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ON THE SPARSE REPRESENTATION OF OPERATORS FOR SOLVING ILL–POSED PROBLEMS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. We propose a new scheme of discretization for solving ill–posed problems and show that combination of this scheme with Morzov's discrepancy principle allows to obtain the best possible order of accuracy of Tikhonov Regularization using an amount of information which is far less than for the standard discretization.

1. The aim of this paper is to describe an economical method for the discretization of ill–posed linear operator equations of the first kind

$$Ax = f\,. \tag{1.1}$$

To construct this method we shall use the relations originally arisen within the framework of Information–Based Complexity research [2], [3].

Let $e_1, e_2, \dots, e_m, \dots$ be some orthonormal basis of Hilbert space X, and let P_m be the orthogonal projector on *span* $\{e_1, e_2, \dots, e_m\}$. We denote by X^r, $r = 1, 2, \dots$, the linear subspace of X which is equipped with the norm

$$\|\varphi\|_{X^r} = \|\varphi\|_X + \sum_{j=1}^{r} \|D_j \varphi\|_X\,,$$

where D_j are some linear operators acting from X^r to X, and for any $m = 1, 2, \dots$

$$\|I - P_m\|_{X^r \to X} \le c_r m^{-r}\,, \tag{1.2}$$

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where I is the identity operator and the constant c_r is independent of m.

Following [4], we consider the class of operators

$$\mathcal{H}_\gamma^r = \left\{ A \, : \, \|A\|_{X\to X^r} + \|A^*\|_{X\to X^r} + \sum_{j=1}^{r} \|(D_j A)^*\|_{X\to X^r} \le \gamma \right\} ,$$

where B^* denotes the adjoint operator of $B : X \to X$, i.e. for any $f, g \in X$ $(f, Bg) = (g, B^* f)$.

As illustrated in [4], the space X^r and the class $\mathcal{H}_\gamma^r$ are a generalization of the spaces of differentiable functions and of the classes of integral operators with kernels having mixed partial derivatives.

Let us introduce some notation: If $N(b)$ and $M(b)$ are functions defined on some set B, we write $N(b) \asymp M(b)$ if there are the constants $c, c_1 > 0$ such that for all $b \in B$ $cM(b) \le N(b) \le c_1 M(b)$. Moreover, for simplicity we often use the same symbol c for possibly different constants.

We shall study the equations (1.1) with $A \in H_\gamma^r$ and $f \in \mathrm{Range}(A)$, i.e. equation (1.1) is solvable, but we assume that only an approximation $f_\delta \in X$ to f is available such that $\|f - f_\delta\|_X \le \delta$, where δ is a known error bound.

The traditional approach to the discretization of the problem (1.1) lies in the application of the Garlekin method. This means that instead of (1.1) we consider now the equation

$$P_m A P_l x = P_m f_\delta \, . \tag{1.3}$$

But if (1.1) is ill–posed in the sense of lack of continuity of its solutions with respect to the data, regularization techniques are required for solving (1.5). The most famous regularization method is the method of Tikhonov. In Tikhonov regularization a solution of (1.3) and hence (1.1) is approximated by a solution $x_{\alpha,m,l}$ of equation

$$\alpha x + P_l A^* P_m A P_l x = P_l A^* P_m f_\delta \, . \tag{1.4}$$

Note that finding an element $x_{\alpha,m,l}$ reduces to solving a system of $\min_{m,l}$ linear algebraic equations.

One of the most widely used strategy for choosing regularization parameter α is Morzov's "discrepancy principle" [1]. Following [5], we shall consider discrepancy principle in the form tailored for discretized version of Tikhonov regularization and $A \in \mathcal{H}_\gamma^r$: Let $1 < d_1 \le d_2$ and $A_{disc} = P_m A P_l$,

$x_\alpha = x_{\alpha,m,l}$. If $\|P_m f_\delta\|_X \le d_1\delta$, then take $x = 0$ as approximation. If $\|P_m f_\delta\|_X > d_1\delta$, then choose $\alpha \ge \alpha_{\min} = (\gamma c_r l^{-r})^2$ such that

$$d_1\delta \le \|P_m f_\delta - A_{disc} x_\alpha\|_X \le d_2\delta\,. \tag{1.5}$$

If there is no $\alpha \ge \alpha_{\min}$, such that (1.5) holds, then choose $\alpha = \alpha_{\min}$.

