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G–INVERSES AND CANONICAL FORMS

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Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. We introduce an useful general representation of $\{2\}$–inverses of an arbitrary real matrix A. This representation is based on the computational scheme $W_1(W_2AW_1)^{-1}W_2$, where W_1 and W_2 are two appropriate matrices, such that W_2AW_1 is invertible matrix which satisfy the condition $\operatorname{rank}(W_2AW_1) \le \operatorname{rank}(A)$. In the case $\operatorname{rank}(W_2AW_1) = \operatorname{rank}(A)$ we obtain well–known general representation for $\{1,2\}$–inverses of A. Using this general representation, we generate two representations of $\{2\}$–inverses by means of the Jordan canonical form and the rational canonical form of the matrix W_2AW_1, respectively. Introduced representation for $\{2\}$–inverses can be simply reduced to analogous representations of $\{1,2\}$ inverses.

1. Introduction

The set of $m \times n$ real (complex) matrices whose rank is r we denote by $\mathbb{R}_r^{m\times n}$ ($\mathbb{C}_r^{m\times n}$). By $\mathbb{O}$ we denote the zero matrix of an appropriate size, and by $\mathbb{I}_k$ the unit matrix of the order k. With $A_{k|}$ and $A_{\underline{k}}$ we denote the first k columns of A and the first k rows of A, respectively. Similarly, $A_{|k}$ and $A_{\overline{k}}$ denote the last k columns and the last k rows of A, respectively.

For any matrix $A \in \mathbb{C}^{m\times n}$ consider the Penrose's equations:

$$\begin{array}{llll} (1) & AXA=A\,, & (2) & XAX=X\,, \\ (3) & (AX)^T=AX\,, & (4) & (XA)^T=XA\,, \end{array}$$

where $*$ denotes conjugate and transpose matrix. If $m=n$ we also consider the equation

$$(5) \quad AX = XA\,.$$

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For any sequence $\mathcal{S} \subseteq \{1,2,3,4,5\}$ the set of matrices satisfying the conditions contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix G in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$–inverse of A and is denoted by $A^{(\mathcal{S})}$. The unique $\{1,2,3,4\}$–inverse of A is said to be the *Moore–Penrose inverse* of A. In the case $m=n$, the *group inverse*, $A^{\#}$, of A is the unique $\{1,2,5\}$–inverse of A.

For the sake of completeness, we present a brief description of the *rational canonical form* and the *Jordan canonical form* (see for example [4], [11]).

The *rational canonical representation* of $A \in \mathbb{R}^{n\times n}$ is given by

$$A = TBT^{-1} = T\,(B_1 \oplus \cdots \oplus B_p)\,T^{-1}\,,$$

where the blocks B_i, $1 \le i \le p$ are the companion matrices of elementary divisors

$$t^{m_i} + a^i_{m_i-1}t^{m_i-1} + \ldots + a^i_1 t + a^i_0$$

of the minimal polynomial of A:

$$B_i = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a^i_0 \\ 1 & 0 & \ldots & 0 & -a^i_1 \\ 0 & 1 & \ldots & 0 & -a^i_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a^i_{m_i-1} \end{bmatrix} = \begin{bmatrix} \mathbb{O} & -a^i_0 \\ \mathbb{I}_{m_i-1} & -\vec{a^i} \end{bmatrix} \in \mathbb{R}^{m_i\times m_i}\,.$$

In this formula the vector $\vec{a^i}$ contains the following elements:

$$\vec{a^i} = \begin{bmatrix} a^i_1 \\ \cdots \\ a^i_{m_i-1} \end{bmatrix}, \quad 1 \le i \le p\,.$$

We can suppose, without loss of generality, that the blocks $B_1,\ldots,B_q$ are invertible, i.e. $a^i_0 \neq 0$, $i=1,\ldots q$ and the blocks $B_{q+1},\ldots,B_p$ are singular, i.e. $a^i_0 = 0$, $i = q+1,\ldots p$ ([4], [11]).

