FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 15 (2000), 15–20

ON WEIGHTED NORM INEQUALITIES FOR OPERATORS OF HARDY TYPE

James Adedayo Oguntuase

Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. In this paper, we use the Jensen's inequality to obtain conditions on the weight functions $u(x)$ and $v(x)$ which ensures that the weighted estimates

$$\int_0^{+\infty} (uTf)^q \leq C \int_0^{+\infty} \left(vf^{\frac{q}{p}} y^{\frac{q-p}{p}}\right)^p$$

holds, where C is a constant independent of f and T is the Hardy's operator.

1. Introduction

In a recent paper Bloom and Kerman [1] proved the following results:

Theorem 1.1. *Let u and v be nonnegative measurable functions on* $(0, +\infty)$ *with* $0 < u, v < +\infty$ *almost everywhere. Then for* $1 < p < +\infty$,

$$\int_0^{+\infty} (uTf)^P \leq C \int_0^{+\infty} (vf)^p \tag{1}$$

for all nonnegative f *if and only if*

$$T^\star \left[\left(v^{-1} T^\star u^P\right)^{p'} \right] \leq C T^\star u^p < +\infty \,, \tag{2}$$

where T *is the Hardy's operator defined by*

$$Tf(x) = \int_0^x f(y)dy, \qquad x > 0 \quad \textit{and} \quad \frac{1}{p} + \frac{1}{p'} = 1 \,.$$

Received January 28, 1999.
1991 *Mathematics Subject Classification*. Primary 26D15, 26A33.

They employed the use of the adjoint operator $T^\star$ defined by

$$T^\star g(x) = \int_x^{+\infty} g(y)dy$$

to obtain conditions on the nonnegative weight functions $u(x)$ and $v(x)$ which ensures that the inequality (1) is satisfied. The objective of this paper is to obtain conditions on the nonnegative weight functions $u(x)$ and $v(x)$ using the Jensen's inequality to obtain a result which is more general than Theorem 1.1 obtained by Bloom and Kerman [1].

2. Main Results

Theorem 2.1. *Let u and v be nonnegative measurable functions on $(0, +\infty)$ with $0 < u, v < +\infty$ almost everywhere. Then for $1 < p \le q < +\infty$,*

$$\int_0^{+\infty} (uTf)^q \le C \int_0^{+\infty} \left(vf^{\frac{q}{p}} y^{\frac{q-p}{p}}\right)^p \tag{3}$$

for all nonnegative f if and only if

$$T^\star \left[\left(v^{-1} T^\star u^q y^{q-p}\right)^{p'}\right] \le C T^\star u^p x^{q-p} < +\infty\,. \tag{4}$$

Proof. Assume that f is nonnegative and that f is compactly supported in $(0, +\infty)$, we have

$$\int_0^{+\infty} (uTf)^q = \int_0^{+\infty} u^q \left(\int_0^x f\right)^q = \int_0^{+\infty} \left[u^{q/p} \left(\int_0^x f\right)^{q/p}\right]^p$$

and, by Jensen's inequality,

$$\begin{aligned} \int_0^{+\infty} (uTf)^q &\le \int_0^{+\infty} \left[u^{q/p} x^{\frac{q-p}{p}} \left(\int_0^x f^{q/p}\right)\right]^p \\ &= \int_0^{+\infty} u^q x^{q-p} \left(\int_0^x f^{q/p}\right)^p . \end{aligned} \tag{5}$$

Integration by parts yields

$$
\begin{aligned}
(6)\qquad \int_0^{+\infty} (uTf)^q &\le \left(\int_0^x f^{q/p}\right)^p \left[\int_0^x u^q y^{q-p}\right]_0^{+\infty} \\
&\quad - p\int_0^{+\infty} \left(\int_0^x f^{q/p}\right)^{p-1} \left(\int_0^x u^q y^{q-p}\right) f^{q/p} \\
&= \left(\int_0^{+\infty} f^{q/p}\right)^p \left(\int_0^{+\infty} u^q y^{q-p}\right) \\
&\quad - p\int_0^{+\infty} \left(\int_0^{+\infty} u^q y^{q-p}\right) \left(\int_0^x f^{q/p}\right)^{p-1} f^{q/p} \\
&\quad + p\int_0^{+\infty} \left(\int_x^{+\infty} u^q y^{q-p}\right) \left(\int_0^x f^{q/p}\right)^{p-1} f^{q/p} \\
&= \left(\int_0^{+\infty} f^{q/p}\right)^p \left(\int_0^{+\infty} u^q y^{q-p}\right) \\
&\quad - \int_0^{+\infty} \left(\int_0^{+\infty} u^q y^{q-p}\right) d\left(\int_0^x f^{q/p}\right)^p \\
&\quad + p\int_0^{+\infty} \left(\int_x^{+\infty} u^q y^{q-p}\right) \left(\int_0^x f^{q/p}\right)^{p-1} f^{q/p} \\
&= \left(\int_0^{+\infty} f^{q/p}\right)^p \left(\int_0^{+\infty} u^q y^{q-p}\right) \\
&\quad - \left(\int_0^{+\infty} u^q y^{q-p}\right) \left(\int_0^{+\infty} f^{q/p}\right)^p \\
&\quad + p\int_0^{+\infty} \left(\int_x^{+\infty} u^q y^{q-p}\right) \left(\int_0^x f^{q/p}\right)^{p-1} f^{q/p} \\
&= p\int_0^{+\infty} f^{q/p} \left(Tf^{q/p}\right)^{p-1} T^* u^q y^{q-p} \,.
\end{aligned}
$$