The usual discussion of order of accuracy of discretized regularization methods for equations (1.1) is done under the assumption that the exact free term f belongs to the set

$$AM_{p,\rho}(A) := \{f \,:\, f = Au,\ u \in M_{p,\rho}(A)\}\,,$$

where $M_{p,\rho}(A) := \{u \,:\, u = |A|^p v,\ \|v\|_X \le \rho\}$, $|A| = (A^*A)^{1/2}$. It is well known that in this case equation (1.1) has a unique solution $x_0 \in M_{p,\rho}(A)$. Moreover, from [6] it follows that $x_{disc}(R, A, f_\delta)$ is an approximation to the solution of (1.1) obtained within the framework of some discretized regularization method R then

$$\inf_R \sup_{f\in AM_{p,\rho}(A)} \inf_{f_\delta:\|f-f_\delta\|_X\le\delta} \|x_0 - x_{disc}(R, A, f_\delta)\|_X \asymp \delta^{p/(p+1)}\,. \tag{1.6}$$

Therefore in the sequel we shall consider the class $\Phi^{r,p}_{\gamma,p}$ of equations (1.1) with $A \in \mathcal{H}^r_\gamma$, $f \in AM_{p,\rho}(A)$.

2. The ensuing theorem allows to estimate the efficiency of traditional approach to discretization (1.3), (1.4).

Theorem 2.1. [5] *Let the parameter α be chosen according to the discrepancy principle. If equation* (1.1) *belongs to the class $\Phi^{r,p}_{\gamma,\rho}$, $0 < p \le 1$, then*

$$\|x_0 - x_{\alpha,m,l}\|_X \le d_p\left(\delta^{p/(p+1)} + l^{-pr} + m^{-pr}\right)\,,$$

where d_p is independent of δ, l, m.

Let us consider the following situation. We have the information that equation (1.1) belongs to the class $\Phi^{r,p}_{\gamma,\rho}$ for some $p \in (0, 1]$, but we don't know the exact value of p. From the Theorem 2.1 it follows that in this situation within the framework of traditional approach (1.3), (1.4) with discrepancy principle of parameter selection we can guarantee the optimal order of accuracy (1.6) in the case when for all $p \in (0, 1]$ $l \ge \delta^{-1/r(p+1)}$, $m \ge \delta^{-1/r(p+1)}$.

It is obvious that the minimal l and m satisfying above conditions for all $p \in (0,1]$ have the order $l \asymp m \asymp \delta^{-1/r}$.

Denote by Card (IP) the number of inner products of the form

$$(e_i, Ae_j), (e_i, f_\delta) \tag{2.1}$$

required to construct an approximate solution $x_{\alpha,m,l}$ realizing the optimal order of accuracy (1.6) for all $p \in (0,1]$. Using above reasons for Card (IP) we have

$$\mathrm{Card}\,(\mathrm{IP}) = ml + m \asymp \delta^{-2/r}\,. \tag{2.2}$$

3. Now we combine Morzov's "discrepancy principle" of parameter selection with some new discretization scheme and show that for all $p \in (0,1]$ this combination allows to obtain the optimal order of accuracy (1.6) using only $O\left(\delta^{-1.5/r}\log(1/\delta)\right)$ values of inner products of the form (2.1).

Within the framework of above mentioned Morzov's discrepancy principle we set $m = l = 2^n$ and as operator A_{disc} we take the operator

$$A_{disc} = A_n = \sum_{i=1}^{n} (P_{2^k} - P_{2^{k-1}})\, AP_{2^{1.5n-k}} + P_1 AP_{2^{1.5n}}\,. \tag{3.1}$$

Lemma 3.1. *For $A \in \mathcal{H}_\gamma^r$ and $p \in (0,1]$ we have*

$$\|(A_n - P_{2^n}A)|A|^p\|_{X\to X} \le c2^{-nr(p+3)/2}\,.$$

Proof. From the definition of operator A_n we find

$$\begin{aligned}
&\|(A_n - P_{2^n}A)|A|^p\|_{X\to X} \\
&\le \sum_{k=1}^{n} \|(P_{2^k} - P_{2^{k-1}})A(P_{2^{1.5n-k}} - I)|A|^p\|_{X\to X} \\
&+\|P_1 A(P_{2^{1.5n}} - I)|A|^p\|_{X\to X}\,.
\end{aligned} \tag{3.2}$$

Using Lemma 4.3 [5] and arguments like that in the proof of Lemma 3.2 [4], we get the estimate

$$\|(P_{2^k} - P_{2^{k-1}})A(P_{2^{1.5n-k}} - I)|A|^p\|_{X\to X} \le c2^{-3nr(p+1)/2}2^{krp}\,, \quad k = 0, 1, \dots, n\,. \tag{3.3}$$