For $A \in \mathbb{R}^{n\times n}$, let $A = TJT^{-1}$ be its Jordan canonical representation. Then, the block diagonal matrix J can be represented in the form

$$J = J_1 \oplus \cdots \oplus J_t \oplus J_{t+1} \oplus \cdots \oplus J_h\,,$$

where $J_1,\ldots,J_t$ are lower Jordan and nonsingular matrices, and $J_{t+1},\ldots,J_h$ are lower Jordan and singular ([4], [11]).

The plan of this paper is as follows. We derive the following general representation of $\{2\}$–inverses of an arbitrary real matrix $A \in \mathbb{C}_r^{m\times n}$:

$$
\begin{aligned}
A\{2\} = \{W_1(W_2AW_1)^{-1}W_2 : \quad W_1 \in \mathbb{C}_q^{n\times q}, \quad W_2 \in \mathbb{C}_q^{q\times m}, \\
\mathrm{rank}(W_2AW_1) = q \le \mathrm{rank}(A)\} \,.
\end{aligned}
\tag{1.1}
$$

In the case $\mathrm{rank}(W_2AW_1) = \mathrm{rank}(A)$ we obtain well–known general representation of $\{1,2\}$–inverses of A.

In section 3. we investigate computations of generalized inverses by means of the introduced general representations and the rational canonical form or the Jordan canonical form. In the paper [6] we introduce an explicit block representation for $\{1\}$–inverses and the group inverse of a real square matrix A, using the *rational canonical form* $B = T^{-1}AT$. In [6] we use the following idea in computation of an arbitrary $\{1\}$–inverse of A: compute an arbitrary $\{1\}$–inverse $Z \in B\{1\}$ of B, and then use the similarity transformation $X = TZT^{-1}$ to obtain $X \in A\{1\}$. Generalized inverses $Z \in B\{1\}$ are generated by splitting the matrices B and Z into the corresponding blocks, and solving the corresponding matrix equations.

Also, representation of generalized inverses of a square matrix A in terms of its Jordan canonical form is the well–known method ([2], [3], [5], [9], [10]). In the papers [3], [9], the corresponding results are obtained by splitting the matrices A and X into the corresponding blocks and solving the corresponding matrix equations.

In this paper we use a new algorithm, applicable to arbitrary, rectangular or square, real matrices. Instead of the rational canonical form (the Jordan canonical form) of A, we use the rational canonical form (the Jordan canonical form) corresponding to W_2AW_1, where W_1 and W_2 are matrices of the corresponding dimensions, such that W_2AW_1 is regular matrix of an arbitrary order $q \le \mathrm{rank}(A)$. Then we compute $(W_2AW_1)^{-1}$ by inverting the corresponding companion submatrices or the corresponding Jordan blocks. Finally, an arbitrary $\{2\}$–inverse of A can be computed using the matrix product $W_1(W_2AW_1)^{-1}W_2$, according to the introduced general representation of $\{2\}$–inverses.

Introduced *canonical form representations* of $\{2\}$–inverses can be simply reduced into the corresponding *canonical form representations* of $\{1,2\}$–inverses, without solving the equation (1).

2. General Representations

In the following lemma we obtain general representation of reflexive g–inverses for complex matrices, using full–rank factorization.

Lemma 2.1. *Let $A = PQ$ be a full–rank decomposition for $A \in \mathbb{C}^{m\times n}$. Also, let W_1, W_2 be arbitrary $n\times r$ and $r\times m$ matrices, respectively, satisfying*

$$\text{rank}(QW_1) = \text{rank}(W_2P) = \text{rank}(A)\,, \tag{2.1}$$

and U, V are $m \times m$ and $n \times n$ matrices, respectively, such that

$$\text{rank}(QVQ^*) = \text{rank}(Q), \quad \text{rank}(P^*UP) = \text{rank}(P)\,. \tag{2.2}$$

Then

$$X = VQ^*(QVQ^*)^{-1}(P^*UP)^{-1}P^*U \Leftrightarrow X = W_1(QW_1)^{-1}(W_2P)^{-1}W_2\,. \tag{2.3}$$

Also, X represents the general solution of the equations (1), (2).

