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APOSTOL SPECTRUM AND GENERALIZATIONS: A BRIEF SURVEY

Vladimir Rakočević

This paper is dedicated to Professor D. S. Mitrinović

Abstract. Let $\mathcal{A}$ be a complex unital Banach algebra. If Ω is an subset of $\mathcal{A}$, then the Ω-spectrum of $a \in \mathcal{A}$ is $\mathrm{Sp}^{\Omega}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega\}$. A natural question is what should be chosen for Ω if we want to make these investigations worthwhile? Of course many applications suggest the choice of Ω. In this paper, among other things, we wont to show that Apostol's paper [3] implicitly suggested several examples of Ω, and we survey some results and problems connected with Apostol's results.

1. Introduction

From the beginning of Banach algebra theory (cf. [40]) the spectrum has played a basic role in the general theory and in its applications. Recall that if $\mathcal{A}$ is a complex unital Banach algebra, $a \in \mathcal{A}$, then the *spectrum,* $\sigma(a)$, of a is the set of complex number λ for which $a-\lambda$ is not invertible in $\mathcal{A}$. It is well known that $\sigma(a)$ is a compact nonempty subset of the set of complex numbers $\mathbb{C}$. Let $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$ be the *spectral radius* of a. The spectrum is an important concept in the standard examples of Banach algebras that occur in applications. For operators on a Banach space the spectrum is the usual operator spectrum, and the Fredholm spectrum (essential spectrum) is the spectrum in the Calkin algebra.

If Ω is an subset of $\mathcal{A}$, then the Ω-*spectrum* of $a \in \mathcal{A}$ is ([114])

$$\mathrm{Sp}^{\Omega}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega\}.$$

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Of course, it may be that $\mathrm{Sp}^{\Omega}(a)$ is empty for some a. If $\mathcal{A}^{-1}$, $\mathcal{A}_l^{-1}$ and $\mathcal{A}_r^{-1}$ are the sets of all invertible, the left invertible and the right invertible elements in $\mathcal{A}$, then $\mathrm{Sp}^{\mathcal{A}^{-1}}(a)$, $\mathrm{Sp}^{\mathcal{A}_l^{-1}}(a)$ and $\mathrm{Sp}^{\mathcal{A}_r^{-1}}(a)$ are the spectrum, left spectrum and right spectrum of a, respectively. It should be clear that if Ω has some properties *close* to the set of all invertible $\mathcal{A}^{-1}$ (e.g. Ω is an open subset in $\mathcal{A}$, or an open semigroup in $\mathcal{A}$), then the set $\mathrm{Sp}^{\Omega}(a)$ is expected to have some properties of the ordinary spectrum. A natural question is what should be chosen for Ω if we want to make these investigations worthwhile? Of course many applications suggest the choice of Ω. In this paper, among other things, we wont to show that Apostol's paper [3] implicitly suggested several examples of Ω. Although Apostol's results were published in 1985., they are very actual nowadays, and here we survey some recent results and problems connected with them.

In Section 2 we gather some results and notations from Fredholm, semi-Fredholm, Browder, and semi-Browder theory connected with Section 4 and Section 5. We primary wont to mention some less known results.

In Section 3 we present some Apostol's result from [3] (see also [1], [2]). These results have been proved for bounded linear operators on Hilbert space, and some of them *mutátis mutándis* are true for bounded (even closed densely defined) linear operators on Banach space.

In Section 4 we consider bounded linear operators on Banach space, connected with Apostol's results, but we don't suppose that any operator has complemented null space or range space.

Finally, Section 5 is a continuation of Section 4, we consider bounded linear operators on Banach space, and we suppose that some operators have complemented null space or range space.

