FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 14 (1999), 109–134

AN ALGORITHM FOR THE EVALUATION OF THE TWO-DIMENSIONAL HILBERT TRANSFORM WITH NON-STANDARD WEIGHT FUNCTIONS

Maria Carmela De Bonis

Abstract. The author proposes an algorithm for the numerical evaluation of two-dimensional Cauchy principal value integrals with respect to generalized Ditzian-Totik weight functions. Convergence results and some numerical examples are given.

1. Introduction

Let $\Phi(f;s,t)$ denote the two-dimensional Cauchy principal value of the function f, which is defined by

$$\begin{aligned}\Phi(f;s,t) &= \int_I \frac{f(x,y)}{(x-s)(y-t)} w_1(x)w_2(y)dxdy \\ &= \lim_{\varepsilon_1,\varepsilon_2\to 0^+} \int\int_{D_{\varepsilon_1\varepsilon_2}} \frac{f(x,y)}{(x-s)(y-t)} w_1(x)w_2(y)dxdy \end{aligned} \tag{1.1}$$

where $I=[-1,1]^2$, $D_{\varepsilon_1,\varepsilon_2}=\{(x,y)\in I \ : \ |x-s|\geq \varepsilon_1,\ |y-t|\geq \varepsilon_2\}$ $(\varepsilon_1,\varepsilon_2>0)$, w_1,w_2 are two generalized Ditzian-Totik weight functions $(w_1,w_2\in GDT)$ of the following type:

$$w_1(x)=\psi(x)(1+x)^{\gamma_0}(1-x)^{\gamma_1}|x-t_2|^{\gamma_2}\log^{\gamma_3}\frac{e}{|x-t_3|}\,, \tag{1.2}$$

$$w_2(y)=\psi(y)(1+y)^{\bar{\gamma}_0}(1-y)^{\bar{\gamma}_1}|y-\bar{t_2}|^{\bar{\gamma}_2}\log^{\bar{\gamma}_3}\frac{e}{|y-\bar{t_3}|}\,, \tag{1.3}$$

where ψ is an Hölder continuous function, $\gamma_i,\bar{\gamma}_i>-1, i=0,1,2,\gamma_3,\bar{\gamma}_3\geq 0, -1\leq t_2<t_3\leq 1, -1\leq \bar{t_2}<\bar{t_3}\leq 1$ and $s\in(-1,1)\setminus\{t_2,t_3\}$, $t\in(-1,1)\setminus\{\bar{t_2},\bar{t_3}\}$.

Received February 16, 1996.
1991 *Mathematics Subject Classification*. Primary 65D30, 41A05.

In [7] the author proposed an algorithm for the numerical evaluation of the integral $\Phi(f;s,t)$ when w_1, w_2 are two Jacobi weight functions, i.e. $\gamma_2 = \gamma_3 = \bar{\gamma_2} = \bar{\gamma_3} = 0$ in (1.2) and (1.3).

The above mentioned algorithm makes use of a Gauss type quadrature rule that results convergent and stable.

However, this Gauss type quadrature rule requires the knowledge of the zeros and the Christoffel numbers of the orthogonal polynomials with respect to the weight functions w_1 and w_2.

Since the zeros and the Christoffel numbers of the orthogonal polynomials with respect to the generalized Ditzian-Totik weight functions w_1 and w_2 are not available in literature, the above mentioned algorithm cannot be used.

Following an idea proposed in [11] for the evaluation of one-dimensional Hilbert transform, in this paper, we propose an algorithm that allow us to overcome the above mentioned numerical difficulties. In fact, we approximate the integral $\Phi(f;s,t)$ by using an *interpolatory product rule* in which, irrespective of w_1 and w_2, the quadrature knots are chosen to be the zeros of the Chebyshev polynomials of first kind.

We show that the numerical amplification factor of the proposed quadrature rule results bounded, except for logarithmic factors. Finally, we prove the convergence of our method and we give error estimates for many classes of function.

This paper is organized as follows. In Section 2 we state the above mentioned algorithm. In Section 3 we give some error estimates, while section 4 contains their proofs. In Section 5 we give some numerical examples and in the Appendix we give the proof of Lemma 4.1.

2. An Algorithm for the Evaluation of Two-Dimensional Hilbert Transform

Let $L_{m,n}(f,x,y)$ be the Lagrange interpolating polynomial of f at the zeros

$$x_{m,i} = \cos\frac{(2i-1)\pi}{2m} \qquad (i = 1, \dots, m)$$

and

$$y_{n,j} = \cos\frac{(2j-1)\pi}{2n} \qquad (j = 1, \dots, n)$$

of the Chebyshev orthonormal polynomials of first kind $T_m(x)$ and $T_n(y)$,

respectively. We have

$$L_{m,n}(f,x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} l_{m,i}(x) l_{n,j}(y) f(x_{m,i}, y_{n,j}), \tag{2.1}$$

where

$$l_{m,i}(s) = \frac{\pi}{m}\sum_{k=0}^{m-1} T_k(x) T_k(x_{m,i}) \tag{2.2}$$

and

$$l_{n,j}(t) = \frac{\pi}{n}\sum_{r=0}^{n-1} T_r(y) T_r(y_{n,j}) \tag{2.3}$$

are the fundamental Lagrange polynomials.

Replacing f by $L_{m,n}(f,x,y)$ into (1.1), we obtain the following Interpolatory Product Rule for the numerical evaluation of the integral $\Phi(f;s,t)$ where $(s,t) \in (-1,1)^2$:

$$\Phi_{m,n}(f;s,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{m,i}^{w_1}(s) B_{n,j}^{w_2}(t) f(x_{m,i}, y_{n,j}), \tag{2.4}$$

where

$$A_{m,i}^{w_1}(s) = \int_{-1}^{1} \frac{l_{m,i}(x)}{x-s} w_1(x) dx \tag{2.5}$$

and

$$B_{n,j}^{w_2}(t) = \int_{-1}^{1} \frac{l_{n,j}(y)}{y-t} w_2(y) dy. \tag{2.6}$$

The degree of exactness of this formula is (m,n) at least, i.e.,

$$\Phi(x^m y^n, s, t) = \Phi_{m,n}(x^m y^n, s, t).$$

In order to describe the algorithm, we give a representation for the coefficients (2.5) and (2.6).

Substituting (2.2) into (2.5) and following an idea used in [11], we obtain

$$A_{m,i}^{w_1}(s) = \frac{\pi}{m}\sum_{k=0}^{m-1} T_k(x_{m,i})\int_{-1}^{1}\frac{T_k(x)}{x-s}w_1(x)dx = \frac{\pi}{m}\sum_{k=0}^{m-1} T_k(x_{m,i})Q_k(w_1;s), \tag{2.7}$$

where

$$Q_k(w_1;s) = \int_{-1}^{1}\frac{T_k(x)}{x-s}w_1(x)dx.$$

Analogously, we obtain

$$B_{n,j}^{w_2}(t) = \frac{\pi}{m}\sum_{r=0}^{n-1} T_r(y_{n,j})Q_r(w_2;t), \tag{2.8}$$

where

$$Q_r(w_2;t) = \int_{-1}^{1}\frac{T_r(y)}{y-y}w_2(y)dy.$$

Applying (2.7) and (2.8) we can rewrite formula (2.4) in the form

$$\Phi_{m,n}(f;s,t) = \frac{\pi}{m}\frac{\pi}{n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=0}^{m-1} T_k(x_{m,i})Q_k(w_1;s)\right)\times \times\left(\sum_{r=0}^{n-1} T_r(y_{n,j})Q_r(w_2;t)\right) f(x_{m,i},y_{n,j}). \tag{2.9}$$

Since the Chebyshev orthonormal polynomials of first kind $T_i(z)$ satisfy the following very simple three-term recurrence relation

$$T_{i+1}(z) = 2zT_i(z) - T_{i-1}(z), \quad i = 2,3,\dots,$$

where

$$T_0(z) = \frac{1}{\sqrt{\pi}}, \quad T_1(z) = \sqrt{\frac{2}{\pi}}z, \quad T_2(z) = 2\sqrt{\frac{2}{\pi}}z^2 - \sqrt{\frac{2}{\pi}}.$$