Proof. Using the results from [8] (Theorem 2.1.1 and Lemma 2.5.2), one can proved that $X \in \mathbb{C}^{n\times m}$ is reflexive g–inverse of A if and only if it can be expressed as

$$X = VQ^*(QVQ^*)^{-1}(P^*UP)^{-1}P^*U\,.$$

To complete the proof we prove the equivalence in (2.3). It is evident that $X=VQ^*(QVQ^*)^{-1}(P^*UP)^{-1}P^*U$ implies $X=W_1(QW_1)^{-1}(W_2P)^{-1}W_2$. On the other hand, the equation $X = W_1(QW_1)^{-1}(W_2P)^{-1}W_2$ implies $X = VQ^*(QVQ^*)^{-1}(P^*UP)^{-1}P^*U$, because of consistency of the equations $W_1 = VQ^*$ and $W_2 = P^*U$. For example, the matrix equation $W_1 = VQ^*$ is consistent if and only if $V(Q^*)^{(1)}Q^* = V$, for some $(Q^*)^{(1)}$. Observe that in the place of $(Q^*)^{(1)}$ we can use an arbitrary left inverse of Q^*. □

In the following theorem we introduce a general representation of $\{2\}$–inverses of a given real matrix.

Theorem 2.1. *For $A \in \mathbb{C}_r^{m\times n}$ the set $A\{2\}$ is equal to*

$$\begin{aligned}\{W_1(W_2AW_1)^{-1}W_2 \;:\; & W_1 \in \mathbb{C}^{n\times t}\,,\ W_2 \in \mathbb{C}^{t\times m}\,,\\ & \text{rank}(W_2AW_1) = t \le r\}\,.\end{aligned} \tag{2.4}$$

Proof. Consider an arbitrary $\{2\}$–inverse X of A. According to [8](Theorem 3.4.1), it can be represented in the form $X = C(DAC)^{(1,2)}D$, where $C \in \mathbb{C}^{n\times u}$, $D \in \mathbb{C}^{v\times m}$ are arbitrary, and $\text{rank}(X) = \text{rank}(DAC)$. Let $\text{rank}(DAC) = q \le \min\{u, v\} \le r$. According to Lemma 2.1, we get the following representation for X:

$$X = CF(HDACF)^{-1}HD,\ \ F \in \mathbb{C}^{u\times q},\ \ H \in \mathbb{C}^{q\times v},\ \ \text{rank}(HDACF) = q\,.$$

After the substitutions $CF = W_1$, $HD = W_2$, it is easy to verify that X is an element of the set defined in (2.4).

Conversely, it is an exercise to verify that an arbitrary element from the set (2.4) satisfies the equation $XAX = X$. □

3. Application of Canonical Forms

Using general representations and the rational canonical form of the matrix W_2AW_1 we develop a *rational canonical form representation* of $\{2\}$–inverses.

Theorem 3.1. *Let $A \in \mathbb{R}_r^{m\times n}$, and let W_1, W_2 be arbitrary $n\times q$ and $q\times m$ matrices, respectively, such that* $\operatorname{rank}(W_2AW_1) = q \le \operatorname{rank}(A)$. *Suppose that the rational canonical form of W_2AW_1 is equal to*

$$W_2AW_1 = TB_{W_2AW_1}T^{-1} = T\cdot B_{W_2AW_1}\cdot T^{-1} = T\cdot B_1\oplus\cdots\oplus B_t\cdot T^{-1},$$

where blocks $B_i = \begin{bmatrix} \mathbb{O} & -a_0^i \\ \mathbb{I}_{m_i-1} & -\vec{a}^i \end{bmatrix}$ *are of the order $m_i\times m_i$, $i=1,\dots,t$. Let the matrix $U = W_1T$ is divided into the blocks $U_{\alpha\beta}$ of the order $m_\alpha\times m_\beta$, $\binom{\alpha=1,\dots,p}{\beta=1,\dots,t}$, and the matrix $V = T^{-1}W_2$ is partitioned into blocks $V_{\gamma\delta}$ of the order $m_\gamma\times m_\delta$, $\binom{\gamma=1,\dots,t}{\beta=1,\dots,s}$. Then (α,β) block $G_{\alpha,\beta}$ from $\{2\}$–inverse G of A corresponding to W_1 and W_2 can be represented as follows:*

$$G_{\alpha\beta} = \sum_{\gamma=1}^{t}\left(-\frac{1}{a_0^\gamma}\left((U_{\alpha\gamma})_{m_\gamma-1|}\vec{a}^\gamma+(U_{\alpha\gamma})_{|1}\right)(V_{\gamma\beta})_{\underline{1}}+(U_{\alpha\gamma})_{m_\gamma-1|}(V_{\gamma\beta})_{\overline{m_\gamma-1}}\right) \tag{3.1}$$

for each $\alpha = 1,\dots,p$,$\beta = 1,\dots,s$.