2. Fredholm Operators and Generalizations

Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ $(K(X))$. For T in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Let $N^{\infty}(T) = \cup_n N(T^n)$ and $R^{\infty}(T) = \cap_n R(T^n)$ be, respectively, the hyperkernel and the hyperrange of T [55]. Set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an *index* $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. It is well known that index is continuous function on the set of semi-Fredholm operators. Let $\Phi_+(X)\,(\Phi_-(X))$ denote the set of *upper (lower) semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty\,(\beta(T) < \infty)$. An operator T is *Fredholm* if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi(X)$ denote the set of Fredholm operators. Set $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$. Recall that $a(T)$ ($d(T)$), the *ascent (descent)* of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ $(R(T^n) = R(T^{n+1}))$. If no such n exists, than $a(T) = \infty$ ($d(T) = \infty$). An operator T is called *upper semi-Browder* if $T \in \Phi_+(X)$ and $a(T) < \infty$; T is called *lower semi-Browder* if $T \in \Phi_-(X)$ and $d(T) < \infty$ [55, Definition 7.9.1]. Let $\mathcal{B}_+(X)$ $(\mathcal{B}_-(X))$ denote the set of upper (lower) semi-Browder operators. An operator T is *Browder* if it is both upper semi-Browder and lower semi-Browder [55, Definition 7.7.1]. Let $\mathcal{B}(X)$ denote the set of Browder operators, i.e., $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. It is well known that $\Phi(X)$, $\Phi_0(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ are open semigroups in $B(X)$, and it is less known that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets in $B(X)$ [71, Satz 4], and that (see [55, Theorem 7.9.2]) $S, T \in \mathcal{B}_\pm(X)$ and $ST = TS$, implies $ST \in \mathcal{B}_\pm(X)$.

The sets of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators are stable under compact perturbation, and for the semi-Browder operators, recall the following Grabiner's result ([50, Theorem 2], [55, Theorem 7.9.2]).

Theorem 2.1. *Suppose that* $T \in B(X)$, $K \in K(X)$ *and* $TK = KT$. *Then*

$$T \in \mathcal{B}_\pm(X) \quad \Longrightarrow \quad T + K \in \mathcal{B}_\pm(X).$$

It is known that the commutativity condition in Theorem 2.1 is essential (see e.g. [136, p. 599]).

The fact that $K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the *Calkin algebra* over X as the quotient algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra in the quotient algebra norm $\|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|$. We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Let $r_e(T) = r(\pi(T)$ be the *essential spectral radius* of T. An operator $T \in B(X)$ is *Riesz* operator if $r_e(T) = 0$. Let $R(X)$ denote the set of Riesz operators in $B(X)$. The *semi-Fredholm radii* of the operator T are

$$r_\pm(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_\pm(X) \text{ for } |\lambda| < \epsilon\},$$

and if both $r_+(T)$ and $r_-(T)$ are positive, then they are equal. The following results ([110], [111]) are generalization of Theorem 2.1.

Theorem 2.2. *Suppose that* T, $K \in B(X)$ *and* $TK = KT$. *Then*

$$T \in \mathcal{B}_\pm(X) \quad \text{and} \quad r_e(K) < r_\pm(T) \quad \Longrightarrow \quad T + K \in \mathcal{B}_\pm(X).$$

Corollary 2.3. *Suppose that $T \in B(X)$, $K \in R(X)$ and $TK = KT$. Then*

$$T \in \mathcal{B}_{\pm}(X) \quad \Longrightarrow \quad T + K \in \mathcal{B}_{\pm}(X).$$

Let $\sigma_a(A)$ and $\sigma_d(A)$ denote, respectively, the *approximate point spectrum* and *approximate defect spectrum* of an element A of $B(X)$. Set

$$\begin{aligned}
\sigma_{ek}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_+(X) \cup \Phi_-(X)\},\\
\sigma_{ew}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi(X)\},\\
\sigma_{em}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_0(X)\},\\
\sigma_{e\beta}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_-(X)\},\\
\sigma_{e\alpha}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_+(X)\},\\
\sigma_{eb}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_0(X)\} \cup \{\text{limit points of } \sigma(A)\}.
\end{aligned}$$

