# COMPLEXITY FOR SOME CLASSES OF EQUATIONS OF THE SECOND KIND 

S. G. Solodky<br>This paper is dedicated to Professor D. S. Mitrinović


#### Abstract

For some classes of equations of the second kind in Hilbert space the exact power order of complexity of the approximate solution is found. We establish that the optimal power order is realized by iterative methods, which use the Galerkin information with numbers from the hyperbolic crosses. These results allow to obtain the exact order of complexity for a class of Volterra equations with differentiable kernels.


Let $\left\{e_{i}\right\}_{i=1}^{\infty}$ be some orthogonal basis in Hilbert space $X$, and let $P_{n}$ be the orthogonal projector on $\operatorname{span}\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$, i.e. $P_{n} \varphi=\sum_{i=1}^{n} e_{i}\left(e_{i}, \varphi\right)$, where $(\cdot, \cdot)$ is an inner product in $X$. We denote by $X^{r}, 0<r<\infty$, the linear subspace of $X$ such that for any $m=1,2, \ldots,\left\|I-P_{n}\right\|_{X^{r} \rightarrow X} \leq \beta_{r} n^{-r}$, where $I$ is the identity operator and the constant $\beta_{r}$ is independent on $m$, and for all $\varphi \in X^{r},\|\varphi\|_{X} \leq\|\varphi\|_{X^{r}}$.

Moreover, let $\mathcal{L}\left(X, X^{r}\right)$ be the space of linear and continuous operators $H$ acting from $X$ to $X^{r}$ with usual norm. We denote by $\Psi_{\mathcal{H}, \Phi}^{r}$ the class of uniquely solvable equations of the second kind

$$
\begin{equation*}
z=H z+f, \tag{1}
\end{equation*}
$$

where $H \in \mathcal{H} \subset \mathcal{L}\left(X, X^{r}\right)$ and $f \in \Phi \subset X^{r}$.

The research described in this publication was made possible in part by Grant UB1000 from the International Science Foundation.

We shall investigate the complexity of finding approximate solution of Eq. (1) for some classes $\Psi_{\mathcal{H}, \Phi}^{r}$. The formulation of the problem and terminology are borrowed from [1], [2].

Let $T=\left\{\delta_{i}\right\}_{i=1}^{m}$ be some collection of continuous functionals $\delta_{i}$, of which $\delta_{1}, \delta_{2}, \ldots, \delta_{\mu}$ are defined on the set $\mathcal{H}$ and $\delta_{\mu+1}, \ldots, \delta_{m}$ on the set $\Phi$, $\operatorname{card}(T)=m$,

$$
\mathcal{T}_{M}=\{T: \operatorname{card}(T) \leq M\} .
$$

To each Eq. (1) of $\Psi_{\mathcal{H}, \Phi}^{r}$ we assign the numerical vector

$$
\begin{equation*}
T(H, f)=\left(\delta_{1}(H), \delta_{2}(H), \ldots, \delta_{\mu}(H), \delta_{\mu+1}(f), \ldots, \delta_{m}(f)\right), \tag{2}
\end{equation*}
$$

which we call the information of Eq. (1), and the collection of functionals $T$ will be called a method of specifying information.

By the algorithm $A$ of an approximate solving of equations from $\Psi_{\mathcal{H}, \Phi}^{r}$ we mean the operator assigning to information (2) as an appropriate solution of Eq. (1) an element $A(T, H, f) \in X$. We assume that every algorithm $A$ connected with the parametric set of elements

$$
F_{A}=\left\{\varphi_{\xi_{1}, \xi_{2}, \ldots, \xi_{n}}: \varphi_{\xi_{1}, \xi_{2}, \ldots, \xi_{n}} \in X, \quad \xi_{i} \in R_{1}, \quad i=1,2, \ldots, n\right\}
$$