In the case $q=r$ block $G_{\alpha\beta}$, defined in (3.1) *represents the corresponding (α,β) block from $G\in A\{1,2\}$.*

Proof. According to Theorem 2.1, $G\in A\{2\}$ if and only if G can be represented in the form $G = W_1(W_2AW_1)^{-1}W_2$, where $W_1\in\mathbb{C}^{n\times q}$, $W_2\in\mathbb{C}^{q\times m}$ and $\operatorname{rank}(W_2AW_1) = q\le\operatorname{rank}(A)$. The matrix W_2AW_1 is regular, which implies the following [1]:

$$(W_2AW_1)^{-1} = TB_{W_2AW_1}^{-1}T^{-1} = T\left(B_1^{-1}\oplus\cdots\oplus B_t^{-1}\right)T^{-1}.$$

Applying $G = W_1(W_2AW_1)^{-1}W_2$, we get $G = U\left(B_1^{-1}\oplus\cdots\oplus B_t^{-1}\right)V$.

It is easy to verify

$$B_\theta^{-1} = \begin{bmatrix} -\frac{1}{a_0^\theta}\vec{a}^\theta & \mathbb{I}_{m_\theta-1} \\ -\frac{1}{a_0^\theta} & \mathbb{O} \end{bmatrix}, \quad \theta = 1,\dots,t.$$

Consequently,

$$U_{\alpha\gamma}B_\gamma^{-1} = \left[\,(U_{\alpha\gamma})_{m_\gamma-1|} \quad (U_{\alpha\gamma})_{|1}\,\right]\cdot\begin{bmatrix} -\frac{1}{a_0^\gamma}\vec{a}^\gamma & \mathbb{I}_{m_\gamma-1} \\ -\frac{1}{a_0^\gamma} & \mathbb{O}\end{bmatrix}$$

$$= \left[\,-\tfrac{1}{a_0^\gamma}\Big((U_{\alpha\gamma})_{m_\gamma-1|}\vec{a}^\gamma + (U_{\alpha\gamma})_{|1}\Big) \quad (U_{\alpha\gamma})_{m_\gamma-1|}\,\right], \quad \gamma\in\{1,\dots,t\}.$$

Now, an arbitrary (α,β) block in G, denoted by $G_{\alpha\beta}$, can be computed as follows:

$$\begin{aligned} G_{\alpha\beta} &= \sum_{\gamma=1}^{t} U_{\alpha\gamma}B_\gamma^{-1}V_{\gamma\beta} \\ &= \sum_{\gamma=1}^{t}\left[-\tfrac{1}{a_0^\gamma}\Big((U_{\alpha\gamma})_{m_\gamma-1|}\vec{a}^\gamma + (U_{\alpha\gamma})_{|1}\Big) \qquad (U_{\alpha\gamma})_{m_\gamma-1|}\right]\cdot\begin{bmatrix}(V_{\gamma\beta})_{\underline{1}} \\ (V_{\gamma\beta})_{\overline{m_\gamma-1}}\end{bmatrix} \\ &= \sum_{\gamma=1}^{t}\left(-\frac{1}{a_0^\gamma}\Big((U_{\alpha\gamma})_{m_\gamma-1|}\vec{a}^\gamma + (U_{\alpha\gamma})_{|1}\Big)(V_{\gamma\beta})_{\underline{1}} + (U_{\alpha\gamma})_{m_\gamma-1|}(V_{\gamma\beta})_{\overline{m_\gamma-1|}}\right). \end{aligned}$$

In the case $q=r$, According to Theorem 3.1, Theorem 2.1 and Lemma 2.1, we conclude $G\in A\{1,2\}$. □

The same principle can be used in representations of $\{2\}$ and $\{1,2\}$–inverses in terms of the Jordan canonical form.