It is easy to verify that the functions $Q_i(w,\lambda)$ satisfy the following relationship

$$Q_{i+1}(w;\lambda) = 2\lambda Q_i(w,\lambda) - Q_{i-1}(w,\lambda) + 2M_i(w), \quad i = 2,3,\dots, \tag{2.10}$$

where

$$M_i(w) = \int_{-1}^{1} T_i(z)w(z)dz, \qquad i = 0,1,\dots,$$

are the so-called "modified moments" (see [8]). Here,

$$\begin{aligned}
Q_0(w;\lambda) &= \frac{1}{\sqrt{\pi}}\int_{-1}^{1}\frac{w(z)}{z-\lambda}dz,\\
Q_1(w;\lambda) &= \lambda\sqrt{2}Q_0(w;\lambda) + \sqrt{2}M_0(w),\\
Q_2(w;\lambda) &= 2\lambda Q_1(w;\lambda) - \sqrt{2}Q_0(w;\lambda) + 2M_1(w).
\end{aligned}$$

Taking into account the above recurrence relationships, we observe that in (2.9) the only non trivial computations are those of $Q_0(w;\lambda)$ and of $M_i(w)$, $i = 0,1,\dots$.

For some weight functions, $M_i(w)$, $i = 0,1,\dots$, may be computed by using the recurrence relationships developed in [12] and in [1]. In [6] it can be found the explicit expressions for $Q_0(w,\lambda)$ when w is one of the following weight functions:

1° $|x-a|^\gamma$, $\gamma > -1$ with $\gamma \neq 0,1,2,\dots$ and $a,\lambda \in (-1,1)$ with $a \neq \lambda$;

2° $(1-x^2)^\alpha|x|^\gamma$, $2\lambda^2 - 1 \in (-1,1)$;

3° $(1-x)^\alpha|x-a|^\gamma$, $\gamma > -1$ and $a,\lambda \in (-1,1)$ with $a \neq \lambda$;

4° $(1+x)^\beta|x-a|^\gamma$, $\gamma > -1$ and $a,\lambda \in (-1,1)$ with $a \neq \lambda$;

5° $(1-x^2)^{-1/2}\log[e/(1+x)]$, $\lambda = 0$.

However, when $Q_0(w,\lambda)$ is not available, it is possible to get its estimate, for example, by numerical integration writing

$$\begin{aligned}
Q_0(w,\lambda) &= \int_{-1}^{1}\frac{w(z)-w(\lambda)}{x-\lambda}dx - w(\lambda)\int_{-1}^{1}\frac{dx}{x-\lambda}\\
&= \int_{-1}^{1}\frac{w(z)-w(\lambda)}{x-\lambda}dx - w(\lambda)\log\frac{1-\lambda}{1+\lambda}.
\end{aligned}$$

3. Stability and Convergence of Quadrature Rule $\Phi_{m,n}$

In this section we want to show that the proposed method converges for many classes of functions and it is numerically efficient. For this purpose, we need the following preliminary definitions and notations.

The symbol $\mathcal{C}$ stands for some positive constant which may take a different value each time it is used.

If A and B are two expressions depending on some variables, then we write $A \sim B$ iff $|AB^{-1}| \leq \mathcal{C}$ and $|A^{-1}B| \leq \mathcal{C}$, uniformly for the considered variables.

We define the *modulus of continuity* of the function $f \in C^0(I)$ by

$$\omega(f;\delta) = \max_{h_1+h_2\leq\delta}\{|\Delta_{h_1,h_2}f(x,y)| \ : \ (x,y)\in I, (x+h_1,y+h_2)\in I\}, \tag{3.1}$$

where $\Delta_{h_1,h_2} = f(x+h_1,y+h_2) - f(x,y)$ and $h_1,h_2,\delta \geq 0$.

Through the modulus of continuity $\omega(f;\delta)$ we may define the space of the $\sigma-$ Hölder continuous functions, $\mathrm{Lip}_M\,\sigma$, by

$$\mathrm{Lip}_M\,\sigma = \left\{f \in C^0(I) \ : \ \omega(f;\delta) \leq M\delta^\sigma, \ 0<\sigma\leq 1, \ M>0\right\}. \tag{3.2}$$

We denote by $C^k(I), k \geq 0$, the space of the continuous functions on $I = [-1,1]^2$ with continuous partial derivatives of order i, where $i = 0,1,\ldots,k$.

Furthermore, if $f \in C^k(I), k \geq 0$, we define the following *uniform modulus of continuity:*

$$\Omega_k(f;\delta) = \max_{i=0,1,\ldots,k}\{\omega(f_i^{k-i};\delta)\},$$

where $f_i^{k-i} = \partial^k f/\partial x^{k-i}\partial y^i$.

We denote by $C^{k+\sigma}(I), k \geq 0$, the following class of functions:

$$C^{k+\sigma}(I) = \left\{f \in C^k(I) \ : \ \Omega_k(f;\delta) \leq M\delta^\sigma, \ 0<\sigma\leq 1, \ M>0\right\}.$$

Obviously, if $k = 0, \Omega_k(f;\delta)$ is equal to $\omega(f;\delta)$, defined by (3.1), and $C^{k+\sigma}$ is equal to $\mathrm{Lip}_M\,\sigma$, defined by (3.2).

Let $u_1(x), u_2(y)$ denote the singular parts of $w_1(x), w_2(y)$ respectively. For example, if $w_1(x) = (1-x)^{-1/2}|x+0.5|^{1/3}$ and $w_2(y) = |y|^{2/3}\log[e/|y|]$, we define $u_1(x) = (1-x)^{-1/2}$ and $u_2(y) = \log[e/|y|]$.

Note that $\Phi_{m,n}$ can be regarded as a discrete operator of which the domain of definition is the space of the σ-Hölder continuous functions. On the other hand, by the following lemma, $\Phi_{m,n}f$ does not appear bounded on the boundary of I. In fact, we have:

Lemma 3.1. *Let $w_1(x), w_2(y)$ as in* (1.2) *and* (1.3)*, respectively. Let $v_1(x)$ and $v_2(y)$ two Chebyshev weight functions of first kind. Let $(s,t) \in (-1,1)^2$. Then the following inequalities*

$$\sum_{i=1}^{m} |A_{m,i}^{w_1}(s)| \leq \mathcal{C} u_1(s) \log m, \tag{3.3}$$

$$\sum_{j=1}^{n} |B_{n,j}^{w_2}(t)| \leq \mathcal{C} u_2(t) \log m \tag{3.4}$$

hold, where $\mathcal{C}$ is a positive constant independent of $m, n, s, t, w_1, w_2, v_1$, and v_2.

Let $u(x,y) = u_1 u_2$. Let m, n be fixed, we have

$$\sup_{s,t \in [-1,1]} |\Phi_{m,n}(f;s,t) u^{-1}(s,t)| < +\infty.$$

Therefore $\Phi_{m,n}(f;s,t)$ is bounded on $C_{u^{-1}}$, where

$$C_{u^{-1}} = \{g \ : \sup_{x,y \in [-1,1]} |g(x,y) u^{-1}(x,y)| < +\infty\}.$$

Then, $\Phi_{m,n}$ is a map from $\mathrm{Lip}_M \sigma$ into $C_{u^{-1}}$ and its norm is

$$\|\Phi_{m,n}\| := \|\Phi_{m,n}\|_{Lip_M \sigma \to C_{u^{-1}}} = \sup_{\|gu^{-1}\|=1} \|(\Phi_{m,n} g) u^{-1}\|.$$

We call $\|\Phi_{m,n}\|$ the "numerical amplification factor" of the quadrature rule $\Phi_{m,n}(f;s,t)$.

The following theorem shows that the "numerical amplification factor" $\|\Phi_{m,n}\|$ results bounded, except for logarithmic factors.