and $A(T, H, f)=\varphi_{\xi_{1}, \xi_{2}, \ldots, \xi_{n}} \in F_{A}$, where value $\xi_{i}, i=1,2, \ldots, n$, depends on the components of the vector $T(H, f)$ and for calculation of these values it is required to execute only finite number of arithmetic operators (A.O.) on the $\delta_{1}(H), \delta_{2}(H), \ldots, \delta_{\mu}(H), \delta_{\mu+1}(f), \ldots, \delta_{m}(f)$. We denote by $\mathcal{A}_{N}(T)$ the set of algorithms $A$ in which it is required to execute no more than $N$ A.O. on the components of the vector (2) to determine $A(T, H, f) \in F_{A}$. In considering algorithms of $\mathcal{A}_{N}(T)$ it is natural to suppose that $T \in \mathcal{T}_{M}$ for $M \leq N$. Otherwise, any algorithm of $\mathcal{A}_{N}$ cannot utilize all information represented by the components of the vector (2). The error of the algorithm $A$ on the class $\Psi_{\mathcal{H}, \Phi}^{r}$ is defined as

$$
e\left(\Psi_{\mathcal{H}, \Phi}^{r}, A\right)=\sup _{\substack{z=H z+f \\ H \in \mathcal{H}, f \in \Phi}}\|z-A(T, H, f)\|_{X} .
$$

The quantity

$$
E_{N}\left(\Psi_{\mathcal{H}, \Phi}^{r}\right)=\inf _{\substack{T \in \mathcal{I}_{M} \\ M \leq N}} \inf _{A \in \mathcal{A}_{N}(T)} e\left(\Psi_{\mathcal{H}, \Phi}^{r}, A\right)
$$

is the minimal error, which we are able to guarnatee on the class $\Psi_{\mathcal{H}, \Phi}^{r}$ after the execution $N$ A.O. on the informational functionals $\delta_{i}$. Thus, the quantity $E_{N}$ characterizes the complexity of approximate solving of equations from $\Psi_{\mathcal{H}, \Phi}^{r}$.

Let $\Pi_{N}$ be the set of all continuous maps $\pi$ from $X^{r}$ into $N$-dimensional Euclidian space $R_{N}$. Moreover, let $\pi^{-1} \circ \pi(\varphi)$ be an inverse image of element $\pi(\varphi) \in R_{N}, \varphi \in X^{r}$. The quantity

$$
\Delta_{N}\left(X_{d}^{r}, X\right)=\inf _{\pi \in \Pi_{N}} \sup _{\varphi \in X_{d}^{r}} \sup _{f, g \in \pi^{-1} \circ \pi(\varphi)}\|f-g\|_{X}
$$

is called [3] a pretabulated width of the set

$$
X_{d}^{r}=\left\{\varphi: \varphi \in X^{r},\|\varphi\|_{X^{r}} \leq d\right\} .
$$

The next lemma ascertains a connection between $E_{N}\left(\Psi_{\mathcal{H}, \Phi}^{r}\right)$ and the pretabulated width $\Delta_{N}\left(X_{d}^{r}, X\right)$.
Lemma 1. ([4]) Let $\mathcal{H}$ be some set of the operators $H \in \mathcal{L}\left(X, X^{r}\right)$ for which $\|H\|_{X \rightarrow X^{r}} \leq \gamma_{1},\left\|(I-H)^{-1}\right\|_{X \rightarrow X} \leq \gamma_{2}$. Then

$$
E_{N}\left(\Psi_{\mathcal{H}, \Phi}^{r}\right) \geq \frac{1}{2} \Delta_{N}\left(X_{d}^{r}, X\right),
$$

where $d=\left(1+\gamma_{1}\right)^{-1}$.
Now consider the set of methods for specifying information which we call the Galerkin information. The so-called Galerkin method of approximate solution of Eq. (1) reduces to the situation where to Eq. (1) there is assigned a uniquely solvable equation

$$
z_{G}=P_{n} H z_{G}+P_{n} f
$$

and $z_{G}$ is taken as an approximate solution of (1). It is clear that $z_{G}=$ $\sum_{i=1}^{n} \eta_{i} e_{i}$, where unknown coefficients $\eta_{i}$ will be found from the following system of linear equations

$$
\eta_{i}=\sum_{j=1}^{n} \eta_{j}\left(e_{i}, H e_{j}\right)+\left(e_{i}, f\right) .
$$

Thus, in the Galerkin method, to construct the approximate solution $z_{G}$ it is necessary to have the information (2), where $\delta_{1}(H)=\left(e_{i_{1}}, H e_{j_{1}}\right), \ldots, \delta_{\mu}(H)=$ $\left(e_{i_{\mu}}, H e_{j_{\mu}}\right), \delta_{\mu+1}(f)=\left(e_{i_{\mu+1}}, f\right), \ldots, \delta_{m}(f)=\left(e_{i_{m}}, f\right)$. Information of this type we call the Galerkin information.