Let us recall that $\sigma_{ek}(A), \sigma_{ew}(A), \sigma_{em}(A)$ and $\sigma_{eb}(A)$ are (the classical essential spectra) respectively called the essential spectrum of A according to Kato [63], Wolf [135], Schechter (Weyl) [117, 118, 119, 120] and Browder [11]. $\sigma_{e\beta}(A)$ and $\sigma_{e\alpha}(A)$ are essential spectrum of A according to Gustafson and Weidmann [53] (for more details see, e.g., [35], [97] or [105], and references therein). The set of upper (lower) semi-Browder operators and Browder operators define, respectively, the corresponding spectra

$$\begin{aligned}
\sigma_{ab}(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{B}_+(X)\},\\
\sigma_{db}(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{B}_-(X)\}.
\end{aligned}$$

It is clear that $\sigma_{eb}(A) = \sigma_{ab}(A) \cup \sigma_{db}(A)$; $\sigma_{ab}(A)$ and $\sigma_{db}(A)$ are respectively called the *Browder´s essential approximate point spectrum* of A and *Browder´s essential defect spectrum* of A ([106], [110], [139], [141]).

Set

$$\begin{aligned}
\Phi_+^-(X) &= \{A \in \Phi_+(X) : i(A) \leq 0\},\\
\Phi_-^+(X) &= \{A \in \Phi_-(X) : i(A) \geq 0\},\\
\sigma_{ea}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_+^-(X)\},\\
\sigma_{ed}(A) &= \{\lambda \in C : A - \lambda I \notin \Phi_-^+(X)\}.
\end{aligned}$$

$\Phi_+^-(X)$ and $\Phi_-^+(X)$ are open semigroups in $B(X)$ ([103], [104], [105], [128]), and

$$\sigma_{ea}(A) = \bigcap_{K \in K(X)} \sigma_a(A + K) \qquad \text{and} \qquad \sigma_{ed}(A) = \bigcap_{K \in K(X)} \sigma_d(A + K).$$

The *essential approximate point spectrum* $\sigma_{ea}(A)$ of A has been studied in ([29], [30], [31], [32], [103], [104], [105], [106], [107], [128], [139], [141]). Clearly, the *essential defect spectrum* of $A \in B(X)$, $\sigma_{ed}(A)$ is a dual version of $\sigma_{ea}(A)$.

The *polynomial hull* $\hat{E}$ of a compact subset E of the complex plane $\mathbb{C}$ is the complement of the unbounded component of $\mathbb{C} \setminus E$. Here and in what follows ∂E denotes the boundary of the set E.

Now we present some results ([106]) which characterize $\sigma_{ab}(A)$, and the corresponding dual results are true for $\sigma_{db}(A)$.

Theorem 2.4. *Suppose that $T \in B(X)$. Then:*

(1)
$$\sigma_{ab}(T) = \bigcap_{\substack{TK=KT \\ K \in K(X)}} \sigma_a(T+K).$$

(2) $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$ *if and only if* λ *is an isolated point of* $\sigma_a(T)$, *an eigenvalue of* T *of finite multiplicity,* $a(T-\lambda) < \infty$ *and* $R(T-\lambda)$ *is closed.*

(3) *Let* $\lambda \in \sigma_a(T)$ *be an isolated point of* $\sigma_a(T)$ *and let* $a(T-\lambda) = \infty$. *Then* $\lambda \in \sigma_{ea}(T)$.

(4) $\sigma_{ab}(T) = \sigma_{ea}(T) \cup \{$ *limit points of* $\sigma_a(T)\}$.

(5) $\sigma_{ea}(T) \subset \sigma_{ab}(T) \subset \sigma_{eb}(T)$.

(6) $\partial\sigma_{eb}(T) \subset \partial\sigma_{ab}(T) \subset \partial\sigma_{ea}(T)$.

(7) $\hat{\sigma}_{ea}(T) = \hat{\sigma}_{ab}(T) = \hat{\sigma}_{eb}(T)$.