Theorem 3.1. *Let $(s,t) \in (-1,1)^2$. Then the following inequality*

$$\sup_{m,n \in \mathbb{N}} \frac{\|\Phi_{m,n}\|}{\log m \log n} < +\infty \tag{3.5}$$

holds.

The proof of the Theorem 3.1 easily follows by applying Lemma 3.1.

Let $R_{m,n} = \Phi - \Phi_{m,n}$, $\gamma(s,t) = \log \dfrac{e^2}{\sqrt{(1-s^2)(1-t^2)}}$, and $W = \gamma^{-1} u^{-1}$.

The convergence of quadrature rule $\Phi_{m,n}$ is established by the following theorem.

Theorem 3.2. *Let $w_1(x), w_2(y)$ as in (1.2) and (1.3), respectively. Let $(s,t) \in (-1,1)^2$. Assume $f \in C^{k+\sigma}(I)$, $k \geq 0$. Then*

$$\sup_{s,t\in(-1,1)} |W(s,t)R_{m,n}(f;s,t)| \leq \mathcal{C}\left(\frac{1}{m}+\frac{1}{n}\right)^{k+\sigma} \log m \log n, \tag{3.6}$$

where $\mathcal{C}$ is a positive constant independent of m, n, s, t and f.

4. Proofs

In order to prove Lemma 3.1 we need the following result:

Lemma 4.1. *Let $w(x)$ as in (1.2) and $t_0 = -1$, $t_1 = 1$. Then the inequalities*

$$\left|\int_{-1}^{1} \frac{w(x)}{x-\lambda}dx\right| \leq \mathcal{C}\left|\log\frac{1-\lambda}{1+\lambda}\right| w(\lambda)\prod_{j=0}^{2}\Psi_j(\lambda), \tag{4.1}$$

$$\int_{-1}^{1}\left|\frac{w(x)-w(\lambda)}{x-\lambda}\right| dx \leq \mathcal{C}w(\lambda)\prod_{j=0}^{2}\Psi_j(\lambda) \tag{4.2}$$

hold, where

$$\Psi_j(\lambda) = \begin{cases} |x-t_j|^{-\gamma_j} & \gamma_j > 0, \\ 1 & \gamma_j < 0, \\ 1 & \gamma_j = 0, \end{cases}$$

for $j = 0, 1, 2$, and $\mathcal{C}$ is a positive constant independent on w, λ, t_j and γ_j.

The proof of the above lemma follows an idea used in [5, Lemma 3.1, p. 24]. Because of completeness we give it in the Appendix.

For any $s \in (-1,1) \setminus \{t_2, t_3\}, m \in N$, we denote by $x_{m,c}$ the closest knots to s, defined by

$$x_{m,c} = \begin{cases} x_{m,d} & if s - x_{m,d} \leq x_{m,d+1} - s \\ x_{m,d+1} & if s - x_{m,d} > x_{m,d+1} - s, \end{cases}$$

where $x_{m,d} \leq s \leq x_{m,d+1}$ for some $d \in \{0, 1, \ldots, m\}$ with $x_{m,0} = -1$, $x_{m,m+1} = 1$.

Analogously, for any $t \in (-1,1) \setminus \{\bar{t}_2, \bar{t}_3\}$, $n \in \mathbb{N}$, we denote by $y_{n,c}$ the closest knots to t.

Let

$$\sigma_m^*(\lambda) = \sum_{\substack{i=1\\ i\neq c}}^{m} \frac{\lambda_{m,i}(w)}{|x_{m,i}-\lambda|}.$$

Lemma 4.2. *If* $w(x) = v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, *then the inequalities*

$$\sigma_m^*(\lambda) \le \mathcal{C} \log m \quad if \ \alpha, \beta \ge 0, \tag{4.3}$$

$$\sigma_m^*(\lambda) \le \mathcal{C} v^{\alpha,\beta}(\lambda) \log m \quad if \ -1 < \alpha, \beta < 0, \tag{4.4}$$

hold uniformly for $\lambda \in (-1, 1)$.

The proof of the above lemma can be found in [4, Lemma 3.4, p. 454].

Let

$$\delta_m^*(\lambda) = \sum_{\substack{i=1 \\ i \ne c}}^{m} \lambda_{m,i}(w) \frac{|T_{m-1}(x_{m,i})|}{|x_{m,i} - \lambda|}.$$

Lemma 4.3. *If* $w(x) = v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, *then the inequality*

$$\delta_m^*(\lambda) \le \mathcal{C} \sqrt{\sigma_m^*(\lambda) \log m} \tag{4.5}$$

holds uniformly for $\lambda \in (-1, 1)$.

The proof of the above lemma can be found in [2, Lemma 3.1, p. 730].

Proof of Lemma 3.1. We first prove (3.3). Since the following decomposition

$$\begin{aligned} A_{m,i}^{w_1}(s) &= \int_{-1}^{1} l_{m,i}(x) \frac{w_1(x) - w_1(s)}{x - s} dx + \frac{w_1(s)}{v_1(s)} \int_{-1}^{1} \frac{l_{m,i}(x)}{x - s} v_1(s) dx \\ &= \int_{-1}^{1} l_{m,i}(x) \frac{w_1(x) - w_1(s)}{x - s} dx \\ &- \frac{w_1(s)}{v_1(s)} \int_{-1}^{1} l_{m,i}(x) \frac{v_1(x) - v_1(s)}{x - s} dx + \frac{w_1(s)}{v_1(s)} A_{m,i}^{v_1}(s). \end{aligned}$$

holds, we have

$$\begin{aligned} \sum_{i=1}^{m} |A_{m,i}^{w_1}(s)| &\le \|L_m(v_1)\| \int_{-1}^{1} \left| \frac{w_1(x) - w_1(s)}{x - s} \right| dx \\ &+ \frac{w_1(s)}{v_1(s)} \|L_m(v_1)\| \int_{-1}^{1} \left| \frac{v_1(x) - v_1(s)}{x - s} \right| dx \\ &+ \frac{w_1(s)}{v_1(s)} \sum_{i=1}^{m} |A_{m,i}^{v_1}(s)|. \end{aligned} \tag{4.6}$$

Let us evaluate $\sum_{i=1}^{m} |A_{m,i}^{v_1}(s)|$. Putting

$$A_m^*(v_1, s) = \int_{-1}^{1} \frac{v_1(x)}{x-s} dx - \frac{\pi}{m} \sum_{\substack{i=1 \\ i \neq c}}^{m} \frac{1}{x_{m,i} - s},$$

we can see at once that

1° if $s \neq x_{m,c}$, then for $k \neq c$

$$A_{m,i}^{v_1}(s) = A_m^*(v_1, s) l_{m,i}(s) + \frac{\pi}{m} \frac{1}{x_{m,i} - s} + \frac{\gamma_{m-1}}{\gamma_m} \frac{\pi^2}{m^2} \frac{T_{m-1}(x_{m,i})}{s - x_{m,i}} T_m'(\tau_c) \tag{4.7}$$

and for $k = c$

$$A_{m,c}^{v_1}(s) = A_m^*(v_1, s) l_{m,c}(s) + l_{m,c}'(\tau_c') \frac{\pi}{m}, \tag{4.8}$$

where $\tau_c, \tau_c' \in (s, x_{m,c})$;

2° if $s = x_{m,c}$, then for $k \neq c$

$$A_{m,i}^{v_1}(x_{m,c}) = \frac{\pi}{m} \frac{1}{x_{m,i} - x_{m,c}} + \frac{\gamma_{m-1}}{\gamma_m} \frac{\pi^2}{m^2} \frac{T_{m-1}(x_{m,i})}{x_{m,c} - x_{m,i}} T_m'(x_{m,c}) \tag{4.9}$$

and for $k = c$

$$A_{m,c}^{v_1}(x_{m,c}) = A_m^*(v_1, x_{m,c}) + l_{m,c}'(x_{m,c}) \frac{\pi}{m}. \tag{4.10}$$

