} + (\delta + \varepsilon)x^2 + \frac{1}{2}(2\delta^2 + 2\delta\varepsilon - \varepsilon^2)x^4 \\
&\quad + \frac{1}{2}(\delta^3 - \varepsilon^3 - 3\delta^2\varepsilon)x^6 + \cdots\big]\,,
\end{aligned}$$

i.e.

$$f(x) = \frac{2}{3} + \frac{2}{3}\alpha x^2 + \frac{1}{3}(\alpha^2 - \beta^2)x^4 + (\frac{1}{9}\alpha^3 - \frac{1}{3}\alpha\beta^2)x^6 + \cdots + \theta(\dots) + \theta^2(\dots)$$

and

$$\begin{aligned}
a(x) &= \frac{2}{3} + \frac{2}{3}\alpha x^2 + \frac{1}{3}(\alpha^2 - \beta^2)x^4 + (\frac{1}{9}\alpha^3 - \frac{1}{3}\alpha\beta^2)x^6 + \cdots \\
&= \frac{2}{3}\big(1 + \alpha x^2 + \frac{1}{2}(\alpha^2 - \beta^2)x^4 + (\frac{1}{6}\alpha^3 - \frac{1}{2}\alpha\beta^2)x^6 + \cdots\big)\,,
\end{aligned}$$

i.e.

$$a(x) = \frac{2}{3}e^{\alpha x^2}\cos\beta x^2\,.$$

After some calculations we obtain (8) (taking into account that $-f$ is also solution of $(1''')$).

In the case of "regular solution" (9) corresponds to $f(0) = \mathbf{1}$ (unit element of algebra), the condition $Aj = jA = A$ from (11) left the constant A arbitrary

$$A = \delta + \theta\varepsilon + \omega\theta^2\,.$$

We have

$$\begin{aligned}
A^2 &= \delta^2 + 2\varepsilon\omega + \theta(\omega^2 + 2\delta\varepsilon) + \theta^2(\varepsilon^2 + 2\delta\omega)\,, \\
A^3 &= \delta^3 + \varepsilon^3 + \omega^3 + 6\delta\varepsilon\omega + 3\theta(\varepsilon^2\omega + \delta^2\varepsilon + \omega^2\delta) + 3\theta^2(\varepsilon^2\delta + \delta^2\omega + \omega^2\varepsilon)\,,
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= 1 + \delta x^2 + \frac{1}{2}(\delta^2 + 2\varepsilon\omega)x^4 + \frac{1}{6}(\delta^3 + \varepsilon^3 + \omega^3 + 6\delta\varepsilon\omega)x^6 \\
&\quad + 3\theta(\varepsilon^2\omega + \delta^2\varepsilon + \omega^2\delta) + 3\theta^2(\varepsilon^2\delta + \delta^2\omega + \omega^2\varepsilon)\,.
\end{aligned}$$

Putting

$$\begin{cases} \delta & = \frac{2}{3}\alpha + \frac{1}{3}\gamma\,, \\ \varepsilon & = \frac{1}{3}\gamma - \frac{1}{3}\alpha + \frac{1}{\sqrt{3}}\beta\,, \\ \omega & = -\frac{1}{3}\alpha - \frac{1}{\sqrt{3}}\beta + \frac{1}{3}\gamma\,, \end{cases}$$

we obtain

$$\begin{aligned} a(x) &= 1 + \left(\frac{2}{3}\alpha + \frac{1}{3}\gamma\right)x^2 + \left(\frac{1}{3}\alpha^2 - \frac{1}{3}\beta^2 + \frac{1}{6}\gamma^2\right)x^4 + \cdots \\ &= \frac{1}{3}e^{\gamma x^2} + \frac{2}{3}e^{\alpha x^2}\cos\beta x^2\,. \end{aligned}$$

Finally, with some calculations we obtain that the solutions of $(1''')$ can be represented in the Hille–Phillips form with x^2 instead of x. (Further we take into account that $-f$ is also solution of $(1''')$).

Remark 2. One can directly verify that the Hille–Phillips series

$$f(x) = j + \sum_{n=1}^{+\infty} \frac{A^n}{n!}x^{2n}$$

satisfies the log–quadratic functional equation

$$f(x+y)f(x-y) = f(x)^2 f(y)^2$$

for functions $f\colon \mathbb{R} \to \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra.

In this paper we have obtained via functional equation elementary techniques the continuous solution of a system of functional equations which cannot easily be obtained in finite terms from the above general result.

Remark 3. In the case of $\mathcal{A}_3$ the solutions of the log–quadratic functional equation $(1''')$ can be expressed in finite terms with aid of so called functions of P. Appell (see [2], [3]).

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