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MULTIPLIERS FROM $H^1(U)$ INTO $BMOA(B)$

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Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. In this paper we characterize multipliers from the Hardy space H^1 on the unit disc U into the space $BMOA$ of analytic functions of bounded mean oscillation on the unit ball B in $\mathbb{C}^n$, $n > 1$.

1. Introduction

Let $B = B_n$ be the open unit ball in $\mathbb{C}^n$, $n \geq 1$ ($U = B_1$ is the open unit disc in $\mathbb{C}$) and ν Lebesgue measure normalized so that $\nu(B) = 1$, while σ is the normalized surface measure on the boundary S of B.

For a measurable function f on B and $0 < r < 1$:

$$f_r(z) = f(rz)\,, \quad z \in B\,,$$

$$M_\infty(r, f) = \sup\Big\{|f_r(\xi)| \,:\, \xi \in S\Big\}\,;$$

$$M_p(r, f) = \left(\int_S |f_r(\xi)|^p d\sigma(\xi)\right)^{1/p}, \quad 0 < p < +\infty\,.$$

A function f holomorphic on B, $f \in H(B)$, is said to belong to the Hardy space $H^p = H^p(B)$ if

$$\|f\|_{H^p} = \sup_r M_p(r, f) < +\infty.$$

We write $BMOA$ for the space consisting of functions in H^2 of bounded mean oscillation.

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For a holomorphic function f on B with homogeneous expansion $f = \sum\limits_{k=0}^{+\infty} f_k$ the radial fractional derivative of order $\beta > 0$ is defined by

$$D^\beta f(z) = \sum_{k=0}^{+\infty} (k+1)^\beta f_k(z).$$

(Thus, for $\beta = 1$, $D^1 f = f + Rf$, where R denotes the radial derivative operator defined by $Rf(z) = \sum\limits_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$.)

A function $f \in H(B)$ belongs to the Bloch space $\mathcal{B} = \mathcal{B}(B)$ if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_r (1-r) M_\infty(r, D^1 f) < +\infty.$$

If $g(z) = \sum\limits_{k=0}^{+\infty} g_k(z)$ is a function holomorphic on the unit ball B and $f(z) = \sum\limits_{k=0}^{+\infty} \widehat{f}_k z^k$ is a function holomorphic on the unit disc U, we define their convolution as

$$(g * f)(z) = \sum_{k=0}^{+\infty} \widehat{f}_k g_k(z).$$

If X is a space of analytic functions in the unit disc U, Y a space of analytic functions in B, then by (X, Y) we define the space of functions $g \in H(B)$, such that $g * f \in Y$ for every $f \in X$.

Theorem 1.1. *Equation* $(H^1(U), BMOA(B)) = \mathcal{B}(B)$ *holds.*

Our work was motivated by the paper [5] where it is shown that $(H^1(U), BMOA(U)) = \mathcal{B}(U)$.

2. Preliminaries

We begin with a Hardy–Littlewood type theorem.

Theorem 2.1. *Following assertions hold:*

a) *If* $f \in H^p$, $0 < p \le 2$, *then* $\int_0^1 (1-r) M_p(r, D^1 f)^2 dr < +\infty$;

b) *If* $f \in H(B)$ *and* $\int_0^1 (1-r)M_p(r,D^1)^2 dr < +\infty$*, where* $2 \le p < +\infty$*, then* $f \in H^p$*.*

Proof. For a function $f \in H(B)$ let

$$g(f)(\eta) = \left\{ \int_0^1 |D^1 f(r\eta)|^2 (1-r)dr \right\}^{1/2}, \quad \eta \in S,$$

be the Littlewood–Paley function of f. It is well–known that a function $f \in H(B)$ belongs to the Hardy space H^p, $0 < p < +\infty$, if and only if $g(f) \in L^p(\sigma)$ (see [1]).

An application of Minkowski's inequality (continuous form) shows that

$$\left(\int_0^1 (1-r)\big[M_p(r,D^1 f)\big]^2 dr \right)^{1/2} \le \left(\int_S \big[g(f)(\eta)\big]^p d\sigma(\eta) \right)^{1/p}, \quad 0 < p \le 2,$$

and

$$\left(\int_S \big[g(f)(\eta)\big]^p d\sigma(\eta) \right)^{1/p} \le \left(\int_0^1 (1-r)\big[M_p(r,D^1 f)\big]^2 dr \right)^{1/2}, \ 2 \le p < +\infty.$$

Now, theorem follows from this. $\square$

The following theorem may be viewed as a limiting case of the previous one (part (b)).

