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ON THE DIVERGENT PROPERTIES OF TWO–SIDED SOR WEIERSTRASS METHOD

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Abstract. In this paper the convergence properties of the two-side successive over-relaxation (TSSOR) method of Weierstrass' type is discussed. Initial conditions under which this method is divergent are stated. The main theorem generalizes some recent results concerning divergent properties of simultaneous methods for finding polynomial zeros.

Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a monic polynomial with simple real roots $\xi_1 < \xi_2 < \ldots < \xi_n$. We assume that the roots are located in n nonintersecting real intervals $X_i^0 = [\underline{x}_i^0, \overline{x}_i^0]$, $i = 1, 2, \ldots, n$, that is, $X_i^0 \cap X_j^0 = \emptyset$ for $i \neq j$ and $\xi_i \in X_i^0$ for $i = 1, 2, \ldots, n$.

In this paper we are concerned with the two-sided method

$$\begin{aligned}
\overline{x}_i^{k+1} &= \overline{x}_i^k - \overline{h}_k \frac{f(\overline{x}_i^k)}{\prod_{j=1}^{i-1}(\overline{x}_i^k - \underline{x}_j^k) \prod_{j=i+1}^{n}(\overline{x}_i^k - \overline{x}_j^k)}, \\
\underline{x}_i^{k+1} &= \underline{x}_i^k - \underline{h}_k \frac{f(\underline{x}_i^k)}{\prod_{j=1}^{i-1}(\underline{x}_i^k - \underline{x}_j^k) \prod_{j=i+1}^{n}(\underline{x}_i^k - \overline{x}_j^k)},
\end{aligned} \tag{1}$$

$$i = 1, 2, \ldots, n; \; k = 0, 1, 2, \ldots,$$

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where $\overline{h}_k,\ \underline{h}_k \in (0,1]$ are acceleration parameters. (The symbols $\prod_{j=1}^{0}$ and $\prod_{j=n+1}^{n}$ is assumed to be equal to one). This method is an obvious generalization of Kjurkchiev-Markov's method [3] which is obtained from (1) for $\overline{h}_k = 1,\ \underline{h}_k = 1$, and has the form of the classical Weierstrass' method

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{j\neq i}(x_i^k - x_j^k)} \quad i = 1,2,\dots,n;\ k = 0,1,2,\dots .$$

Denote

$$\overline{h}^*_{k,i} = \overline{h}_k \prod_{j=1}^{i-1} \frac{\overline{x}_i^k - \overline{x}_j^k}{\overline{x}_i^k - \underline{x}_j}, \quad \underline{h}^*_{k,i} = \underline{h}_k \prod_{j=i+1}^{n} \frac{\underline{x}_i^k - \underline{x}_j^k}{\underline{x}_i^k - \overline{x}_j^k}.$$

It is easy to find

$$\overline{x}_i^{k+1} = \overline{x}_i^k - \overline{h}^*_{k,i} \frac{f(\overline{x}_i^k)}{\prod\limits_{j\neq i}^{n}(\overline{x}_i^k - \overline{x}_j^k)}, \qquad \underline{x}_i^{k+1} = \underline{x}_i^k - \underline{h}^*_{k,i} \frac{f(\underline{x}_i^k)}{\prod\limits_{j\neq i}^{n}(\underline{x}_i^k - \underline{x}_j^k)}.$$

The method (1) is two-side successive over-relaxation (TSSOR) iteration, i.e.,

$$\underline{x}_1^k \le \xi_1 \le \overline{x}_1^k < \underline{x}_2^k \le \xi_2 \le \overline{x}_2^k < \dots < \underline{x}_n^k \le \xi_n \le \overline{x}_n^k,$$

and also $0 < \overline{h}^*_{k,i} \le 1,\ 0 < \underline{h}^*_{k,i} \le 1$ for each $k = 0,1,2,\dots$. The details are tedious and lengthy, and are therefore omitted. We refer to [3] for more details.

It is well known that the simultaneous methods of Weierstrass' type are globally convergent for almost all initial approximations to the roots. The aim of this paper is to present some divergent properties of the TSSOR method (1). For this purpose we give first Theorem 1.