Denote by $\Psi_{\gamma}^{r, s}$ the calss of Eq. (1) whose free terms $f$ belong to the ball $X_{1}^{r}$ and operators $H$ belong to the class

$$
\begin{aligned}
\mathcal{H}_{\gamma}^{r, s}= & \left\{H: H \in \mathcal{L}\left(X, X^{r}\right), \quad H^{*} \in \mathcal{L}\left(X, X^{s}\right), \quad\|H\|_{X \rightarrow X^{r}} \leq \gamma_{1},\right. \\
& \left.\left\|(I-H)^{-1}\right\|_{X \rightarrow X} \leq \gamma_{2}, \quad\left\|H^{*}\right\|_{X \rightarrow X^{s}} \leq \gamma_{3}\right\} .
\end{aligned}
$$

Note, that the complexity of classes $\Psi_{\gamma}^{r, s}$ was investigated already in [4], but in [4] one assumed that $s \leq r \leq 2 s$. Our purpose is to find the exact power-order of the complexity for the whole scale of classes $\Psi_{\gamma}^{r, s}$.

Let us introduce some notation: we write $a_{n} \prec b_{n}$, if there is a constant $c_{0}$ such that for all $n \geq n_{0} a_{n} \leq c_{0} b_{n}$. We write $a_{n} \asymp b_{n}$, if $a_{n} \prec b_{n}$ and $b_{n} \prec a_{n}$.

Let $\Gamma_{m}$ be the set of the form

$$
\Gamma_{m}=\{1\} \times\left[1,2^{2 m}\right] \bigcup_{j=1}^{2 m}\left(2^{j-1}, 2^{j}\right] \times\left[1,2^{2 m-j}\right]
$$

We consider the method of specifying information $T_{m}^{\Gamma}$, determined by Galerkin functionals $\left(e_{i}, H e_{j}\right)$ with numbers from $\Gamma_{m}$. Namely, $T_{m}^{\Gamma}(H, f)$ is Galerkin information of the form

$$
T_{m}^{\Gamma}(H, f)=\left(\left(e_{i}, H e_{j}\right), \quad\left(e_{k}, f\right) ; \quad(i, j) \in \Gamma_{m}, \quad k=1,2, \ldots, 2^{2 m}\right) .
$$

It is easy to show that

$$
\begin{equation*}
T_{m}^{\Gamma} \in \mathcal{T}_{M}, \quad M \asymp m 2^{2 m} . \tag{3}
\end{equation*}
$$

Let us assign to each operator $H \in \mathcal{H}_{\gamma}^{r, s}$ the finite-dimensional operator

$$
H^{\Gamma}=H^{\Gamma}(H):=\sum_{j=1}^{2 m}\left(P_{2^{j}}-P_{2^{j-1}}\right) H P_{2^{2 m-j}}+P_{1} H P_{2^{2 m}} .
$$

For each Eq. (1) we determine the sequence of elements

$$
\begin{array}{r}
z_{0}=0, \quad z_{k}=z_{k+1}+\left(I-H^{\Gamma} P_{2^{n}}\right)^{-1}\left(H^{\Gamma} z_{k-1}-z_{k-1}+P_{2^{2 m}} f\right),  \tag{4}\\
k=1,2, \ldots ; \quad n=[2 n / 3] .
\end{array}
$$