We observe that by [5, Lemma 3.2, p. 26], we have

$$|A_m^*(v_1, s)| \leq \mathcal{C} v_1(s). \tag{4.11}$$

Applying Bernstein's inequality for a derivative of a polynomial (see [10, p. 92]) and taking into account that $x_{m,c} \sim s$ and $\|l_{m,c}\| \sim 1$, we have

(4.12)

$$\begin{aligned} \left| \frac{\pi}{m} l_{m,c}'(x_{m,c}) \right| &= \frac{\pi}{\sqrt{1 - x_{m,c}^2}} \left| \frac{\sqrt{1 - x_{m,c}^2}}{m} l_{m,c}'(x_{m,c}) \right| \\ &\leq \frac{\pi}{\sqrt{1 - x_{m,c}^2}} \|l_{m,c}\| \\ &\leq \mathcal{C} v_1(s). \end{aligned}$$

Moreover, taking into account the expression of the derivative of the Chebyshev polynomial of first kind, we get

$$\left|\frac{\pi}{m}T'_m(x_{m,c})\right| = \left|\frac{\pi}{m}\frac{m(-1)^{c+1}}{\sqrt{1-x_{m,c}^2}}\right| \leq \mathcal{C}v_1(s). \tag{4.13}$$

Since $\tau_c, \tau'_c \sim x_{m,c}$, in the same manner we can prove that

$$\left|\frac{\pi}{m}l'_{m,c}(\tau_c)\right| \leq \mathcal{C}v_1(s) \tag{4.14}$$

and

$$\left|\frac{\pi}{m}T'_m(\tau_c)\right| \leq \mathcal{C}v_1(s). \tag{4.15}$$

Combining (4.7), (4.8), (4.9) and (4.10) with (4.11), (4.12), (4.13), (4.14), and (4.15), we obtain

$$|A^{v_1}_{m,c}(s)| \leq \mathcal{C}v_1(s) \tag{4.16}$$

and

$$|A^{v_1}_{m,i}(s)| \leq \mathcal{C}v_1(s)|l_{m,i}(s)| + \frac{\pi}{m}\frac{1}{|x_{m,i}-s|} + \mathcal{C}v_1(s)\frac{\pi}{m}\frac{T_{m-1}(x_{m,i})}{|s-x_{m,i}|}. \tag{4.17}$$

Thus, we have

$$\begin{aligned}\sum_{i=1}^{m}|A^{v_1}_{m,i}(s)| &\leq \sum_{\substack{i=1\\ i\neq c}}^{m}|A^{v_1}_{m,i}(s)| + |A^{v_1}_{m,c}(s)| \\ &\leq \sigma^*_m(s) + \mathcal{C}v_1(s)[1 + \|L_m(v_1)\| + \delta^*_m(s)].\end{aligned}$$

By (4.4), (4.5) and taking into account that $\|L_m(v_1)\| \sim 1$ (see [13, Theorem 14.4, p. 335]), we have

$$\sum_{i=1}^{m}|A^{v_1}_{m,i}(s)| \leq \mathcal{C}v_1(s)\log m. \tag{4.18}$$

Finally, combining (4.6) with (4.2) and (4.18), we complete the proof of (3.3).

In the same manner we can prove (3.4). □

In order to prove Theorem 3.2 we need the following auxiliary results.

Lemma 4.4. *For any function $f \in C^k(I), k \geq 0$, and for any $(m,n) \in \mathbb{N} \times \mathbb{N}$, there exists an algebraic polynomial $P_{m,n}(x,y)$ such that,*

$$\|r_{m,n}\|_I \leq \mathcal{C}(m^{-1}+n^{-1})^k \Omega_{k,I}(f; m^{-1}+n^{-1}) \tag{4.19}$$

and

$$\omega(r_{m,n};\delta) \leq \mathcal{C}\omega(f;\sqrt{\delta}) \tag{4.20}$$

where $r_{m,n} = f - P_{m,n}$ and $\mathcal{C}$ is a positive constant independent of m, n, and f. Moreover, for any function $f \in \mathrm{Lip}_M\, \sigma$ the following inequalities

$$|r_{m,n}(s,y) - r_{m,n}(s,t)| \leq \mathcal{C}\ |y-t|^{\xi}\ (m^{-1}+n^{-1})^{\sigma-2\xi}, \tag{4.21}$$

$$|r_{m,n}(x,t) - r_{m,n}1(s,t)| \leq \mathcal{C}\ |x-s|^{\xi}\ (m^{-1}+n^{-1})^{\sigma-2\xi} \tag{4.22}$$

hold, where $2\xi \leq \sigma, x, y, s, t \in [-1,1]$ and $\mathcal{C}$ is a positive constant independent of m, n, s, t, and f.

The proof of Lemma 4.4 can be found in [3].

Let

$$H_1(g;s,y) = \int_{-1}^{1} \frac{g(x,y)-g(s,y)}{x-s}dx,\ \ H_2(g;x,t) = \int_{-1}^{1} \frac{g(x,y)-g(x,t)}{y-t}dy,$$

and $H = H_1H_2 = H_2H_1$, we have the following result:

Lemma 4.5. *For any function $f \in C^{k+\sigma}(I)$, $k \geq 0$, and for sufficiently large m and n, the following inequalities*

$$|H_1(r_{m,n};s,t)| \leq \mathcal{C}(m^{-1}+n^{-1})^{k+\sigma}\log n, \tag{4.23}$$

$$|H_2(r_{m,n};s,t| \leq \mathcal{C}(m^{-1}+n^{-1})^{k+\sigma}\log m \tag{4.24}$$

hold uniformly for $(s,t) \in (-1,1)^2$, where $\mathcal{C}$ is a positive constant independent of m, n, s, t, and f.

Proof. We first prove (4.24). Note that, for every ε sufficiently small,

$$\begin{aligned} |H_2(r_{m,n};s,t)| \leq & \left|\left\{\int_{-1}^{t-m^{-\lambda}} + \int_{t+m^{-\lambda}}^{1}\right\} \frac{(r_{m,n}(s,y)-r_{m,n}(s,t))}{y-t}dy\right| \\ & + \left|\int_{t-m^{-\lambda}}^{t+m^{-\lambda}} \frac{(r_{m,n}(s,y)-r_{m,n}(s,t))^{1-\varepsilon}}{y-t}(r_{m,n}(s,y)-r_{m,n}(s,t))^{\varepsilon}dy\right| \end{aligned} \tag{4.25}$$

holds for $\lambda \geq 0$.

By choosing $\lambda = 2$, we give the proof for the case $k = 0$.

By (4.25) we have

$$|H_2(r_{m,n}; s, t)| \leq 2\|r_{m,n}\| \left|\log m^2(1+t) + \log m^2(1-t)\right| + 2\|r_{m,n}\|^{1-\varepsilon} \int_{t-m^{-\lambda}}^{t+m^{-\lambda}} \frac{|r_{m,n}(s,y) - r_{m,n}(s,t)|^{\varepsilon}}{|y-t|} dy$$

and applying Lemma 4.4, we obtain:

$$\begin{aligned} |H_2(r_{m,n}; s, t)| &\leq \mathcal{C}(m^{-1} + n^{-1})^{\sigma} \log m \\ &\quad + \mathcal{C}(m^{-1} + n^{-1})^{\varepsilon(\sigma - 2\xi) + \sigma(1-\varepsilon)} \int_{t-m^{-\lambda}}^{t+m^{-\lambda}} |y-t|^{\xi\varepsilon - 1} dy \\ &\leq \mathcal{C}(m^{-1} + n^{-1})^{\sigma} \log m \\ &\quad + \mathcal{C}(m^{-1} + n^{-1})^{\varepsilon(\sigma - 2\xi) + \sigma(1-\varepsilon) + 2\xi\varepsilon} \\ &\leq \mathcal{C}(m^{-1} + n^{-1})^{\sigma} \log m \end{aligned}$$

which proves (4.24). In the same manner we can prove (4.23). The case $k > 0$ follows by the same method, by choosing $\lambda = (2\xi + k)/\xi$. $\square$

Lemma 4.6. *For any function $f \in C^{k+\sigma}(I)$, $k \geq 0$, and for sufficiently large m and n, the following inequality*

$$|H(r_{m,n}; s, t)| \leq \mathcal{C}(m^{-1} + n^{-1})^{k+\sigma} \log n \log m$$

holds uniformly for $(s,t) \in (-1,1)^2$, where $\mathcal{C}$ is a positive constant independent of m, n, s, t, and f.