Theorem 1. *For* $\underline{x}_i^k,\ \overline{x}_i^k,\ i = 1,\dots,n,$ *determined by* (1)*, and for* $k = 1,2,\dots$ *, the following relations are valid*

$$\sum_{i=1}^{n} \frac{1}{\overline{h}^*_{k,i}} \overline{x}_i^{k+1} = \sum_{i=1}^{n} \left(\frac{1}{\overline{h}^*_{k,i}} - 1 \right) \overline{x}_i^k - a_{n-1},$$

$$(2) \qquad \sum_{i=1}^{n} \frac{1}{\overline{h}^*_{k,i}} \overline{x}_i^{k+1} \sum_{j\neq i}^{n} \overline{x}_j^k = \sum_{\nu<s}^{n} \left(\frac{1}{\overline{h}^*_{k,\nu}} + \frac{1}{\overline{h}^*_{k,s}} - 1 \right) \overline{x}_\nu^k \overline{x} + a_{n-2},$$

$$\dots$$

$$\sum_{i=1}^{n} \frac{1}{\overline{h}^*_{k,i}} \overline{x}_i^{k+1} \prod_{j\neq i}^{n} \overline{x}_j^k = \left(\sum_{i=1}^{n} \frac{1}{\overline{h}^*_{k,i}} - 1 \right) \prod_{j\neq i}^{n} \overline{x}_j^k + (-1)^n a_0$$

and

$$
(3)\quad
\begin{aligned}
&\sum_{i=1}^{n}\frac{1}{\underline{h}^{*}_{k,i}}\underline{x}_i^{k+1}=\sum_{i=1}^{n}\left(\frac{1}{\underline{h}^{*}_{k,i}}-1\right)\underline{x}_i^{k}-a_{n-1},\\
&\sum_{i=1}^{n}\frac{1}{\underline{h}^{*}_{k,i}}\underline{x}_i^{k+1}\sum_{j\neq i}^{n}\underline{x}_j^{k}=\sum_{\nu<s}^{n}\left(\frac{1}{\underline{h}^{*}_{k,\nu}}+\frac{1}{\underline{h}^{*}_{k,s}}-1\right)\underline{x}_\nu^{k}\underline{x}_s^{k}+a_{n-2},\\
&\dots\\
&\sum_{i=1}^{n}\frac{1}{\underline{h}^{*}_{k,i}}\underline{x}_i^{k+1}\prod_{j\neq i}^{n}\underline{x}_j^{k}=\left(\sum_{i=1}^{n}\frac{1}{\underline{h}^{*}_{k,i}}-1\right)\prod_{j\neq i}^{n}\underline{x}_j^{k}+(-1)^n a_0.
\end{aligned}
$$

The proof follows the ideas given in [1], [2], [4] and [5] and will be omitted.

Now we state initial conditions under which the TSSOR method (1) is divergent.

Theorem 2. *The iteration* (1) *will fail if the endpoints* $\underline{x}_1^0,\dots\underline{x}_n^0,\ \overline{x}_1^0,\dots,\overline{x}_n^0$ *of the inclusion initial intervals* $X_1^0,\dots,X_n^0$ *satisfy the systems of nonlinear equations*

$$
(4)\quad
\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{1}{\overline{h}^{*}_{0,i}}-1\right)\overline{x}_i^{0}-a_{n-1}=0,\\
&\sum_{\nu<s}^{n}\left(\frac{1}{\overline{h}^{*}_{0,\nu}}+\frac{1}{\overline{h}^{*}_{0,s}}-1\right)\overline{x}_\nu^{0}\overline{x}_s^{0}+a_{n-2}=0,\\
&\dots\\
&\left(\sum_{i=1}^{n}\frac{1}{\overline{h}^{*}_{0,i}}-1\right)\prod_{j\neq i}^{n}\overline{x}_j^{0}+(-1)^n a_0=0
\end{aligned}
$$

and

$$
(5)\quad
\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{1}{\underline{h}^{*}_{0,i}}-1\right)\underline{x}_i^{0}-a_{n-1}=0,\\
&\sum_{\nu<s}^{n}\left(\frac{1}{\underline{h}^{*}_{0,\nu}}+\frac{1}{\underline{h}^{*}_{0,s}}-1\right)\underline{x}_\nu^{0}\underline{x}_s^{0}+a_{n-2}=0,\\
&\dots\\
&\left(\sum_{i=1}^{n}\frac{1}{\underline{h}^{*}_{0,i}}-1\right)\prod_{j\neq i}^{n}\underline{x}_j^{0}+(-1)^n a_0=0.
\end{aligned}
$$