The assertion of the lemma follows from (3.2), (3.3). □

Corollary 3.1. *Let $A \in \mathcal{H}_\gamma^r$ and $f \in AM_{p,\rho}(A)$, $0 < p \leq 1$. If $x_0 \in M_{p,\rho}(A)$ is the solution of equation* (1.1) *then*

$$\|A_n x_0 - P_{2^n} f_\delta\|_X \leq \delta + c2^{-nr(p+3)/2}\,.$$

Indeed, for $x_0 \in M_{p,\rho}(A)$ we have $x_0 = |A|^p z$, $\|z\|_X \leq \rho$. Then

$$\begin{aligned}\|A_n x_0 - P_{2^n} f_\delta\|_X &\leq \|A_n x_0 - P_{2^n} A x_0\|_X + \|P_{2^n}(f - f_\delta)\|_X \\ &\leq \delta + \|(A - n - P_{2^n} A)|A|^p z\|_X \leq \delta + c2^{-nr(p+3)/2}\,.\end{aligned}$$

Lemma 3.2. *For $A \in \mathcal{H}_\gamma^r$ and $p \in (0,1]$ we have*

$$\|P_{2^{1.5n}}|A|^p - |A_n|^p\|_{X \to X} \leq c2^{-nrp}\,.$$

Proof. By analogy with Lemma 3 [3] we can obtain the inequality

$$\|A - A_n\|_{X \to X} \leq c2^{-nr}\,. \tag{3.4}$$

Moreover, from Lemma 4.3 [5] and from the definition of A_n it follows that $P_{2^{1.5n}}|A_n|^p = |A_n|^p$. Then, using (3.4) and Lemma 4.1 [5], we have

$$\begin{aligned}\|P_{2^{1.5n}}|A|^p - |A_n|^p\|_{X \to X} &\leq \||A|^p - |A_n|^p\|_{X \to X} \leq \\ &\leq \frac{4}{\pi}\|A - A_n\|_{X \to X}^p \leq c2^{-nrp}\,. \quad \square\end{aligned}$$

4. Finally, we give the following result:

Theorem 4.1. *Let $x_{\alpha,n}$ be the approximate solution of equation* (1.1) *obtained within the framework of Morzov's "discrepancy principle" for $A_{disc} = A_n$ and $l = m = 2^n$. If equation* (1.1) *belongs to the class $\Phi_{\gamma,p}^{r,p}$ $0 < p \leq 1$, and $x_0 \in M_{p,\rho}(A)$ is the solution of* (1.1) *then*

$$\|x_0 - x_{\alpha,n}\|_X \leq c\left(2^{-nrp} + \delta^{p/(p+1)}\right)\,.$$

Proof. We put $R_{\alpha,n} = (\alpha I + A_n^* A_n)^{-1} A_n^*$, $S_{\alpha,n} = I - R_{\alpha,n} A_n$. From [5] one sees that for $\alpha > 0$

$$\|R_{\alpha,n}\|_{X \to X} \leq c_1 \alpha^{-1/2}\,, \quad \|S_{\alpha,n}\|_{X \to X} \leq c_2\,, \tag{4.1}$$

$$
\begin{aligned}
&\|I - A_n R_{\alpha,n}\|_{X\to X} \le 1\,, \\
&\|A_n S_{\alpha,n}|A_n|^p\|_{X\to X} \le c_p \alpha^{(p+1)/2}\,, \quad p \in [0,1]\,.
\end{aligned} \tag{4.2}
$$

Using (4.1) and Corollary 3.1, from the definition $x_{\alpha,n}$ we find

$$
\begin{aligned}
\|x_0 - x_{\alpha,n}\|_X &= \|S_{\alpha,n}x_0 + R_{\alpha,n}(A_n x_0 - P_{2^n} f_\delta\|_X \\
&\le \|S_{\alpha,n}x_0\|_X + c_1\alpha^{-1/2}\left(\delta + c2^{-nr(p+3)/2}\right) \\
&\le c_1\alpha^{-1/2}\delta + \|S_{\alpha,n}x_0\|_X + c2^{-nr(p+1)/2}\,.
\end{aligned} \tag{4.3}
$$

Now following [5], we consider the element

$$
A_n S_{\alpha,n} x_0 = (P_{2^n} f_\delta - A_n x_{\alpha,n}) + (I - A_n R_{\alpha,n})(A_n x_0 - P_{2^n} f_\delta)\,.
$$

From (4.2) and (1.5) we have

$$
d_1\delta - \|A_n x_0 - P_{2^n} f_\delta\|_X \le \|A_n S_{\alpha,n} x_0\|_X \le d_2\delta + \|A_n x_0 - P_{2^n} f_\delta\|_X\,. \tag{4.4}
$$

Thus, from Corollary 3.1 one sees that

$$
\alpha^{-1/2}\delta \le (d_1 - 1)^{-1}\left(\alpha^{-1/2}\|A_n S_{\alpha,n}x_0\|_X + c2^{-nr(p+1)/2}\right)\,. \tag{4.5}
$$