Recall that if (G_n) is a sequence of compact subset of $\mathbb{C}$, then the *limit inferior* , $\liminf G_n$, is the set of all λ in $\mathbb{C}$ such that every neighbourhood of λ has a non-empty intersection with all but finitely many G_n. The *limit superior,* $\limsup G_n$, is the set of all λ in $\mathbb{C}$ such that every neighbourhood of λ intersects infinitely many G_n. If $\liminf G_n = \limsup G_n$, then $\lim G_n$ is said to exist and is equal to this common limit.

A mapping s defined on $B(X)$ whose values are compact subset of $\mathbb{C}$ is said to be *upper* (*lower*) *semi-continuous* at A when if $A_n \to A$ then $\limsup s(A_n) \subset s(A)(s(A) \subset \liminf s(A_n))$. If s is both upper and lower semi-continuous at A, then it is said to be *continuous* at A and in this case $\lim s(A_n) = s(A)$.

There are extremely numerous papers on the continuity of the spectrum, its parts and the spectral radius (see e.g. [15], [16], [18], [19], [25], [26], [27],

[28], [30], [31], [32], [59], [93], [95], [96], [101], [102], [106], [109]). Recall that the mappings $A \to \sigma_{em}(A)$ and $A \to \sigma_{eb}(A)$ are upper semi-continuous at A ([95, Theorem 1], [96, Theorem 2]), and ([106, Theorem 3.2]) the mappings $T \mapsto \sigma_{ab}(T)$ and $T \mapsto \sigma_{ad}(T)$ are upper semi-continuous at T. S. Djordjević has obtained for operators on separable Hilbert space necessary and sufficient conditions for the continuity of the mappings $A \to \sigma_{ea}(A)$ [31] and $A \to \sigma_{ab}(A)$ [30].

The usual spectral mapping theorem for linear operators may be generalized to the theorems of the form

$$\sigma_i(f(A)) = f\{\sigma_i(A)\}, \tag{2.1}$$

where $\sigma_i(A)$ is a certain subset of the $\sigma(A)$, and f is an analytic function defined on a neighbourhood of $\sigma(A)$. It is well-known that if $i \in \{ek, ew, eb, e\alpha, e\beta, ab, ad\}$, then (2.1) is valid ([52], [94], [96], [14], [33], [104], [106], [128]), and that

$$\begin{aligned}
\sigma_{ek}(f(A)) &\supset f\{\sigma_{ek}(A)\},\\
\sigma_{em}(f(A)) &\subset f\{\sigma_{em}(A)\},\\
\sigma_{ea}(f(T)) &\subset f\{\sigma_{ea}(T)\},\\
\sigma_{ed}(f(T)) &\subset f\{\sigma_{ed}(T)\},
\end{aligned}$$

and the inclusions may be proper. C. Schmoeger [128] has described the set of all $A \in B(X)$ such that $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$, for all analytic function f defined on a neighbourhood of $\sigma(A)$, and the set of all $A \in B(X)$ such that $\sigma_{ed}(f(A)) = f(\sigma_{ed}(A))$, for all analytic function f defined on a neighbourhood of $\sigma(A)$. Then he considers necessary and sufficient conditions on A which ensure that $\sigma_{em}(f(A)) = f(\sigma_{em}(A))$, for all analytic function f defined on a neighbourhood of $\sigma(A)$.

3. Apostol Spectrum

Let X be an infinite-dimensional complex Banach space. For an element T in $B(X)$ the *reduced minimum modulus* of T, $\gamma(T)$, is defined by $\gamma(T) = \inf\{\|Tx\|/\mathrm{dist}\,(x, N(T)) : \mathrm{dist}\,(x, N(T)) > 0\}$; if $T = 0$ then we set $\gamma(T) = \infty$. Recall that $R(T)$ is closed if and only if $\gamma(T) > 0$ [63, p. 231]. If $N(T) = \{0\}$, then $\gamma(T) = \mu(T)$, where $\mu(T)$ is the *minimum modulus* of T, defined by $\mu(T) = \inf\{\|Tx\| : \|x\| = 1\}$. While the minimum modulus is continuos function, the reduced minimum modulus is not continuous function in general. Even the function $\lambda \mapsto \gamma(T-\lambda)$ defined for complex λ is not continuous in general (see [62, pp. 282, Remark 3]).