Proof. Our proof starts with the observation that

$$|H(r_{m,n}; s, t)| = \left| \int_{-1}^{1} \int_{-1}^{1} \frac{r_{m,n}(x,y) - r_{m,n}(s,y) - r_{m,n}(x,t) + r_{m,n}(s,t)}{(x-s)(y-t)} dx dy \right|$$

$$\leq \left| \int\int_{|x-s| > \frac{1}{n^2}, |y-t| > \frac{1}{m^2}} + \cdots + \int_{-1}^{1} \int_{|x-s| < \frac{1}{n^2}} \cdots + \int_{-1}^{1} \int_{|y-t| < \frac{1}{m^2}} \cdots \right|,$$

i.e.,

$$
\begin{aligned}
(4.26)\quad |H(r_{m,n};s,t)| \le 4\|r_{m,n}\| &\int_{|x-s|>\frac{1}{n^2}} \frac{dx}{|x-s|} \int_{|y-t|>\frac{1}{m^2}} \frac{dy}{|y-t|} \\
&+ \int_{|x-s|<\frac{1}{n^2}} \frac{|H_2(r_{m,n};x,t) - H_2(r_{m,n};s,t)|}{|x-s|} dx \\
&+ \int_{|y-t|<\frac{1}{m^2}} \frac{|H_1(r_{m,n};s,y) - H_1(r_{m,n};s,t)|}{|y-t|} dy \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

We consider now the case $k = 0$. Applying (4.19) we obtain

$$
(4.27)\qquad A_1 \le C\|r_{m,n}\| \log m \log n \le C(m^{-1} + n^{-1})^{\sigma} \log m \log n.
$$

In order to estimate A_2 we need the following estimate. Namely, since $f \in \mathrm{Lip}_M\, \sigma$, using (4.20) we have

$$
\begin{aligned}
&|H_2(r_{m,n};x,t) - H_2(r_{m,n};s,t)| = \\
&\quad = \left| \int_{-1}^{1} \frac{r_{m,n}(s,y) - r_{m,n}(x,t) - r_{m,n}(s,y) - r_{m,n}(s,t)}{(y-t)} dy \right| \\
&\quad \le 2 \int_{|y-t|\le|x-s|} \frac{\omega(r_{m,n};|y-t|)}{|y-t|} dy + 2 \int_{|y-t|\ge|x-s|} \frac{\omega(r_{m,n};|x-s|)}{|y-t|} dy \\
&\quad \le \mathcal{C} \int_0^{|x-s|} \omega(f;\sqrt{u}) u^{-1} du + \mathcal{C}\omega(f;|x-s|^{1/2}) \log |x-s|^{-1} \\
&\quad \le \mathcal{C}|x-s|^{\sigma/2} \log |x-s|^{-1}
\end{aligned}
$$

holds. From what has already been proved it follows that

$$
\begin{aligned}
(4.28)\qquad A_2 &\le \mathcal{C} \int_{|x-s|<\frac{1}{n^2}} |x-s|^{\sigma/2-1} \log |x-s|^{-1} dx \\
&\le \mathcal{C} \int_0^{n^{-2}} u^{\sigma/2-1} \log u^{-1} du \\
&\le \frac{\mathcal{C}}{n^{\sigma}} \log n.
\end{aligned}
$$

In the same manner we can see that

$$A_3 \leq \frac{\mathcal{C}}{m^\sigma} \log m \tag{4.29}$$

Finally, combining (4.27), (4.28) and (4.29) with (4.26) we prove the lemma for $k = 0$.

We can proceed analogously to the proof of lemma for $k > 0$. $\square$

Lemma 4.7. *Let $w_1(x), w_2(y)$ as in* (1.2) *and* (1.3)*, respectively. Supposing that $|s| < 1,\ |t| < 1$, we have*

$$\left|\int_{-1}^{1}\int_{-1}^{1} \frac{w_1(x)w_2(y) - w_1(s)w_2(t)}{(x-s)(y-t)}\, dxdy\right| \leq \mathcal{C}\left|\log\frac{1-t}{1+t}\right|\left|\log\frac{1-s}{1+s}\right| u(s,t), \tag{4.30}$$

where u is the singular part of $w_1 w_2$ and $\mathcal{C}$ is a positive constant independent of s and t.

Proof. Adding and subtracting $w_1(x)w_2(t)$ we get

$$\begin{aligned}\left|\int_{-1}^{1}\int_{-1}^{1} \frac{w_1(x)w_2(y) - w_1(s)w_2(t)}{(x-s)(y-t)}\, dxdy\right| &= \left|\int_{-1}^{1}\int_{-1}^{1} \frac{w_1(x)w_2(y) - w_1(x)w_2(t)}{(x-s)(y-t)}\, dxdy\right| \\ &\quad + \left|\int_{-1}^{1}\int_{-1}^{1} \frac{w_1(x)w_2(t) - w_1(s)w_2(t)}{(x-s)(y-t)} dxdy\right| \\ &= A + B.\end{aligned} \tag{4.31}$$

Applying Lemma 4.1, we have

$$A \leq \left|\int_{-1}^{1} \frac{w_1(x)}{x-s} dx\right| \int_{-1}^{1} \left|\frac{w_2(y) - w_2(t)}{y-t}\right| dy \leq \mathcal{C}\left|\log\frac{1-s}{1+s}\right| u(s,t) \tag{4.32}$$

and

$$B \leq w_2(t)\left|\int_{-1}^{1} \frac{dy}{y-t}\right| \int_{-1}^{1} \left|\frac{w_1(x) - w_1(s)}{x-s}\right| dx \leq \mathcal{C}\left|\log\frac{1-t}{1+t}\right| u(s,t). \tag{4.33}$$

The combination of (4.32) and (4.33) with (4.31) proves (4.30). □

We are now able to prove Theorem 3.2.

Proof of Theorem 3.2. The proof is based on the following observation:

$$
\begin{aligned}
(4.34)\quad R_{m,n}(f;s,t) &= R_{m,n}(r_{m,n}-r_{m,n}(s,t);s,t)\\
&= \Phi(r_{m,n}-r_{m,n}(s,t);s,t)-\Phi_{m,n}(r_{m,n}-r_{m,n}(s,t);s,t)\\
&= \int_{-1}^{1}\int_{-1}^{1}\frac{r_{m,n}(x,y)-r_{m,n}(s,t)}{(x-s)(y-t)}w_1(x)w_2(y)dxdy\\
&\quad -\sum_{i=1}^{m}\sum_{j=1}^{n}A_{m,i}^{w_1}(s)B_{n,j}^{w_2}(t)(r_{m,n}(x_{m,i},y_{n,j})-r_{m,n}(s,t))\\
&= A+B.
\end{aligned}
$$

We begin by proving that (3.6) holds for $k=0$, i.e., $f\in \mathrm{Lip}_M\,\sigma$. Note that by (4.19) and Lemma 3.1, we have

$$
\begin{aligned}
(4.35)\qquad |B| &\le 2\|r_{m,n}\|\sum_{i=1}^{m}\sum_{j=1}^{n}|A_{m,i}^{w_1}(s)||B_{n,j}^{w_2}(t)|\\
&\le \mathcal{C}\left(\frac{1}{m}+\frac{1}{n}\right)^{\sigma}u(s,t)\log m\log n,
\end{aligned}
$$

where u is the singular part of w_1w_2.