Theorem 3.2. *Let $A\in\mathbb{R}_r^{m\times n}$, and let W_1, W_2 be arbitrary $n\times q$ and $q\times m$ matrices, respectively, such that* $\operatorname{rank}(W_2AW_1) = q \le \operatorname{rank}(A)$. *Suppose that*

$$J_{W_2AW_1} = J_1\oplus\cdots\oplus J_t$$

is the Jordan canonical form of W_2AW_1, where the lower Jordan blocks J_i are of the order $m_i\times m_i$, $i=1,\dots,t$. Let the matrix $U=W_1T$ is divided into the blocks $U_{\alpha\beta}$ of the order $m_\alpha\times m_\beta$, $\left(\substack{\alpha=1,\dots,p\\ \beta=1,\dots,t}\right)$, and the matrix $V=T^{-1}W_2$ is partitioned into the blocks $V_{\gamma\delta}$ of the order $m_\gamma\times m_\delta$, $\left(\substack{\gamma=1,\dots,t\\ \beta=1,\dots,s}\right)$. Also, let (i,j)th element of (α,β) block in U and V is denoted by $u_{ij}^{\alpha\beta}$ and $v_{ij}^{\alpha\beta}$, *respectively. Then (i,j)th element of the (α,β) block from an arbitrary g–inverse $G \in A\{2\}$ can be represented as follows:*

$$g_{ij}^{\alpha\beta} = \sum_{\gamma=1}^{t} \sum_{k=1}^{m_\gamma} \sum_{l=1}^{k} u_{il}^{\alpha\gamma} (-1)^{k-l} \frac{1}{\lambda_\gamma^{k-l+1}} v_{kj}^{\gamma\beta}, \tag{3.2}$$
$$\left({\alpha=1,\dots,p \atop \beta=1,\dots,s} \right), \quad \left({i=1,\dots,m_\alpha \atop j=1,\dots,m_\beta} \right), \quad r,s \geq t\,.$$

In the case $q = r$, the real number $g_{ij}^{\alpha\beta}$, defined in (3.2), represents the corresponding (i,j)th element contained in (α,β) block from $G \in A\{1,2\}$.

Proof. The matrix W_2AW_1 is regular, which means the following ([1]):

$$(W_2AW_1)^{-1} = TJ_{W_2AW_1}^{-1}T^{-1} = T\left(J_1^{-1} \oplus \cdots \oplus J_t^{-1}\right)T^{-1}\,.$$

An application of Theorem 2.1 gives $G = UJ_{W_2AW_1}^{-1}V$.

It is easy to verify the following:

$$\left(J_\theta^{-1}\right)_{ij} = \begin{cases} 0, & i > j, \\ (-1)^{j-i}\frac{1}{\lambda_\theta^{j-i+1}}, & i \leq j, \end{cases} \qquad \theta = 1,\dots,t\,.$$

Consequently,

$$\left(U_{\alpha\gamma}J_\gamma^{-1}\right)_{ij} = \sum_{l=1}^{j} u_{il}^{\alpha\gamma}(-1)^{j-l}\frac{1}{\lambda_\alpha^{j-l+1}}, \quad \gamma \in \{1,\dots,t\}\,.$$

The (α,β) block in G, denoted by $G_{\alpha\beta}$, is equal to

$$G_{\alpha\beta} = \sum_{\gamma=1}^{t} U_{\alpha\gamma}J_\gamma^{-1}V_{\gamma\beta}, \quad \left({\alpha=1,\dots,p \atop \beta=1,\dots,s} \right)\,.$$

Now, for arbitrary $i \in \{1,\dots,m_\alpha\}$, $j \in \{1,\dots,m_\beta\}$, we obtain

$$\begin{aligned} g_{ij}^{\alpha\beta} &= \left(\sum_{\gamma=1}^{t} U_{\alpha\gamma}J_\gamma^{-1}V_{\gamma\beta} \right)_{ij} = \sum_{\gamma=1}^{t}\sum_{k=1}^{m_\gamma} \left(U_{\alpha\gamma}J_\gamma^{-1}\right)_{ik} v_{kj}^{\gamma\beta} \\ &= \sum_{\gamma=1}^{t} \sum_{k=1}^{m_\gamma} \sum_{l=1}^{k} u_{il}^{\alpha\gamma}(-1)^{k-l}\frac{1}{\lambda_\gamma^{k-l+1}} v_{kj}^{\gamma\beta}\,. \quad \square \end{aligned}$$

Using the result of Theorem 3.2, we obtain representation of an arbitrary element from $G \in A\{2\}$.