In this section we shall present some Apostol's results [3]. These results are connected with the point of continuity of the function $\lambda \mapsto \gamma(T-\lambda)$, $\lambda \in \mathbb{C}$.

Let H be a Hilbert space and $T \in B(H)$. If M is a closed subspaces of H, then P_M will denote the orthogonal projection of H onto M. Recall that $\mu \in \mathbb{C}$ is a $T-$ *regular* point if the function $\lambda \mapsto P_{N(T-\lambda)}$, $\lambda \in \mathbb{C}$, is norm-continuous at μ. If $\mu \in \mathbb{C}$ is not T–regular, then it is T–*singular*. Set

$$\begin{aligned}
\sigma_{c.r}(T) &= \{\lambda \in \sigma(T) : R(T-\lambda) \quad \text{is closed}\},\\
\sigma_{c.r}^{r}(T) &= \{\lambda \in \sigma_{c.r}(T) : \lambda \quad \text{is } T\text{–regular}\},\\
\sigma_{c.r}^{s}(T) &= \{\lambda \in \sigma_{c.r}(T) : \lambda \quad \text{is } T\text{–singular}\}.
\end{aligned}$$

Theorem 3.1. *Let H be a Hilbert space and $T \in B(H)$. For every $\mu \in \mathbb{C}$, $\lim_{\lambda\to\mu}\gamma(T-\lambda)$ exists and the following implications hold true:*

$$\mu \textit{ is } T\textit{–regular} \Longrightarrow \gamma(T-\lambda) \textit{ is continuous at } \mu; \tag{1}$$

$$\lim_{\lambda\to\mu}\gamma(T-\lambda) > 0 \Longrightarrow \mu \textit{ is } T\textit{–regular.} \tag{2}$$

Definition 3.2. Let H be a Hilbert space and $T \in B(H)$. Set

$$\sigma_\gamma(T) = \{\lambda \in \mathbb{C} : \lim_{\lambda\to\mu}\gamma(T-\lambda) = 0\}, \qquad \rho_\gamma(T) = \mathbb{C} \setminus \sigma_\gamma(T).$$

Let us call $\sigma_\gamma(T)$ the *Apostol spectrum* of T (see [67], [68], [100]).

Theorem 3.3. *The set $\sigma_\gamma(T)$ is closed and*

$$\partial\sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T), \tag{1}$$

$$\sigma_\gamma(T) = \sigma_{c.r}^{s}(T) \cup \{\mu \in \mathbb{C} : \gamma(T-\mu) = 0\}, \tag{2}$$

$$\rho_\gamma(T) = \sigma_{c.r}^{r}(T) \cup \rho(T). \tag{3}$$

Theorem 3.4. *There exists an analytic function $F : \rho_\gamma(T) \mapsto B(H)$ such that*

$$(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda, \quad F(\lambda)(T-\lambda)F(\lambda) = F(\lambda), \quad \lambda \in \rho_\gamma(T).$$

Theorem 3.5. *Let G be an open subset of $\mathbb{C}$ and let $F : G \mapsto B(H)$ be an analytic function such that $(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda, \quad \lambda \in G$. Then $G \subset \rho_\gamma(T)$ and we have*

$$(T-\lambda)^{n+1}\frac{d^nF(\lambda)}{d\lambda^n}(T-\lambda)^{n+1} = n!(T-\lambda)^{n+1}, \quad \lambda \in G, \quad n \geq 0.$$

$$\gamma((T-\lambda)^{n+1}) \geq n!\Big\|\frac{d^nF(\lambda)}{d\lambda^n}\Big\|^{-1}, \quad \lambda \in G, \quad n \geq 0.$$