Theorem 2.2. *Let* $f \in H(B)$*. If* $\int_0^1 (1-r)M_\infty(r,D^1 f)^2 dr < +\infty$*, then* $f \in BMOA(B)$*.*

Proof. Recall that a positive measure μ on B is called a Carleson measure if

$$\mu(Q(\xi,\delta)) \le C\delta^n, \quad \xi \in S, \quad \delta > 0,$$

where $Q(\xi,\delta) = \{z \in B : |1 - z\overline{\xi}| < \delta\}$ are nonisotropic balls. (Here and elsewhere constants are denoted by C, which may indicate a different constant from one occurrence to the next.)

It is easy to see that a positive measure μ on B is a Carleson measure if and only if

$$\sup_{z\in B} \int_B \frac{(1-|z|^2)^n}{|1-z\overline{w}|^{2n}} d\mu(w) < +\infty.$$

On the other hand, in [4] it is shown that a holomorphic function $f \in BMOA$ if and only if a measure $d\mu(w) = (1-|w|^2)|D^1 f(w)|^2 d\nu(w)$ is a Carleson measure. Thus, to prove Theorem 2.2 it suffices to show that

$$\sup_{z\in B} \int_B \frac{(1-|z|^2)^n(1-|w|^2)|D^1 f(w)|^2}{|1-z\overline{w}|^{2n}} d\nu(w) < +\infty\,.$$

This follows from the following estimates

$$\begin{aligned}
&\int_B \frac{(1-|z|^2)^n}{|1-z\overline{w}|^{2n}}(1-|w|^2)|D^1 f(w)|^2 d\nu(w)\\
&\qquad \le C\int_0^1 (1-|z|^2)^n(1-\rho^2)M_\infty(\rho, D^1 f)^2 d\rho \cdot \int_S \frac{d\sigma(\xi)}{|1-z\rho\overline{\xi}|^{2n}}\\
&\qquad \le C\int_0^1 (1-\rho)M_\infty(\rho, D^1)^2 d\rho < +\infty\,.
\end{aligned}$$

Here, we use the standard estimate (see [6])

$$\int_S \frac{d\sigma(\xi)}{|1-z\rho\overline{\xi}|^{2n}} \le \frac{C}{(1-|z|\rho)^n}\,. \quad \Box$$

3. Proof of Theorem 1.1

Let $g(z) = \sum_{k=0}^{+\infty} g_k(z)$ belongs to $\mathcal{B}(B)$ and$f(z) = \sum_{k=0}^{+\infty} \widehat{f}_k z^k \in H^1(U)$. Then we have

$$|D^2(g*f)(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi} D^1 f(re^{i\theta})D^1 g(ze^{-i\theta})d\theta\right| \le \frac{C}{1-|z|}M_1(r, D^1 f)\,.$$

From this it follows that

$$(1-r)^3 M_\infty^2(r, D^2(g*f)) \le C(1-r)M_1^2(r, D^1 f)\,.$$

Since $f \in H^1(U)$, we see that $\int_0^1 (1-r)M_1^2(r, D^1 f)dr < +\infty$, by Theorem 2.1. Thus,

$$\int_0^1 (1-r)^3 M_\infty^2(r, D^2(g*f))dr < +\infty$$

which in turn implies that (see [2], [3])

$$\int_0^1 (1-r)M_\infty^2(r, D^1(g*f))dr < +\infty\,.$$

Now, by Theorem 2.2 we have that $g*f \in BMOA(B)$.

Conversely, let $g \in (H^1(U), BMOA(B))$. Then we have that

$$\|g*f\|_{\mathcal{B}(B)} \le C\|f\|_{H^1(U)}\,,$$

for every $f \in H^1(U)$. Let $f(z) = \sum\limits_{k=0}^{+\infty}(k+1)z^k$, $z \in U$. Then

$$\|D^1 g_r\|_{\mathcal{B}(B)} \le \frac{C}{1-r}\,.$$

From this it follows that

$$M_\infty(r\rho, D^2 g) \le C(1-\rho)^{-1}(1-r)^{-1}\,, \quad 0<\rho, r<1\,.$$

In particular, we have

$$M_\infty(r, D^2 g) \le C(1-r)^{-2}\,.$$

Finally, we conclude that

$$\sup_{0<r<1}(1-r)M_\infty(r, D^1 g) < +\infty\,.$$

Thus, $g \in \mathcal{B}(B)$. $\square$

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