All these elemnts belong to the subspace span $\left\{e_{1}, e_{2}, \ldots, e_{22 m}\right\}$. To construct the element $z_{k}$ it suffices to have the information $T_{m}^{\Gamma}(H, f)$ and to solve the equations

$$
\begin{aligned}
& \varepsilon_{l}=H^{\Gamma} P_{2^{n}} \varepsilon_{l}\left(H^{\Gamma} z_{l-1}-z_{l-1}+P_{2^{2 m}} f\right), \quad l=1,2, \ldots, k, \\
& z_{l}=z_{l-1}+\varepsilon_{l} .
\end{aligned}
$$

The discrepancy $\varepsilon_{l}$ is sought in the form

$$
\varepsilon_{l}=\sum_{i=1}^{2^{n}} q_{i} H^{\Gamma} e_{i}+H^{\Gamma} z_{l-1}-z_{l-1}+P_{2^{2 m}} f
$$

where the unknown coefficients $q_{i}, i=1,2, \ldots, 2^{n}$, will be found from the following system of linear equations

$$
q_{i}=\sum_{j=1}^{2^{n}} q_{j}\left(e_{i}, H e_{j}\right)+\left(e_{i}, H^{\Gamma} z_{l-1}-z_{l-1}+P_{2^{2 m}} f\right), \quad i=1,2, \ldots, 2^{n} .
$$

Now we consider the algorithm $A_{m}$ from $\mathcal{A}_{N}\left(T_{m}^{\Gamma}\right)$ for which $A_{m}\left(T_{m}^{\Gamma}, H, f\right)=$ $z_{k_{*}}$, where $k_{*}=l, l-1<2 r / s \leq l$.
Theorem 1. If for the pretabulated width of the ball $X_{d}^{r}$ we have the estimate

$$
\begin{equation*}
\Delta_{N}\left(X_{d}^{r}, X\right) \succ N^{-r}, \tag{5}
\end{equation*}
$$

then

$$
\begin{equation*}
N^{-r} \prec E_{N}\left(\Psi_{\gamma}^{r, s}\right) \prec N^{-r} \log ^{r+1} N . \tag{6}
\end{equation*}
$$

The algorithm $A_{m}$ and Galerkin information $T_{m}^{\Gamma}(H, f), m 2^{2 m} \asymp N$, are order-optimal in the power scale in the sense of the quantity $E_{N}\left(\Psi_{\gamma}^{r, s}\right)$.

The proof of Theorem 1 is based on the following lemmas.
Lemma 2. ([4]) In the algorithm $A_{m}$, to represent a solution

$$
A_{m}\left(T_{m}^{\Gamma}, H, f\right)=z_{k_{*}},
$$

where $k_{*}=l, l-1<2 r / s \leq l$, of any Eq. (1) of the class $\Psi_{\gamma}^{r, s}$ in the form

$$
A_{m}\left(T_{m}^{\Gamma}, H, f\right)=\sum_{i=1}^{2^{2 m}} \alpha_{i} e_{i}
$$

it suffices to perform no more than $O\left(m 2^{2 m}\right)$ A.O. on the component of the vector $T_{m}^{\Gamma}(H, f)$.

Lemma 3. For $H \in \mathcal{H}_{\gamma}^{r, s}$ and $H^{\Gamma}=H^{\Gamma}(H)$ we have

$$
\begin{align*}
\left\|H-H^{\Gamma}\right\|_{X \rightarrow X} & \prec 2^{-2 r s m /(r+s)}  \tag{7}\\
\left\|H-H^{\Gamma}\right\|_{X^{r} \rightarrow X} & \prec m 2^{-2 r m} . \tag{8}
\end{align*}
$$

Proof. First of all we note that for $H \in \mathcal{H}_{\gamma}^{r, s}$ and for any $k=1,2, \ldots$,

$$
\begin{align*}
& \left\|H-P_{2^{k}} H\right\|_{X \rightarrow X} \leq \gamma_{1} \beta_{r} 2^{-k r} \\
& \left\|H-H P_{2^{k}}\right\|_{X \rightarrow X} \leq \gamma_{3} \beta_{s} 2^{-k s} \tag{9}
\end{align*}
$$