Theorem 3.6. *Let $T \in B(H)$ and f be an analytic function defined on a neighbourhood of the spectrum of T. Then*

$$\sigma_\gamma(f(T)) = f\{\sigma_\gamma(T)\}.$$

Theorem 3.7. *Suppose $0 \in \rho_\gamma(T)$ and let r denote the radius of the largest open disk centered at 0 and included in $\rho_\gamma(T)$. Then $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists and we have $\lim_{n\to\infty}\gamma(T^n)^{1/n} = r$*

Corollary 3.8. *Suppose $0 \in \sigma_\gamma(T)$ and the connected component of $\sigma_\gamma(T)$ containing 0 is singleton. Then $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists and, if strictly positive, coincides with the radius of the largest open disk centered at 0 and included in $\rho_\gamma(T) \cup \{0\}$.*

Remark 3.9. For further generalizations, and results of the independent interest, but connected with above results, see e.g., ([72], [73], [74], [81], [82], [83], [84], [85], [90]) and references therein.

4. $\mathcal{V}_0(X), \mathcal{V}(X)$ and Corresponding Spectra

If M and N are two closed subspaces of the Banach space X, set $\delta(M,N) = \sup\{\text{dist}\,(u,N) : u \in M, \|u\| = 1\}$; (if $M = \{0\}$ then $\delta(M,N) = 0$) and $\hat{\delta}(M,N) = \max\,[\delta(M,N), \delta(N,M)]$. $\hat{\delta}$ is called the *gap* (or *opening*) between the M and N ([63, p. 197], [98]).

For the convenience of the reader, recall the following well known result of A. S. Markus [80, Theorem 2 and Remark 1].

Theorem 4.1. *Suppose that $A, A_n \in B(X)$, $R(A)$ and $R(A_n)$ are closed, $n = 1, 2, \ldots$, and let $A_n \to A$. Then the following conditions are equivalent:*

(1) $$\inf_n \gamma(A_n) > 0,$$

(2) $$\lim_{n\to\infty}\gamma(A_n) = \gamma(A),$$

(3) $$\lim_{n\to\infty}\hat{\delta}(N(A_n), N(A)) = 0,$$

(4) $$\lim_{n\to\infty}\hat{\delta}(R(A_n), R(A)) = 0.$$

Theorem 4.1 gives a nice connection between the gap function and the reduced minimum modulus, when the continuity of the reduced minimum modulus is considered. For the recent applications of Theorem 4.1 see J. Koliha and V. Rakočević [66], and V. Rakočević [112]. We applied Markus theorem to study the continuity of the generalized Drazin inverse (see [64], [65]) and the continuity of the Drazin inverse for elements of Banach algebras and bounded linear operators on Banach spaces.

Following Grabiner [51], for each $n \geq 0$, $T \in B(X)$ induces a linear transformations from $R(T^n)/R(T^{n+1}) \mapsto R(T^{n+1})/R(T^{n+2})$. Denote by $k_n(T)$ the dimension of its null space and let

$$k(T) = \sum_{n=0}^{\infty} k_n(T).$$

Lemma 4.2. ([51], Lemma 3.7) *Let $T \in B(X)$ and $n \geq 0$. Then:*

$$k(T) = \sup_n \dim \frac{N(T)}{N(T) \cap R(T^{n+1})}, \tag{1}$$

$$k(T) = \dim \frac{N(T)}{N(T) \cap R^{\infty}(T)}, \tag{2}$$

$$k(T) = \sup_n \dim \frac{R(T) + N(T^n)}{R(T)}, \tag{3}$$

$$k(T) = \dim \frac{R(T) + N^{\infty}(T)}{R(T)}. \tag{4}$$

Let us remark that $k(T) = n < \infty$ precisely when T has Kaashoek's property $P(I, n)$ [60, pp. 452-453], or when T has *almost uniform descent* ([51, Definition 1.3]). In particular $k(T) = 0$ if and only if $N(T) \subset R^{\infty}(T)$, or when T is *hyperexact* ([56], [57], [58]).