Corollary 3.1. *Under the suppositions of Theorem* 3.2*, the elements from* $G \in A\{2\}$ *can be represented as follows:*

$$\begin{aligned} & g_{m_0+m_1+\ldots+m_{\alpha-1}+i,m_0+m_1+\ldots+m_{\beta-1}+j} \\ = & \sum_{\gamma=1}^{t}\sum_{k=1}^{m_\gamma}\sum_{l=1}^{k} u_{m_0+m_1+\ldots+m_{\alpha-1}+i,\, m_0+m_1+\ldots+m_{\gamma-1}+l} \\ \times & \ (-1)^{k-l}\frac{1}{\lambda_\gamma^{k-l+1}} v_{m_0+m_1+\ldots+m_{\gamma-1}+k,\, m_0+m_1+\ldots+m_{\beta-1}+j}\,, \end{aligned}$$

for each $\alpha = 1,\ldots,p$*,* $\beta = 1,\ldots,s$*,* $i = 1,\ldots,m_\alpha$ *and* $j = 1,\ldots,m_\beta$*, where* $m_0 = 0$*.*

Proof. The proof follows from Theorem 3.2 and from the following fact, valid for arbitrary (i,j)th element inside of any (α,β) block in a matrix H:

$$(H_{\alpha,\beta})_{ij} = h_{ij}^{\alpha\beta} = h_{m_0+m_1+\ldots+m_{\alpha-1}+i,\, m_0+m_1+\ldots+m_{\beta-1}+j}\,. \qquad \square$$

4. Conclusion

We can suggest at least four advantages contained in our approach in representations of $\{2\}$ and $\{1,2\}$–inverses:

1. Simplification of derivations used in [3] and [6], which are based on adequate splitting of the matrices B, Z and solving corresponding system of matrix equations. Moreover, the splitting technique used in these papers is valid only for the set of $\{1,2\}$–inverses of square matrices. Also, the method used in [3], [6] and [10] is not convenient in solving of the matrix equation (2) without the matrix equation (1).

2. The method developed in Section 3. is applicable to arbitrary, square and rectangular, real matrices.

3. Introduced representation for all the reflexive g–inverses can be simply reduced into the corresponding representations of different classes of generalized inverses, without solving the corresponding set of matrix equations, contained in (1)–(5). For this purpose we can use the following general representations of different generalized inverses:

Proposition 2.1. *If* $A = PQ$ *is a full–rank factorization of* A *and* $W_1 \in \mathbb{C}^{n\times r}$*,* $W_2 \in \mathbb{C}^{r\times m}$ *satisfy the conditions* (2.1)*, then:*

general solution of the system of matrix equations (1), (2), (3) *is* ([8])

$$W_1(QW_1)^{-1}(P^*P)^{-1}P^* = W_1(P^*AW_1)^{-1}P^*\,,$$

general solution of the system of matrix equations (1), (2), (4) *is* ([8])

$$Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2 = Q^*(W_2AQ^*)^{-1}W_2\,,$$

and the Moore–Penrose inverse of A is equal to

$$A^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*\,.$$

4. General representations of $\{2\}$–inverses, introduced in Theorem 2.1, is practical and universal pattern for computation of generalized inverses. But it is also of interest a theoretical power of these representations.