In order to estimate A, we note that

$$
\begin{aligned}
(4.36)\qquad |A| &\le \mathcal{C}\|r_{m,n}\|\left|\int_{-1}^{1}\int_{-1}^{1}\frac{w_1(x)w_2(y)-w_1(s)w_2(t)}{(x-s)(y-t)}dxdy\right|\\
&\quad + w_1(s)w_2(t)\Bigg\{|H(r_{m,n};s,t)|+\left|\log\frac{1-s}{1+s}\right||H_2(r_{m,n};s,t)|\\
&\quad + \left|\log\frac{1-t}{1+t}\right||H_1(r_{m,n};s,t)|\Bigg\}.
\end{aligned}
$$

Hence, by (4.19), (4.30) and Lemmas 4.5 and 4.6, we have

$$
(4.37)\qquad |A|\le \mathcal{C}\left(\frac{1}{m}+\frac{1}{n}\right)^{\sigma}\left|\log\frac{1-t}{1+t}\log\frac{1-s}{1+s}\right|u(s,t)\log m\log n.
$$

Finally, observing that

$$\frac{1}{\gamma(s,t)}\left|\log\frac{1-t}{1+t}\log\frac{1-s}{1+s}\right| \le \mathcal{C},$$

where $\gamma(s,t) = \log\dfrac{e^2}{\sqrt{(1-s^2)(1-t^2)}}$, and combining (4.35) and (4.37) with (4.34), we complete the proof for $k=0$. The same proof works for $k>0$. $\square$

5. Numerical Evaluations

In this section we show some approximate values for the integral $\Phi(f;s,t)$, $(s,t)\in(-1,1)^2$, obtained by using the algorithm described in Section 2.

Since in the following examples the exact answer is not known, the value in the last line of each table that follow is thought to be correct to the number of figures shown.

Example 1. Let

$$f(x,y)=e^{xy}, \quad w_1(x)=\frac{1}{\sqrt{1-x^2}}\log\frac{e}{(1+x)}, \quad w_2(y)=\frac{|y|^{-1/2}}{\sqrt{1-y^2}}.$$

Let us evaluate the following integral

$$\Phi(f;0,t)=\int_{-1}^{1}\int_{-1}^{1}\frac{e^{xy}|y|^{-1/2}}{x(y-t)\sqrt{1-x^2}\sqrt{1-y^2}}\log\frac{e}{(1+x)}dxdy.$$

Since, when $w(x)=(1-x^2)^{-1/2}\log e/(1+x)$ the explicit expression for $Q_0(w,\lambda)$ is available only for $\lambda=0$, we give the approximations of the above integral for $s=0$ and some selected values of t (see Tables 5.1 and 5.2).

Table 5.1: Approximations $\Phi_{m,n}$ in Example 1

m	n	$s=0,\ t=0.1$	$s=0,\ t=0.99$
4	4	71.	25.
8	8	71.4488	25.0177
12	12	71.4488804461	25.017773973
16	16	71.4488804461501287	25.017773973233205
20	20	71.44888044615012879 3911	25.01777397323320554 94400
24	24	71.4488804461501287939112183712	25.0177739732332055494003732408

Table 5.2: Approximations $\Phi_{m,n}$ in Example 1

m	n	$s=0,\ t=-0.99$	$s=0,\ t=-0.5$
4	4	8.21	1.
8	8	8.217	1.2608
12	12	8.2177020368	1.26086110111
16	16	8.217702036892015	1.2608611011106082
20	20	8.217702036892015633672	1.26086110111060823083358
24	24	8.21770203689201563367231464 25	1.2608611011106082308335891 7389

Example 2. Let $f(x,y) = \sin(x+y)$ and $w_1(x) = w_2(y) = 1$. Let us evaluate the following integral

$$\Phi(f;s,t) = \int_{-1}^{1}\int_{-1}^{1} \frac{\sin(x+y)}{(x-s)(y-t)}dxdy.$$

The corresponding numerical results for some selected values of s and t are presented in Tables 5.3 and 5.4.

Table 5.3: Approximations $\Phi_{m,n}$ in Example 2

m	n	$s=0.1,\ t=0.1$	$s=0.99,\ t=0.99$
4	4	$-1.$	21.5
8	8	-1.10958	21.51974
12	12	-1.1095876430	21.519746844
16	16	-1.1095876430890965	21.5197468440861562
20	20	$-1.10958764308909659978 47$	21.5197468440861562386179
24	24	$-1.109587643089096599784 75010769$	21.5197468440861562386179778436

Table 5.4: Approximations $\Phi_{m,n}$ in Example 2

m	n	$s=0.5,\ t=-0.99$	$s=-0.99,\ t=0.1$
4	4	8.9	7.9
8	8	8.95737	7.925821
12	12	8.95737672093	7.9258219285
16	16	8.9573767209349373	7.9258219285504369
20	20	8.95737672093493733377544	7.92582192855043697557251
26	26	8.95737672093493733377544060 3043	7.9258219285504369755725190 16997

In the above example w_1 and w_2 are two Legendre weight functions. As we said in the introduction, since the zeros and the Christoffel numbers of the

orthogonal Legendre polynomials are known, for the numerical evaluation of the above integral it is possible to use a Gauss type quadrature rules [7]. Comparing the above tables with some of those showed in [7, Example 2], we can note that the Gauss type quadrature rule proposed in [7] converges faster than the interpolatory product rule proposed in this paper. More precisely, by the Gauss type quadrature rule proposed in [7], the results written in the last line of each table showed in the above example are obtained by using only 12 points. Because of this observation, when the zeros and the Christoffel numbers of the orthogonal polynomials associated with the weights functions w_1 and w_2 are available, it is convenient to use the algorithm proposed in [7].

Example 3. Let

$$f(x,y) = \frac{1}{x^2+y^2+4},$$

$w_1(x) = (1-x)^{-3/4}|x+0.999|^{-1/4}$, and $w_2(y) = (1-y)^{-1/2}|y+0.999|^{-1/2}$.

Let us evaluate the following integral

$$\Phi(f;s,t) = \int_{-1}^{1}\int_{-1}^{1} \frac{(1-x)^{-3/4}|x+0.999|^{-1/4}(1-y)^{-1/2}|y+0.999|^{-1/2}}{(x^2+y^2+4)(x-s)(y-t)}dxdy.$$

Numerical values $\Phi_{m,n}$ for selected values of s and t are given in Tables 5.5 and 5.6.

Table 5.5: Approximations $\Phi_{m,n}$ in Example 3

m	n	$s=0.1,\ t=0.1$	$s=0.99,\ t=0.99$
4	4	−0.06	−8.
8	8	−0.0642	−8.07
16	16	−0.0642501079	−8.07115948
32	32	−0.06425010794615205128	−8.071159486314938872
48	48	−0.0642501079461520512828730 62	−8.071159486314938872422636422

Table 5.6: Approximations $\Phi_{m,n}$ in Example 3

m	n	$s=0.5,\ t=-0.99$	$s=-0.99,\ t=0.1$
4	4	−3.2	−0.11
8	8	−3.249	−0.1178
16	16	−3.249577755	−0.117887698
32	32	−3.249577755461490754	−0.1178876987017843238
48	48	−3.249577755461490754868423 38	−0.1178876987017843238566917 32

Finally, we show an example of approximating the integral $\Phi(f;s,t)$ in which the function f is less regular than those previously considered.

Example 4. Let

$$f(x,y)=\sqrt{x^2y^2-x^2-y^2+1},\quad w_1(x)=|x|^{1/2}\quad w_2(y)=|y|^{1/2}.$$

Let us evaluate the following integral

$$\Phi(f;s,t)=\int_{-1}^{1}\int_{-1}^{1}\frac{\sqrt{x^2y^2-x^2-y^2+1}}{(x-s)(y-t)}|x|^{1/2}|y|^{1/2}dxdy.$$

Note that if we take $v_1(x)=\sqrt{|x|}\sqrt{1-x^2}$ and $v_2(y)=\sqrt{|y|}\sqrt{1-y^2}$, the above integral is the product of $Q_0(v_1;s)$ and $Q_0(v_2;t)$ of which the exact values are known [6].