Using these estimates, we obtain

$$
\begin{align*}
& \left\|\left(P_{2^{J}}-P_{2^{J-1}}\right) H\left(I-P_{2^{2 m-j}}\right)\right\|_{X \rightarrow X}  \tag{10}\\
\leq & \min \left\{\left\|\left(P_{2^{J}}-P_{2^{J-1}}\right) H\right\|_{X \rightarrow X},\left\|H\left(I-P_{2^{2 m-j}}\right)\right\|_{X \rightarrow X}\right\} \\
\prec & \begin{cases}2^{-(2 m-j)^{s}}, & 0 \leq j \leq j_{0}, \\
2^{-j r}, & j_{0}+1 \leq j \leq 2 m,\end{cases}
\end{align*}
$$

where $j_{0}=[2 s m /(r+s)]$. Since by virtue of the definition of $H^{\Gamma}$ the following representation holds

$$
\begin{equation*}
P_{2^{2 m}} H-H^{\Gamma}=\sum_{j=1}^{2 m}\left(P_{2^{J}}-P_{2^{J-1}}\right) H\left(I-P_{2^{2 m-j}}\right)+P_{1} H\left(I-P_{2^{2 m}}\right), \tag{11}
\end{equation*}
$$

taking into account (10) and (11), we have

$$
\begin{equation*}
\left\|P_{2^{2 m}} H-H^{\Gamma}\right\|_{X \rightarrow X} \prec \sum_{j=0}^{j_{0}} 2^{-(2 m-j) s}+\sum_{j=j_{0}+1}^{2 m} 2^{-j r} \prec 2^{-2 r s m /(r+s)} \tag{12}
\end{equation*}
$$

Reasoning similarly and using (10) and (11), we find

$$
\begin{align*}
\left\|P_{2^{2 m}} H-H^{\Gamma}\right\|_{X^{r} \rightarrow X} & \prec 2^{-2 r m}\left(2^{-2 s m} \sum_{j=0}^{j_{0}} 2^{j(r+s)}+\sum_{j=j_{0}+1}^{2 m} 1\right)  \tag{13}\\
& \prec m 2^{-2 r m} .
\end{align*}
$$

The estimate (7) follows from (9), (12) and the inequality

$$
\left\|H-H^{\Gamma}\right\|_{X \rightarrow X} \leq\left\|H-P_{2^{2 m}} H\right\|_{X \rightarrow X}+\left\|P_{2^{2 m}} H-H^{\Gamma}\right\|_{X \rightarrow X}
$$

In a similar manner, taking into account (13), we obtain (8). The lemma is proved.

Lemma 4. For $H \in \mathcal{H}_{\gamma}^{r, s}, H^{\Gamma}=H^{\Gamma}(H)$ and $n=\left[\frac{2}{3} n\right]$ we have

$$
\begin{align*}
& \left\|H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right\|_{X \rightarrow X} \prec \begin{cases}2^{-2 r s m /(r+s)}, & r / s \leq 1 / 2, \\
2^{-2 s m / 3}, & r / s \geq 1 / 2,\end{cases}  \tag{14}\\
& \left\|H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right\|_{X^{r} \rightarrow X} \prec \begin{cases}m 2^{-2 r m}, & r / s \leq 1 / 2, \\
2^{-2(r+s) m / 3}, & r / s>1 / 2 .\end{cases} \tag{15}
\end{align*}
$$

Proof. We prove relation (15) (relation (14) is proved in the same way). It follows from (9) for $H \in \mathcal{H}_{\gamma}^{r, s}$ that

$$
\left\|H-H P_{2^{n}}\right\|_{X^{r} \rightarrow X} \leq \beta_{r} \beta_{s} \gamma_{3} 2^{-(r+s) n} \asymp 2^{-2(r+s) m / 3} .
$$

Now from Lemma 3 we find

$$
\begin{aligned}
\left\|H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right\|_{X^{r} \rightarrow X} & \leq\left\|H^{\Gamma}-H P_{2^{n}}\right\|_{X^{r} \rightarrow X}+\left\|\left(H-H^{\Gamma}\right) P_{2^{n}}\right\|_{X^{r} \rightarrow X} \\
& \leq 2\left\|H-H^{\Gamma}\right\|_{X^{r} \rightarrow X}+\left\|H-H P_{2^{n}}\right\|_{X^{r} \rightarrow X} \\
& +\left\|H-H^{\Gamma}\right\|_{X \rightarrow X}\left\|I-P_{2^{n}}\right\|_{X^{r} \rightarrow X} \\
& \prec m 2^{-2 r m}+2^{-2(r+s) m / 3}+2^{-2 r s m /(r+s)} 2^{-2 r m / 3} .
\end{aligned}
$$

It means that we obtain the desired estimate. The lemma is proved.
Proof of Theorem 1. The required lower estimate (6) follows from Lemma 1 and (5).