Hence

$$\int_0^{+\infty} (uTf)^q \le p\int_0^{+\infty} f^{q/p} \left(Tf^{q/p}\right)^{p-1} T^\star u^q y^{q-p} \,.$$

Now if we assume that

$$\int_0^{+\infty} \left(vf^{q/p}\right)^p = 1 \,,$$

then by Holder's inequality, (6) gives

$$\begin{aligned}
\int_0^{+\infty} (uTf)^q &\leq p\int_0^{+\infty} vf^{q/p}\left(Tf^{q/p}\right)^{p-1} v^{-1}T^\star\left(u^q y^{q-p}\right)\\
&\leq p\left(\int_0^{+\infty}\left(vf^{q/p}\right)^p\right)^{1/p}\\
&\times\left(\int_0^{+\infty}\left(Tf^{q/p}\right)^{(p-1)p'}\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\right)^{1/p'}\\
&= p\left[\int_0^{+\infty}\left(Tf^{q/p}\right)^p\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\right]^{1/p'}.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\int_0^{+\infty} (uTf)^q &\leq p\left[\left(Tf^{q/p}\right)^p\int_0^x\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\right]_0^{+\infty}\\
&-\int_0^{+\infty}\left\{\int_0^x\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'} d\left(Tf^{q/p}\right)^p\right\}^{1/p'}\\
&= p\left[\left(\int_0^{+\infty} f^{q/p}\right)^p\left(\int_0^{+\infty}\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\right)\right.\\
&-\int_0^{+\infty}\int_0^{+\infty}\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\left(\int_0^{+\infty} f^{q/p}\right)^p\\
&\left.+\int_0^{+\infty}\left(\int_x^{+\infty}\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'}\right) d\left(Tf^{q/p}\right)^p\right]^{1/p'}\\
&= p\left[\int_0^{+\infty} T^\star\left(v^{-1}T^\star u^q y^{q-p}\right)^{p'} x d\left(Tf^{q/p}\right)^p\right]^{1/p'}.
\end{aligned}$$

We have from (4) that

$$\int_0^{+\infty} (uTf)^q \leq p\left[C\int_0^{+\infty}\left(T^\star v^p y^{q-p}\right) d\left(Tf^{q/p}\right)^p\right]^{1/p'}$$

Integration by parts again gives

$$\begin{aligned}
\int_0^{+\infty} (uTf)^q &\le C\left[\left(T^\star v^p y^{q-p}\right)\left(Tf^{q/p}\right)^p \Big|_0^{+\infty} \right. \\
&\qquad \left. - \int_0^{+\infty} \left(Tf^{q/p}\right)^p d\left(T^\star v^p y^{q-p}\right)\right]^{1/p'} \\
&\le -C\left[\int_0^{+\infty}\left(\int_0^x f^{q/p}\right)^p d\left(\int_x^{+\infty} v^p y^{q-p}\right)\right]^{1/p'} \\
&\le C\left[\int_0^{+\infty}\left(\int_0^x f^{q/p}\right)^p v^p x^{q-p}\right]^{1/p'} \\
&\le C\left[\int_0^{+\infty}\left(vf^{q/p}x^{\frac{q-p}{p}}\right)^p\right]^{1/p'} < +\infty\,.
\end{aligned}$$

It follows from this that

$$\int_0^{+\infty} (uTf)^q \le C\,.$$

To show that (3) is implied by (4), we observe that for any of our f with

$$\int_0^{+\infty}\left(vf^{q/p}x^{\frac{q-p}{p}}\right)^p < +\infty\,,$$

we have

$$\begin{aligned}
\int_0^{+\infty} (uTf)^q &\le p\int_0^{+\infty} f^{q/p}\left(Tf^{q/p}\right)^{p-1} T^\star u^q y^{q-p} \\
&\le C\int_0^{+\infty}\left(vf^{q/p}y^{\frac{q-p}{p}}\right)^p < +\infty\,.
\end{aligned}$$

From this, we have that

$$T^\star u^q y^{q-p} < +\infty \quad \text{on} \quad (0,+\infty)\,.$$

If we substitute

$$h(.) = u(.)^q y^{q-p}\chi_{(x,+\infty)}(.)$$

into the equivalent inequality

$$\int_0^{+\infty} \left(v^{-1}T^\star h\right)^{p'} \le C \int_0^{+\infty} \left(u^{-1}h\right)^{p'} ,$$

dual to (3), we obtain (4). This completes the proof of our result. □

Remark. If we set $p = q$ in Theorem 2.1, then we shall obtain Theorem 1.1 obtained by Bloom and Kerman [1].

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University of Agriculture
Department of Mathematical Sciences
Abeokuta
Nigeria