Proof. Let us rewrite the equations (2), (3) in the vector form for $k = 0$ as

$$A\overline{\mathbf{x}}^{\mathbf{1}} = \mathbf{a}, \quad B\underline{\mathbf{x}}^{\mathbf{1}} = \mathbf{b},$$

where $\overline{\mathbf{x}}^{\mathbf{1}} = \{\overline{x}_1^1, \ldots, \overline{x}_n^1\}^T$, $\underline{\mathbf{x}}^{\mathbf{1}} = \{\underline{x}_1^1, \ldots, \underline{x}_n^1\}^T$ are the vectors of approximations in the first step, $\mathbf{a}$ and $\mathbf{b}$ are the vectors which components are the terms on the left-hand side of (4) and (5), respectively, and

$$A = \begin{pmatrix} \frac{1}{\overline{h}^*_{0,1}} & \frac{1}{\overline{h}^*_{0,2}} & \cdots & \frac{1}{\overline{h}^*_{0,n}} \\ \frac{1}{\overline{h}^*_{0,1}} \sum\limits_{j\neq 1} \overline{x}_j^0 & \frac{1}{\overline{h}^*_{0,2}} \sum\limits_{j\neq 2} \overline{x}_j^0 & \cdots & \frac{1}{\overline{h}^*_{0,n}} \sum\limits_{j\neq n} \overline{x}_j^0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\overline{h}^*_{0,1}} \prod\limits_{j\neq 1} \overline{x}_j^0 & \frac{1}{\overline{h}^*_{0,2}} \prod\limits_{j\neq 2} \overline{x}_j^0 & \cdots & \frac{1}{\overline{h}^*_{0,n}} \prod\limits_{j\neq n} \overline{x}_j^0 \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{\underline{h}^*_{0,1}} & \frac{1}{\underline{h}^*_{0,2}} & \cdots & \frac{1}{\underline{h}^*_{0,n}} \\ \frac{1}{\underline{h}^*_{0,1}} \sum\limits_{j\neq 1} \underline{x}_j^0 & \frac{1}{\underline{h}^*_{0,2}} \sum\limits_{j\neq 2} \underline{x}_j^0 & \cdots & \frac{1}{\underline{h}^*_{0,n}} \sum\limits_{j\neq n} \underline{x}_j^0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\underline{h}^*_{0,1}} \prod\limits_{j\neq 1} \underline{x}_j^0 & \frac{1}{\underline{h}^*_{0,2}} \prod\limits_{j\neq 2} \underline{x}_j^0 & \cdots & \frac{1}{\underline{h}^*_{0,n}} \prod\limits_{j\neq n} \underline{x}_j^0 \end{pmatrix}$$

are the matrices of the systems. Evidently,

$$\det A = A_1 \frac{1}{\prod\limits_{\mu=1}^{n} \overline{h}_{0,\mu}}, \quad \det B = B_1 \frac{1}{\prod\limits_{\mu=1}^{n} \underline{h}_{0,\mu}},$$

where

$$A_1 = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum\limits_{j\neq 1} \overline{x}_j^0 & \sum\limits_{j\neq 2} \overline{x}_j^0 & \cdots & \sum\limits_{j\neq n} \overline{x}_j^0 \\ \vdots & \vdots & \ddots & \vdots \\ \prod\limits_{j\neq 1} \overline{x}_j^0 & \prod\limits_{j\neq 2} \overline{x}_j^0 & \cdots & \prod\limits_{j\neq n} \overline{x}_j^0 \end{pmatrix},$$

$$B_1 = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum\limits_{j\neq 1} \underline{x}_j^0 & \sum\limits_{j\neq 2} \underline{x}_j^0 & \cdots & \sum\limits_{j\neq n} \underline{x}_j^0 \\ \vdots & \vdots & \ddots & \vdots \\ \prod\limits_{j\neq 1} \underline{x}_j^0 & \prod\limits_{j\neq 2} \underline{x}_j^0 & \cdots & \prod\limits_{j\neq n} \underline{x}_j^0 \end{pmatrix}.$$

We show that $A_1 = \prod_{i<j}^{n}(\overline{x}_i^0 - \overline{x}_j^0)$ and $B_1 = \prod_{i<j}^{n}(\underline{x}_i^0 - \underline{x}_j^0)$. For simplicity, we will omit the iteration index $k = 0$ and write x_i instead of $\overline{x}_i^0$ or $\underline{x}_i^0$, $i = 1, \ldots, n$.