To obtain the upper estimate (6), we calculate the error of the algorithm $A_{m}$ on the class $\Psi_{\gamma}^{r, s}$. Let us assign to each Eq. (1) the equation

$$
\begin{equation*}
\tilde{z}=H^{\Gamma}(H) \tilde{z}+P_{2^{2 m}} f . \tag{16}
\end{equation*}
$$

From relations (7), (14), and from the theorem on the invertibility of a linear operator that is close to an invertible operator it follows that for $H \in \mathcal{H}_{\gamma}^{r, s}$

$$
\begin{equation*}
\left\|\left(I-H^{\Gamma}\right)^{-1}\right\|_{X \rightarrow X} \leq \frac{\left\|(I-H)^{-1}\right\|_{X \rightarrow X}}{1-\left\|(I-H)^{-1}\right\|_{X \rightarrow X}\left\|H-H^{\Gamma}\right\|_{X \rightarrow X}} \leq c_{1}, \tag{17}
\end{equation*}
$$

(18) $\left\|\left(I-H^{\Gamma} P_{2^{n}}\right)^{-1}\right\|_{X \rightarrow X} \leq \frac{\left\|\left(I-H^{\Gamma}\right)^{-1}\right\|_{X \rightarrow X}}{1-\left\|\left(I-H^{\Gamma}\right)^{-1}\right\|_{X \rightarrow X}\left\|H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right\|_{X \rightarrow X}} \leq c_{2}$,
where $c_{1}$ and $c_{2}$ are dependent only on $\gamma_{1}, \gamma_{2}, \gamma_{3}, r, s$. Moreover, for $H \in \mathcal{H}_{\gamma}^{r, s}$ and $f \in X_{1}^{r}$ there is the estimate

$$
\begin{equation*}
\|z\|_{X^{r}}=\left\|H(I-H)^{-1} f+f\right\|_{X^{r}} \leq \gamma_{1} \gamma_{2}+1 . \tag{19}
\end{equation*}
$$

Taking into account (8), (17), (19), we have

$$
\begin{equation*}
\|z-\tilde{z}\|_{X}=\left\|\left(I-H^{\Gamma}\right)^{-1}\left(f-P_{2^{2 m}} f+\left(H-H^{\Gamma}\right) z\right)\right\|_{X} \prec m 2^{-2 m r} . \tag{20}
\end{equation*}
$$

Since for the solution $\tilde{z}$ of equation (16) there is the representation

$$
\tilde{z}=z_{k-1}+\left(I-H^{\Gamma}\right)^{-1}\left(H^{\Gamma} z_{k-1}-z_{k-1}+P_{2^{2 m}} f\right),
$$

by virtue of (4), for any $k=1,2, \ldots, k_{*}$, the following relation holds

$$
\begin{equation*}
\tilde{z}-z_{k}=\left(\left(I-H^{\Gamma} P_{2^{n}}\right)^{-1}\left(H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right)\right)^{k-1}\left(\tilde{z}-z_{1}\right) . \tag{21}
\end{equation*}
$$

Let $r / s>1 / 2$. We put in (21) $k=1$ and by means of the Lemma 4 and (16), (18)-(20) we obtain

$$
\begin{aligned}
\left\|\tilde{z}-z_{1}\right\|_{X} & \leq c_{2}\left(\left\|\left(H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right) z\right\|_{X}+\left\|H^{\Gamma}-H^{\Gamma} P_{2^{n}}\right\|_{X \rightarrow X}\|z-\tilde{z}\|_{X}\right) \\
& \prec 2^{-2(r+s) m / 3} .
\end{aligned}
$$