5. Examples

Example 5.1. Consider

$$A = \begin{bmatrix} 3 & 5 & 1 & 2 \\ 0 & -2 & -1 & 1 \end{bmatrix}$$

and

$$W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify $J_{W_2AW_1} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. This implies

$$U = W_1T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = T^{-1}W_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

According to Theorem 3.2 and Corollary 3.1, the elements g_{ij} of the reflexive

g–inverse of A can be computed as follows:

$$
\begin{aligned}
g_{11}^{11} &= g_{11} = u_{11}\frac{1}{\lambda_1}v_{11} + u_{12}\frac{1}{\lambda_2}v_{21} = \frac{1}{3}\,;\\
g_{11}^{12} &= g_{12} = u_{11}\frac{1}{\lambda_1}v_{11} + u_{12}\frac{1}{\lambda_2}v_{22} = \frac{5}{6}\,;\\
g_{11}^{21} &= g_{21} = u_{21}\frac{1}{\lambda_1}v_{11} + u_{22}\frac{1}{\lambda_2}v_{21} = 0\,;\\
g_{11}^{22} &= g_{22} = u_{21}\frac{1}{\lambda_1}v_{12} + u_{22}\frac{1}{\lambda_2}v_{22} = -\frac{1}{2}\,;\\
g_{11}^{31} &= g_{31} = u_{31}\frac{1}{\lambda_1}v_{11} + u_{32}\frac{1}{\lambda_2}v_{21} = 0\,;\\
g_{11}^{32} &= g_{32} = u_{31}\frac{1}{\lambda_1}v_{12} + u_{32}\frac{1}{\lambda_2}v_{22} = 0\,;\\
g_{11}^{41} &= g_{41} = u_{41}\frac{1}{\lambda_1}v_{11} + u_{42}\frac{1}{\lambda_2}v_{21} = 0\,;\\
g_{11}^{42} &= g_{42} = u_{41}\frac{1}{\lambda_1}v_{12} + u_{42}\frac{1}{\lambda_2}v_{22} = 0\,.
\end{aligned}
$$

Example 5.2. In this example we compute the reflexive g–inverse of the matrix

$$A = \begin{bmatrix} 23 & -2 & 4 & 8 \\ 4 & 20 & 5 & 0 \\ -2 & 8 & 20 & 2 \end{bmatrix}$$

determined by the following two matrices

$$W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Jordan matrix for W_2AW_1 and the corresponding similarity matrix are

$$J_{W_2AW_1} = \left[\begin{array}{c|cc} 27 & 0 & 0 \\ \hline 0 & 18 & 1 \\ 0 & 0 & 18 \end{array}\right], \quad T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Consequently,

$$U = W_1T = \left[\begin{array}{c|cc} 1 & 2 & 2 \\ \hline 2 & 1 & -2 \\ 2 & -2 & 1 \\ \hline 0 & 0 & 0 \end{array}\right], \quad V = T^{-1}W_2 = \frac{1}{9}\left[\begin{array}{c|cc} 1 & 2 & 2 \\ \hline 2 & 1 & -2 \\ 2 & -2 & 1 \end{array}\right].$$

According to Corollary 3.1, we get

$$g_{11}^{11} = g_{11} = \sum_{\gamma=1}^{2}\sum_{k=1}^{m_\gamma}\sum_{k=1}^{l} u_{1,m_0+\ldots+m_{\gamma-1}+l}(-1)^{k-l}\frac{1}{\lambda_\gamma^{k-l+1}}v_{1,m_0+\ldots+m_{\gamma-1}+k,1}$$

$$= u_{11}\frac{1}{\lambda_1}v_{11} + u_{12}\frac{1}{\lambda_2}v_{21} + \sum_{l=1}^{2} u_{1,1+l}(-1)^{2-l}\frac{1}{\lambda_\gamma^{2-l+1}}v_{1+2,1}$$

$$= u_{11}\frac{1}{\lambda_1}v_{11} + u_{12}\frac{1}{\lambda_2}v_{21} - u_{12}\frac{1}{\lambda_2^2}v_{31} + u_{13}\frac{1}{\lambda_2}v_{31} = \frac{1}{81}\cdot\frac{10}{3}\,.$$