In the following tables, we give the absolute error $E_{m,n}$, where m and n are the total number of knots used by the rule (2.4), for different value of s and t.

Table 5.7: The errors $E_{m,n}$ of approximations $\Phi_{m,n}$ in Example 4 for some selected pairs (s,t)

m	n	$(0.1,0.1)$	$(0.99,0.99)$	$(0.5,-0.99)$	$(-0.99,0.1)$
40	40	$3.097\,\mathrm{E}-5$	$1.450\,\mathrm{E}-2$	$1.293\,\mathrm{E}-3$	$1.645\,\mathrm{E}-3$
80	80	$4.876\,\mathrm{E}-6$	$3.302\,\mathrm{E}-3$	$2.120\,\mathrm{E}-4$	$3.474\,\mathrm{E}-4$
160	160	$1.860\,\mathrm{E}-7$	$2.896\,\mathrm{E}-4$	$2.482\,\mathrm{E}-5$	$3.189\,\mathrm{E}-5$

The corresponding exact values are

$$\begin{aligned}
\Phi(f;0.1,0.1) &= 0.260288410134936951527808145510 1\ldots,\\
\Phi(f;0.99,0.99) &= 5.515334426099947078604629536499 0\ldots,\\
\Phi(f;0.5,-0.99) &= -0.800941885175568159384379382809 0\ldots,\\
\Phi(f;-0.99,0.1) &= 1.198155928555227798355543523206 6\ldots.
\end{aligned}$$

In agree with the theoretical expectation, when the smoothness of the function f decreases the approximation error $E_{m,n}$ increases.

All the computations were done in Q-arithmetic on the VAX 6510 computer (with machine precision $\approx 1.93\times 10^{-34}$).

6. Appendix

Proof of Lemma 4.1. We first prove (4.2). We define

$$\phi_j(x) = |x - t_j|^{-\gamma_j} w(x) \quad (j = 0, 1, 2), \qquad \phi_3(x) = \log^{-\gamma_3} \frac{e}{|x - t_3|} w(x).$$

If $-1 < \lambda < -1 - \varepsilon$, we can consider the following decomposition:

$$\begin{aligned} \int_{-1}^{1} \left| \frac{w(x) - w(\lambda)}{x - \lambda} \right| dx &= \int_{-1}^{2\lambda+1} \cdots + \int_{2\lambda+1}^{t_2 - 2\varepsilon} \cdots + \int_{t_2 - 2\varepsilon}^{t_2 + 2\varepsilon} \cdots \\ &\quad + \int_{t_2 + 2\varepsilon}^{t_3 - 2\varepsilon} \cdots + \int_{t_3 - 2\varepsilon}^{t_3 + 2\varepsilon} \cdots + \int_{t_3 + 2\varepsilon}^{1} \cdots \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{6.1}$$

Now, we have

$$\begin{aligned} I_1 &= \int_{-1}^{2\lambda+1} \phi_0(x) \left| \frac{(1+x)^{\gamma_0} - (1+\lambda)^{\gamma_0}}{x - \lambda} \right| dx \\ &\quad + (1+\lambda)^{\gamma_0} \int_{-1}^{2\lambda+1} \left| \frac{\phi_0(x) - \phi_0(\lambda)}{x - \lambda} \right| dx \\ &\le \mathcal{C} \int_{-1}^{2\lambda+1} \left| \frac{(1+x)^{\gamma_0} - (1+\lambda)^{\gamma_0}}{x - \lambda} \right| dx + \mathcal{C}(1+\lambda)^{\gamma_0} |\phi_0'(\xi)|. \end{aligned} \tag{6.2}$$

Let $1 + x = y(1 + \lambda)$. Then we obtain

$$\begin{aligned} \int_{-1}^{2\lambda+1} \left| \frac{(1+x)^{\gamma_0} - (1+\lambda)^{\gamma_0}}{x - \lambda} \right| dx &\le (1+\lambda)^{\gamma_0} \int_0^2 \left| \frac{y^{\gamma_0} - 1}{y - 1} \right| dy \\ &\le (1+\lambda)^{\gamma_0} \left\{ \int_0^1 + \int_1^2 \right\} y^{\gamma_0} \frac{|1^{-\gamma_0} - y^{-\gamma_0}|}{|1 - y|} dy \\ &\le (1+\lambda)^{\gamma_0} \left\{ \int_0^1 + \int_1^2 \right\} y^{\gamma_0} (1 - y)^{-1-\gamma_0} dy \\ &\le \mathcal{C}(1+\lambda)^{\gamma_0}. \end{aligned} \tag{6.3}$$

Combining (6.2) with (6.3) we get

$$I_1 \le \mathcal{C}(1+\lambda)^{\gamma_0} \Psi_0(\lambda). \tag{6.4}$$

For I_2 and I_4, we have the following estimates

$$I_2 \leq \mathcal{C}|w'(\xi)| \leq \mathcal{C} \tag{6.5}$$

and

$$I_4 \leq \mathcal{C}. \tag{6.6}$$

Moreover, since $x - \lambda > \mathcal{C}$, we have

$$\begin{aligned} I_3 &= \int_{t_2-2\varepsilon}^{t_2+2\varepsilon} \phi_2(x) \left| \frac{|x-t_2|^{\gamma_2} - |\lambda - t_2|^{\gamma_2}}{x-\lambda} \right| dx + \mathcal{C}|\lambda - t_2|^{\gamma_2} |\phi_2'(\xi)| \\ &\leq \mathcal{C} \int_{t_2-2\varepsilon}^{t_2+2\varepsilon} |x-t_2|^{\gamma_2} dx + \mathcal{C}|\lambda - t_2|^{\gamma_2} \\ &\leq \mathcal{C}|\lambda - t_2|^{\gamma_2} \Psi_2(\lambda) \end{aligned} \tag{6.7}$$

and similarly

$$I_5 \leq \mathcal{C} \log^{\gamma_3} \frac{e}{|\lambda - t_3|} \Psi_3(\lambda), \tag{6.8}$$

$$I_6 \leq \mathcal{C}(1-\lambda)^{\gamma_1} \Psi_1(\lambda). \tag{6.9}$$

Combining (6.4), (6.5), (6.6), (6.7), (6.8) and (6.9) with (6.1) we obtain (4.2). Analogously, we do if $1 - \varepsilon < \lambda < 1$.

If $t_2 - \varepsilon < \lambda < t_2$, we can consider the following decomposition:

$$\begin{aligned} \int_{-1}^{1} \left| \frac{w(x) - w(\lambda)}{x - \lambda} \right| dx &= \int_{-1}^{t_2-2\varepsilon} \cdots + \int_{t_2-2\varepsilon}^{2\lambda - t_2} \cdots + \int_{2\lambda-t_2}^{t_2} \cdots \\ &+ \int_{t_2}^{t_2+2\varepsilon} \cdots + \int_{t_2+2\varepsilon}^{t_3-2\varepsilon} \cdots + \int_{t_3-2\varepsilon}^{t_3+2\varepsilon} \cdots \\ &+ \int_{t_3+2\varepsilon}^{1} \cdots \\ &= I_1' + I_2' + I_3' + I_4' + I_5' + I_6' + I_7'. \end{aligned} \tag{6.10}$$

Here, we have

$$
\begin{aligned}
I_3' &= \int_{2\lambda-t_2}^{t_2} \phi_2(x) \left| \frac{|x-t_2|^{\gamma_2} - |\lambda-t_2|^{\gamma_2}}{x-\lambda} \right| dx \\
&\quad + |\lambda-t_2|^{\gamma_2} \int_{2\lambda-t_2}^{t_2} \left| \frac{\phi_2(x)-\phi_2(\lambda)}{x-\lambda} \right| dx \\
&\le \mathcal{C} \int_{2\lambda-t_2}^{t_2} \left| \frac{|x-t_2|^{\gamma_2} - |\lambda-t_2|^{\gamma_2}}{x-\lambda} \right| dx + \mathcal{C}|\lambda-t_2|^{\gamma_2}|\phi_2'(\xi)|.
\end{aligned} \tag{6.11}
$$