Now if we set in (21) $k=k_{*}$, then analogously we find

$$
\left\|\tilde{z}-z_{k_{*}}\right\|_{X} \prec 2^{-2 s k_{*} m / 3} 2^{-2 r m / 3} \prec 2^{-2 r m} .
$$

From the last inequality it follows that
(22) $\left\|z-A_{m}\left(T_{m}^{\Gamma}, H, f\right)\right\|_{X}=\left\|z-z_{k_{*}}\right\|_{X} \leq\|z-\tilde{z}\|_{X}+\left\|\tilde{z}-z_{k_{*}}\right\|_{X} \prec m 2^{-2 r m}$.

It is easy to see that if $r / s \leq 1 / 2$ then the estimate (22) holds for $k_{*}=1$. Using (3), (22) and Lemma 2, we obtain the upper estimate (6) for $N \asymp$ $m 2^{2 m}$

$$
E\left(\Psi_{\gamma}^{r, s}\right) \leq e\left(\Psi_{\gamma}^{r, s}, A_{m}\right) \prec N^{-r} \log ^{r+1} N .
$$

Thus, the theorem 1 is proved.
Now we consider the class $\bar{\Psi}_{\gamma}^{r}$ of Eq. (1) with free terms $f \in X_{1}^{r}$ and smoothing operators $H$ from

$$
\overline{\mathcal{H}}_{\gamma}^{r}=\left\{H: H \in \mathcal{H}_{\gamma}^{r, r} \cap \mathcal{L}\left(X^{r}, X^{2 r}\right), \quad\|H\|_{X^{r} \rightarrow X^{2 r}} \leq \gamma_{4}\right\} .
$$

Denote by $T_{m}^{G}$ the method of specifying information, determined by Galerkin functionals with numbers from the set

$$
G_{m}=\{1\} \times\left[1,2^{2 m}\right] \bigcup_{j=1}^{m}\left(2^{j-1}, 2^{j}\right] \times\left[1,2^{2 m-j}\right]
$$

Let us assign to each operator $H \in \overline{\mathcal{H}}_{\gamma}^{r}$ the finite-dimensional operator

$$
H^{G}=\sum_{j=1}^{m}\left(P_{2^{j}}-P_{2^{j-1}}\right) H P_{2^{2 m-j}}+P_{1} H P_{2^{2 m}} .
$$

Further, we consider the algorithm $\bar{A}_{m}\left(T_{m}^{G}\right)$ in which as an approximate solution of Eq. (1) from $\bar{\Psi}_{\gamma}^{r}$ we take the element

$$
z\left(\bar{A}_{m}\right)=z_{n}+\left(I-H^{G} P_{2^{n}}\right)^{-1}\left(H^{G} z_{n}-z_{n}+P_{2^{2 m}} f\right),
$$

where $z_{n}$ is a solution of the equation

$$
z_{n}=H^{G} P_{2^{n}} z_{n}+P_{2^{2 m}} f, \quad n=\left[\frac{2}{3} m\right]
$$

Theorem 2. If for the pretabulated width of the ball $X_{d}^{r}$ we have the estimate $\Delta_{N}\left(X_{d}^{r}, X\right) \succ N^{-r}$, then

$$
N^{-r} \prec E_{N}\left(\bar{\Psi}_{\gamma}^{r}\right) \prec N^{-r} \log ^{r} N .
$$

The algorithm $\bar{A}_{m}$ and Galerkin information $T_{m}^{G}(H, f), m 2^{2 m} \asymp N$, are order-optimal in the power scale in the sense of the quantity $E_{N}\left(\bar{\Psi}_{\gamma}^{r}\right)$.

Theorem 2 is proved in a way analogous to Theorem 1.
In conclusion of present paper we consider the Volterra integral equations with differentiable kernels.