Note that $A_n S_{\alpha,n} = A_n S_{al,n} P_{2^{1.5n}}$. Then, using (4.2) and Lemma 3.2 for $x_0 \in M_{p,\rho}(A)$ we have

$$
\begin{aligned}
\alpha^{-1/2}\|A_n S_{\alpha,n} x_0\|_X &\le \alpha^{-1/2}\,(\|A_n S_{\alpha,n}|A_n|^p v\|_X \\
&\quad + \|A_n S_{\alpha,n}(P_{2^{1.5n}}|A|^p - |A_n|^p)v\|_X) \\
&\le \alpha^{-1/2}\left(c_p\alpha^{(p+1)/2} + c_0\alpha^{1/2}c2^{-nrp}\right) \\
&\le c_p\alpha^{p/2} + c2^{-nrp}\,.
\end{aligned} \tag{4.6}
$$

Let us estimate $\|S_{\alpha,n}x_0\|_X$. Note that

$$
\|S_{\alpha,n}x_0\|_X \le \|S_{\alpha,n}P_{2^{1.5n}}x_0\|_X + \|S_{\alpha,n}(x_0 - P_{2^{1.5n}}x_0)\|_X\,. \tag{4.7}
$$

It is easy to see that for $x_0 \in M_{p,\rho}(A)$

$$
\|S_{\alpha,n}(x_0 - P_{2^{1.5n}}x_0)\|_X = \|x_0 - P_{2^{1.5n}}x_0\|_X \le (c_r\gamma)^p\rho 2^{-1.5nrp}\,. \tag{4.8}
$$

Moreover, the same steps like in the proof of (4.6) lead to the estimate

$$\|S_{\alpha,n}P_{2^{1.5n}}x_0\|_X \le c\left(\alpha^{p/2} + 2^{-nrp}\right) . \tag{4.9}$$

If $\alpha \le \delta^{2/(p+1)} + 2^{-2nr}$, the assertion of the theorem follows from (4.3), (4.5), (4.6) and (4.7)–(4.8).

Assume now that $\alpha > \beta = \delta^{2/(p+1)} + 2^{-2nr}$. With an argument like that in the proof of Theorem 3.3 [5] we get the estimate

$$\|S_{\alpha,n}P_{2^{1.5n}}x_0\|_X^2 \le c\left(\|S_{\beta,n}P_{2^{1.5n}}x_0\|_X^2 + \beta^{-1}\|A_nS_{\alpha,n}P_{2^{1.5n}}x_0\|_X^2\right) . \tag{4.10}$$

On the other hand, from (4.4) and Corollary we know that

$$\begin{aligned}\beta^{-1}\|A_nS_{\alpha,n}P_{2^{1.5n}}x_0\|_X^2 &\le \beta^{-1}\left[(d_2+1)\delta + c2^{-nr(p+3)/2}\right] \\ &\le c\left(\delta^{2p/(p+1)} + 2^{-nr(p+1)}\right) .\end{aligned} \tag{4.11}$$

Again using the inequality (4.9) we obtain

$$\|S_{\beta,n}P_{2^{1.5n}}x_0\|_X^2 \le c\left(\delta^{2p/(p+1)} + 2^{-2nrp}\right) . \tag{4.12}$$

Uniting (4.3), (4.7), (4.8) and (4.10)–(4.12) for $\alpha > \delta^{2/(p+1)} + 2^{-2nr}$ we have

$$\|x_0 - x_{\alpha,n}\|_X \le c_1\delta\alpha^{-1/2} + c\left(\delta^{p/(p+1)} + 2^{-nrp}\right) \le c\left(\delta^{p/(p+1)} + 2^{-nrp}\right) .$$

The theorem is proved. □

Let now Card (IP) be the number of inner products of the form (2.1) required to construct an approximate solution $x_{\alpha,n}$ From the Theorem 4.1 it follows that within the framework of combination of Morzov's "discrepancy principle" with discretization scheme (3.1) we can guarantee on the classes $\Phi^{r,p}_{\gamma,\rho}$ the optimal order of accuracy (1.6) for all $p \in (0,1]$ in case when $2^n = \delta^{-1/r}$ (it is clear that for such n $\|x_0 - x_{\alpha,n}\|_X = O\left(\delta^{p/(p+1)}\right)$ for all $p \in (0,1]$). Now from (3.1) it follows that

$$\text{Card (IP)} \asymp \sum_{k=0}^{n} 2^{1.5n} \asymp n2^{1.5n} \asymp \delta^{-1.5/r}\log\frac{1}{\delta} .$$

When this relation is compared with (2.2) it is apparent that for the classes $\Phi^{r,p}_{\gamma,\rho}$, $0 < p \leq 1$, the discretization scheme (3.1) is more economical then traditional approach (1.3), (1.4).

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