First, we consider the determinant $D^{n+1-r}_{j_1^{(1)},\ldots,j_r^{(1)};\ldots;j_1^{(n+1-r)},\ldots,j_r^{(n+1-r)}}$ defined by

$$\begin{vmatrix} 1 & \cdots & 1 \\ \sum\limits_{\substack{l_1\neq j_k^{(1)}\\ k=1,\ldots,r}} x_{l_1} & \cdots & \sum\limits_{\substack{l_1\neq j_k^{(n+1-r)}\\ k=1,\ldots,r}} x_{l_1} \\ \sum\limits_{\substack{l_1,l_2\neq j_k^{(1)}\\ l_1<l_2\\ k=1,\ldots,r}} x_{l_1}x_{l_2} & \cdots & \sum\limits_{\substack{l_1,l_2\neq j_k^{(n+1-r)}\\ l_1<l_2\\ k=1,\ldots,r}} x_{l_1}x_{l_2} \\ \vdots & & \vdots \\ \sum\limits_{\substack{l_t\neq j_k^{(1)}\\ l_1<\cdots<l_{n-r}\\ k=1,\ldots,r\\ t=1,\ldots,n-r}} x_{l_1}\cdots x_{l_{n-r}} & \cdots & \sum\limits_{\substack{l_t\neq j_k^{(n+1-r)}\\ l_1<\cdots<l_{n-r}\\ k=1,\ldots,r\\ t=1,\ldots,n-r}} x_{l_1}\cdots x_{l_{n-r}} \end{vmatrix}.$$

Evidently, for $r = 1$, $D^n_{1;2;\ldots;n}$ is identical with A_1. Subtract the second column from the first, the third column from the second, and so on, the last from the $(n-1)$th. The result is

$$D^n_{1;2;\ldots;n} = \begin{vmatrix} 1 & \cdots & 1 \\ \sum\limits_{l_1\neq 1} x_{l_1} & \cdots & \sum\limits_{l_1\neq n} x_{l_1} \\ \sum\limits_{\substack{l_1,l_2\neq 1\\ l_1<l_2}} x_{l_1}x_{l_2} & \cdots & \sum\limits_{\substack{l_1,l_2\neq n\\ l_1<l_2}} x_{l_1}x_{l_2} \\ \vdots & & \vdots \\ \sum\limits_{\substack{l_1,\ldots,l_{n-1}\neq 1\\ l_1<\cdots<l_{n-1}}} x_{l_1}\cdots x_{l_{n-1}} & \cdots & \sum\limits_{\substack{l_1,\ldots,l_{n-1}\neq n\\ l_1<\cdots<l_{n-1}}} x_{l_1}\cdots x_{l_{n-1}} \end{vmatrix}$$

$$
= \begin{vmatrix}
0 & \dots & 0 & U_{ij}^{(n)} \\
1 & \dots & 1 & \sum\limits_{l_1 \neq n} x_{l_1} \\
\sum\limits_{l_1 \neq 1,2} x_{l_1} & \dots & \sum\limits_{l_1 \neq n-1,n} x_{l_1} & \sum\limits_{\substack{l_1, l_2 \neq n \\ l_1 < l_2}} x_{l_1} x_{l_2} \\
\vdots & & \vdots & \vdots \\
\sum\limits_{\substack{l_1 < \cdots < l_{n-2} \\ l_j \neq 1,2 \\ j=1,\dots,n-2}} x_{l_1} \cdots x_{l_{n-2}} & \dots & \sum\limits_{\substack{l_1 < \cdots < l_{n-2} \\ l_j \neq n-1,n \\ j=1,\dots,n-2}} x_{l_1} \cdots x_{l_{n-2}} & \sum\limits_{\substack{l_1 < \cdots < l_{n-1} \\ l_j \neq n \\ j=1,\dots,n-1}} x_{l_1} \cdots x_{l_{n-1}}
\end{vmatrix}
$$

where $U_{ij}^{(n)} = (-1)^{n-1} \prod_{j=1}^{n-1} (x_i - x_j)$. From this it follows that