Let $L_{2}$ be the space of functions which are square-summable on $(0,1)$ with usual norm, and let $W_{2}^{r}, r=1,2, \ldots$, be a normed space of continuous functions $f(t)$ whose $f^{(r-1)}$ are absolutely continuous on $[0,1]$ and $f^{(r)} \in L_{2}$. Therewith $\|f\|_{W_{2}^{r}}=\|f\|_{L_{2}}+\left\|f^{(r)}\right\|_{L_{2}}$. We denote by $\chi_{1}(t)$, $\chi_{2}(t), \ldots, \chi_{n}(t), \ldots$ the Haar orthonormal basis. Let $S_{n}$ be the orthogonal projector on $\operatorname{span}\left\{\chi_{1}, \chi_{2}, \ldots, \chi_{n}\right\}$. It is known [5, p. 82] that for $f \in W_{2}^{r}$
$\left\|f-S_{n} f\right\|_{L_{2}} \prec n^{-r}\|f\|_{W_{2}^{r}}$. We consider the class $\overline{\mathcal{H}}_{V}$ of integral operators $H z(t)=\int_{0}^{t} h(t, \tau) z(\tau) d \tau$ for which

$$
\max _{0 \leq t, \tau \leq 1}|h(t, \tau)|+\left(\int_{0}^{1} \int_{0}^{1}\left(\left|\frac{\partial h(t, \tau)}{\partial t}\right|^{2}+\left|\frac{\partial h(t, \tau)}{\partial \tau}\right|^{2}+\left|\frac{\partial^{2} h(t, \tau)}{\partial \tau^{2}}\right|^{2}\right) d t d \tau\right)^{1 / 2} \leq \gamma_{0}
$$

We denote by $\bar{\Psi}_{V}$ the class of the Volterra equations

$$
z(t)=H z(t)+f(t) \equiv \int_{0}^{t} h(t, \tau) z(\tau) d \tau+f(t)
$$

whose free terms $f(t)$ belong to the unit ball of the space $W_{2}^{r}$ and $H \in \overline{\mathcal{H}}_{V}$. Note that

$$
\begin{equation*}
\bar{\Psi}_{V} \subset \bar{\Psi}_{\gamma}^{r} \tag{23}
\end{equation*}
$$

for $X=L_{2}, r=1, X^{1}=W_{2}^{1}, X^{2}=W_{2}^{2}, P_{m}=S_{m}$ and some parameters $\gamma_{1}-\gamma_{4}$ depending only on $\gamma_{0}$.

Moreover, it is known (see [6, pp.220,247]) that

$$
\begin{equation*}
\Delta_{N}\left(W_{2, d}^{r}, L_{2}\right) \succ N^{-r}, \tag{24}
\end{equation*}
$$

where $W_{2, d}^{r}$ is the ball of the space $W_{2}^{r}$ whose radius is equal to $d$.
Thus, from Theorem 2, Lemma 1 and (23), (24) we obtain the following statement.

## Theorem 3.

$$
N^{-1} \prec E_{N}\left(\bar{\Psi}_{V}\right) \prec N^{-1} \log _{N} .
$$

The indicated optimal order in the power scale of the class $\bar{\Psi}_{V}$ is realized by the algorithm $\bar{A}_{m}$ and Galerkin information $T_{m}^{G}(H, f), m 2^{2 m} \asymp N$, which are constructed on the basis of Haar's orthonormal system.

## REFERENCES

1. J.F. Traub and H. Wozniakowski: General Theory of Optimal Algorithms. Academic Press, New York, 1980.
2. S. V. Pereverzev: Optimization of Methods for Approximate Solution of Operator Equations. Nova, New York, 1995.
3. K. I. Babenko: Estimating the quality of computational algorithms: 1. Comput. Methods Appl. Mech. Engrg. 7 (1976), 47-63.
4. S. V. Pereverzev and C. C. Scharipov: Information complexity of equations of the second kind with compact operators in Hilbert space. J. of Complexity 8 (1992), 176-202.
5. B. S. Kashin and A. A. Saakian: Orthogonal Series. Nauka, Moscow, 1984 (Russian).
6. V. M. Tikhomirov: Some Problems on Approximate Theory. Moscow State Univ. Pub., Moscow, 1976 (Russian).

Institute of Mathematics
Ukrainian Academy of Sciences
Tereschenkivska str. 3
252601 Kiev
Ukraine