Similarly,

$$g_{11}^{12} = g_{12} = u_{11}\frac{1}{\lambda_1}v_{12} + u_{12}\frac{1}{\lambda_2}v_{22} - u_{12}\frac{1}{\lambda_2^2}v_{32} + u_{13}\frac{1}{\lambda_2}v_{32} = \frac{1}{81}\cdot\frac{2}{3}\,;$$

$$g_{11}^{22} = g_{22} = u_{21}\frac{1}{\lambda_1}v_{12} + u_{22}\frac{1}{\lambda_2}v_{22} - u_{22}\frac{1}{\lambda_2^2}v_{32} + u_{23}\frac{1}{\lambda_2}v_{32} = \frac{1}{81}\cdot\frac{13}{3}\,;$$

$$g_{12}^{22} = g_{23} = u_{21}\frac{1}{\lambda_1}v_{13} + u_{22}\frac{1}{\lambda_2}v_{23} - u_{22}\frac{1}{\lambda_2^2}v_{33} + u_{23}\frac{1}{\lambda_2}v_{33} = \frac{1}{81}\cdot\left(-\frac{11}{12}\right)\,;$$

$$g_{22}^{22} = g_{33} = u_{31}\frac{1}{\lambda_1}v_{13} + u_{32}\frac{1}{\lambda_2}v_{23} - u_{32}\frac{1}{\lambda_2^2}v_{33} + u_{33}\frac{1}{\lambda_2}v_{33} = \frac{1}{81}\cdot\frac{13}{3}\,;$$

$$g_{12}^{32} = g_{43} = u_{41}\frac{1}{\lambda_1}v_{13} + u_{42}\frac{1}{\lambda_2}v_{23} - u_{42}\frac{1}{\lambda_2^2}v_{33} + u_{43}\frac{1}{\lambda_2}v_{33} = 0\,.$$

Continuing in the same manner, we obtain

$$A^{(1,2)} = \frac{1}{81}\left[\begin{array}{c|cc} \frac{10}{3} & \frac{2}{3} & -\frac{5}{6} \\ \hline -\frac{5}{6} & \frac{13}{3} & -\frac{11}{12} \\ \frac{2}{3} & -\frac{5}{3} & \frac{13}{3} \\ \hline 0 & 0 & 0 \end{array}\right].$$

Example 5.3. Consider

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & -5 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 3 & 4 & -3 & -1 & 1 \end{bmatrix},$$

$$W_2=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix},\quad W_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}.$$

The *rational canonical representation* of the matrix

$$W_2AW_1=\begin{bmatrix}2&1&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$$

is determined by the matrices

$$B=B_1\oplus B_2=\left[\begin{array}{ccc|c}0&0&8&0\\1&0&-12&0\\0&1&6&0\\\hline 0&0&0&2\end{array}\right],\quad T=\begin{bmatrix}0&0&1&0\\0&1&4&0\\1&2&4&1\\0&0&0&1\end{bmatrix}.$$

Using

$$B_1^{-1}=\begin{bmatrix}12&8&0\\-6&0&8\\1&0&0\end{bmatrix},\quad B_2^{-1}=[\,\tfrac{1}{2}\,],$$

$$U=W_1T=\begin{bmatrix}U_{11}&U_{12}\\U_{21}&U_{22}\end{bmatrix}=\left[\begin{array}{ccc|c}0&0&1&0\\0&1&4&0\\1&2&4&0\\0&0&0&1\\\hline 0&0&0&0\end{array}\right],$$

$$V=T^{-1}W_2=\begin{bmatrix}V_{11}&V_{12}&V_{13}\\V_{21}&V_{22}&V_{23}\end{bmatrix}=\left[\begin{array}{ccc|c|c}4&-2&1&0&0\\-4&1&0&0&0\\0&0&0&0&0\\\hline 0&0&0&1&0\end{array}\right].$$

According to Theorem 3.1, we obtain the following $\{2\}$–inverse of A:

$$G=\begin{bmatrix}G_{11}&G_{12}&G_{13}\\G_{21}&G_{22}&G_{23}\end{bmatrix}=\left[\begin{array}{ccc|c|c}g_{11}&g_{12}&g_{13}&g_{14}&g_{15}\\g_{21}&g_{22}&g_{23}&g_{24}&g_{25}\\g_{31}&g_{32}&g_{33}&g_{34}&g_{35}\\g_{41}&g_{42}&g_{43}&g_{44}&g_{45}\\\hline g_{51}&g_{52}&g_{53}&g_{54}&g_{55}\end{array}\right]$$

i.e.,

$$G = \frac{1}{8}\begin{bmatrix} 4 & 2 & 1 & 0 & 0 \\ 0 & 4 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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