$$
\begin{aligned}
D^n_{1;2;\dots;n} &= \prod_{j=1}^{n-1} (x_j - x_{j+1}) D^{n-1}_{1,2;2,3;\dots;n-1,n} \\
&= \prod_{j=1}^{n-1} (x_j - x_{j+1}) \prod_{j=1}^{n-2} (x_j - x_{j+2}) D^{n-2}_{1,2,3;2,3,4;\dots;n-2,n-1,n}
\end{aligned}
$$

so that

$$
\begin{aligned}
D^n_{1;2;\dots;n} &= \prod_{j=1}^{n-1} (x_j - x_{j+1}) \prod_{j=1}^{n-2} (x_j - x_{j+2}) \times \\
&\quad \times \cdots \times \prod_{j=1}^{2} (x_j - x_{j+n-2}) D^2_{1,2,\dots,n-1;2,3,\dots,n} \\
&= \prod_{s=2}^{n-1} \prod_{j=1}^{s} (x_j - x_{j+n-s}) \cdot \begin{vmatrix} 1 & 1 \\ \sum\limits_{l_1 \neq 1,2,\dots,n-1} x_{l_1} & \sum\limits_{l_1 \neq 2,3,\dots,n} x_{l_1} \end{vmatrix} \\
&= \prod_{s=2}^{n-1} \prod_{j=1}^{s} (x_j - x_{j+n-s}) \cdot (x_1 - x_n) = \prod_{i<j}^{n} (x_i - x_j).
\end{aligned}
$$

Evidently, $\det A \neq 0$, $\det B \neq 0$. According to this fact and from $A\overline{\mathbf{x}}^{\mathbf{1}} = \mathbf{a}$, $B\underline{\mathbf{x}}^{\mathbf{1}} = \mathbf{b}$ (with $\mathbf{a}, \mathbf{b} = 0$ due to (4) and (5)) we get $\overline{\mathbf{x}}^{\mathbf{1}} = \{0, \ldots, 0\}^T$, $\underline{\mathbf{x}}^{\mathbf{1}} = \{0, \ldots, 0\}^T$ and the method (1) cannot be defined at the second step. This proves Theorem 2. $\square$

Remark. Wang and Zhao [6] constructed the following *prediction-correction* method

$$x_i^{k+1} = x_i^k - h_k \frac{f(x_i^k)}{\prod\limits_{j \neq i}^{n} (x_i^k - x_j^k)}.$$

The optimal value of h_k in the sense of a guaranteed convergence is not known. Concerning Wang-Zhao's acceleration parameter (see [6])

$$h_k = \min_i \left(1, : 0.204378 d_k \left(\sum_{i=1}^{n} \left| \frac{f(x_i^k)}{\prod_{j \neq i}^{n} (x_i^k - x_j^k)} \right| \right)^{-1} \right),$$

$$d_k = \min_{i \neq j} |x_i^k - x_j^k|, \quad i = 1, \ldots, n; \quad k = 0, 1, 2, \ldots,$$

at each iteration a new value of h_k can be evaluated taking into account the new information.

In view of the last remark, we consider the following two-sided prediction-correction method

$$\overline{x}_i^{k+1} = \overline{x}_i^k - \overline{h}_k^* \frac{f(\overline{x}_i^k)}{\prod\limits_{j \neq i}^{n} (\overline{x}_i^k - \overline{x}_j^k)},$$

$$\underline{x}_i^{k+1} = \underline{x}_i^k - \underline{h}_k^* \frac{f(\underline{x}_i^k)}{\prod\limits_{j \neq i}^{n} (\underline{x}_i^k - \underline{x}_j^k)},$$

$$\overline{h}_k^* = \min_i \overline{h}_{k,i}^*, \quad \overline{h}_{k,i}^* = \overline{h}_k \prod_{j=1}^{i-1} \frac{\overline{x}_i^k - \overline{x}_j^k}{\overline{x}_i^k - \underline{x}_j^k},$$

$$\underline{h}_k^* = \min_i \underline{h}_{k,i}^*, \quad \underline{h}_{k,i}^* = \underline{h}_k \prod_{j=i+1}^{n} \frac{\underline{x}_i^k - \underline{x}_j^k}{\underline{x}_i^k - \overline{x}_j^k},$$

$$j = 1, 2, \ldots, n; \ k = 0, 1, 2, \ldots .$$

As approximations approach to the zeros, $\overline{h}_k^*$, $\underline{h}_k^*$ become larger and larger, until they get the value 1 defining two-sided variant of Weierstrass' method in the continuation of the iterative procedure.

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