Let $x - t_2 = y(\lambda - t_2)$. Then, we obtain

$$
\begin{aligned}
\int_{2\lambda-t_2}^{t_2} \left| \frac{|x-t_2|^{\gamma_2} - |\lambda-t_2|^{\gamma_2}}{x-\lambda} \right| dx &\le |\lambda-t_2|^{\gamma_2} \int_0^2 \left| \frac{y^{\gamma_2}-1}{y-1} \right| dy \\
&\le |\lambda-t_2|^{\gamma_2} \left\{ \int_0^1 + \int_1^2 \right\} y^{\gamma_2} \frac{|1^{-\gamma_2} - y^{-\gamma_2}|}{|1-y|} dy \\
&\le |\lambda-t_2|^{\gamma_2} \left\{ \int_0^1 + \int_1^2 \right\} y^{\gamma_2} (1-y)^{-1-\gamma_2} dy \\
&\le \mathcal{C}|\lambda-t_2|^{\gamma_2}.
\end{aligned} \tag{6.12}
$$

Combining (6.11) with (6.12) we get

$$
I_3' \le \mathcal{C}|\lambda-t_2|^{\gamma_2} \Psi_2(\lambda). \tag{6.13}
$$

The rest of the proof runs as in the previous case. Analogously, we do if $t_2 < \lambda < t_2 + \varepsilon$.

If $t_3 - \varepsilon < \lambda < t_3$, we can consider the following decomposition:

$$
\begin{aligned}
\int_{-1}^{1} \left| \frac{w(x)-w(\lambda)}{x-\lambda} \right| dx &= \int_{-1}^{t_2-2\varepsilon} \cdots + \int_{t_2-2\varepsilon}^{t_2+2\varepsilon} \cdots + \int_{t_2+2\varepsilon}^{t_3-2\varepsilon} \cdots \\
&\quad + \int_{t_3-2\varepsilon}^{2\lambda-t_3} \cdots + \int_{2\lambda-t_3}^{t_3} \cdots + \int_{t_3}^{t_3+2\varepsilon} \cdots \\
&\quad + \int_{t_3+2\varepsilon}^{1} \cdots \\
&= I_1'' + I_2'' + I_3'' + I_4'' + I_5'' + I_6'' + I_7''.
\end{aligned} \tag{6.14}
$$

Then, we have

$$
\begin{aligned}
(6.15)\quad I_5'' &= \int_{2\lambda-t_3}^{t_3} \phi_3(x) \left| \frac{\log^{\gamma_3} \frac{e}{|x-t_3|} - \log^{\gamma_3} \frac{e}{|\lambda-t_3|}}{x-\lambda} \right| dx \\
&\qquad + \log^{\gamma_3} \frac{e}{|\lambda-t_3|} \int_{2\lambda-t_3}^{t_3} \left| \frac{\phi_3(x)-\phi_3(\lambda)}{x-\lambda} \right| dx \\
&\le \mathcal{C} \int_{2\lambda-t_3}^{t_3} \left| \frac{\log^{\gamma_3} \frac{e}{|x-t_3|} - \log^{\gamma_3} \frac{e}{|\lambda-t_3|}}{x-\lambda} \right| dx + \mathcal{C} \log^{\gamma_3} \frac{e}{|\lambda-t_3|} |\phi_3'(\xi)|.
\end{aligned}
$$

Let $x - t_3 = y(\lambda - t_3)$. We obtain

$$
\begin{aligned}
(6.16)\qquad & \int_{2\lambda-t_3}^{t_3} \left| \frac{\log^{\gamma_3} \frac{e}{|x-t_3|} - \log^{\gamma_3} \frac{e}{|\lambda-t_3|}}{x-\lambda} \right| dx \\
&\le \int_0^2 \left| \frac{\log^{\gamma_3} \frac{e}{|y||\lambda-t_3|} - \log^{\gamma_3} \frac{e}{|\lambda-t_3|}}{y-1} \right| dy \\
&\le \int_0^{1/2} \log^{\gamma_3} \frac{1}{y} \frac{dy}{1-y} + \int_{1/2}^2 \left| \frac{\log^{\gamma_3} \frac{1}{y} - 0}{y-1} \right| dy \\
&\le \int_0^{1/2} \log^{\gamma_3} \frac{1}{y}\, dy + \int_{1/2}^2 \frac{1}{y}\, dy \le \mathcal{C}.
\end{aligned}
$$

Combining (6.15) with (6.16) we get

$$
(6.17)\qquad I_5'' \le \mathcal{C} \log^{\gamma_3} \frac{e}{|\lambda-t_3|} \Psi_3(\lambda).
$$

The rest of the proof runs as in the previous case. Analogously, we do if $t_3 < \lambda < t_3 + \varepsilon$.

Now, we are ready to prove (4.1). Since

$$
\left| \int_{-1}^1 \frac{w(x)}{x-\lambda} dx \right| \le \int_{-1}^1 \left| \frac{w(x)-w(\lambda)}{x-\lambda} \right| dx + w(\lambda) \left| \int_{-1}^1 \frac{dx}{x-\lambda} \right|,
$$

applying (4.2), we prove (4.1). $\square$

Acknowledgment: This paper was written at the Department of Mathematics, University of Basilicata, while the author was working with a scholarship of C.N.R (Bando N. 201.19.1). The author thanks Professor G. Mastroianni for his helpful suggestions and supervision.

REFERENCES

1. M. Branders and R. Piessens: *Numerical solution of integral equations of mathematical physics using Chebyshev polynomials.* J. J. Comput. Phys. **21** (1976), 178–196.
2. G. Criscuolo and G. Mastroianni: *On the convergence of an interpolatory product rule for evaluating Cauchy principal value integrals.* Math. Comp. **48** (1987), 725–735.
3. G. Criscuolo and G. Mastroianni: *Convergence of Gauss type product formulas for the evaluation of two-dimensional Cauchy principal value integrals.* BIT **27** (1987), 72–84.
4. G. Criscuolo and G. Mastroianni: *On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals.* Numer. Math. **54** (1989), 445–461.
5. G. Criscuolo, G. Mastroianni, and P. Nevai: *Associated generalized Jacobi functions and polynomials.* J. Math. Anal. Appl. **158** (1991), 15–33.
6. G. Criscuolo and L. Scuderi: *A new algorithm for the numerical evaluation of Cauchy principal value integrals with non-standard weight functions.* BIT **38** (2) (1998), 256–274.
7. M. C. De Bonis: *An algorithm for the evaluation of two-dimensional Hilbert transform.* J. Electrotechn. Math. (Priština) **4** (1999), 1–34.
8. W. Gautschi: *On the construction of Gaussian rules from modified moments.* Math. Comp. **24** (1970), 245–260.
9. W. Gautschi: *A survey of Gauss-Christoffel quadrature formulae.* In: E.B. Christoffel: The Influence of his Work in Mathematics and Physical Sciences. A collection of articles in honour of Christoffel on the 50th anniversary of his birth (P.L. Butzer, F. Fehér, eds.), Birkhäuser Verlag, Basel, 1981, pp. 72–147.
10. G. Monegato: *The numerical evaluation of one-dimensional Cauchy principal value integral.* Computing **29** (1982), 337–354.
11. R. Piessens: *Modified Clenshaw-Curtis integration and applications to numerical computation of integral transform.* In: Numerical Integration (P. Keast and G. Fairweather, eds.), 1987, pp. 35–51.

12. G. Szegő: *Orthogonal Polynomials.* Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th ed., Amer. Math. Soc., Providence, R. I., 1975.

Department of Mathematics
University of Basilicata
Campus Macchia Romana
85100 Potenza, Italy