The following theorem gives several equivalent conditions for the continuity of the function $\lambda \mapsto \gamma(T - \lambda)$ (e.g. see [5], [79], [88], [92], [129], [130]).

Theorem 4.3. *Let $T \in B(X)$ be an operator with closed range. Then the following conditions are equivalent:*

(1) *the function $\lambda \mapsto \gamma(T - \lambda)$ is continuous at $\lambda = 0$,*

(2) *there exists $\epsilon > 0$ such that* $\inf\{\gamma(T - \lambda) : |\lambda| < \epsilon\} > 0$,

(3) $\lim_{\lambda \to 0} \hat{\delta}(R(T), R(T - \lambda)) = 0$,

(4) $\lim_{\lambda \to 0} \hat{\delta}(N(T), N(T - \lambda)) = 0$,

(5) $N(T) \subset R^\infty(T)$,

(6) $N^\infty(T) \subset R(T)$,

(7) $N^\infty(T) \subset R^\infty(T)$.

Now, set

$$\mathcal{V}_0(X) = \{A \in B(X) : R(A) \quad \text{is closed and} \quad k(A) = 0\},$$
$$\mathcal{V}(X) = \{A \in B(X) : R(A) \quad \text{is closed and} \quad k(A) < \infty\}.$$

It is well known that $\Phi_+(X) \cup \Phi_-(X) \subset \mathcal{V}(X)$; $\mathcal{V}_0(X)$ and $\mathcal{V}(X)$ are neither semigroups nor open or closed subset of $B(X)$ (see e.g., [42], [56], [57]). From the papers of M. A. Goldman [42] and C. Schmoeger [125] we get

$$\text{int}(\mathcal{V}(X)) = \Phi_+(X) \cup \Phi_-(X), \tag{4.1}$$

$$\text{int}(\mathcal{V}_0(X)) = \{A \in \Phi_\pm(X) : \alpha(A) = 0 \ \text{ or} \beta(A) = 0\}. \tag{4.2}$$

An operator $T \in \mathcal{V}_0(X)$ $(\mathcal{V}(X))$ is called *semi-regular, s-regular, Kato regular, Kato non-singular, ... (essential semi regular, essential s-regular, ...)*. The semi-Fredholm and semi-Browder operators are closely related with semi-regular and essentially semi-regular operators which (under various names) were intensively studied, (see e. g. [9], [23], [43], [44], [45], [46], [51], [55], [56], [57], [58], [62], [67], [68], [69], [70], [72], [73], [76], [82], [83], [84], [85], [86], [87], [88], [89], [92], [100], [109], [115], [116], [122], [124], [125], [126], [127]). From a number of equivalent properties, for the beginning we point out the following Kato–type decomposition theorem ([92], [109]) for operators in $\mathcal{V}(X)$ which is related to Kato's theorem for semi-Fredholm operators ([62, Theorem 4], [134, Proposition 2.5]).

Let $T_{|M}$ denotes the restriction of T to the subspace M of X.

Theorem 4.4. (Kato decomposition) $T \in \mathcal{V}(X)$ *if and only if* $R(T)$ *is closed and there exist closed subspaces* $X_1, X_2 \subset X$ *invariant with respect to* T *such that* $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, $T|_{X_1}$ *is nilpotent and* $T|_{X_2} \in \mathcal{V}_0(X_2)$.

Let us remark that if $T \in B(X)$ is a lower semi-Browder operator then the space X_2 in the Kato decomposition is determined uniquely and $X_2 = R^\infty(T)$. Thus $T|_{X_2}$ is onto.

If $M \subset X$, then $\overline{M}$ denotes the closure of M in X.

Theorem 4.5. ([51, Theorem 4.10], [46, Theorem 3]) *Suppose that $T \in \mathcal{V}(X)$, $S \in B(X)$ and $TS = ST$. If $T - S$ is sufficiently small, then:*

(1) $S \in \mathcal{V}(X)$.

(2) *$R(S^n)/R(S^{n+1})$ and $R(T^m)/R(T^{m+1})$ have the same dimension for all sufficiently large m and n.*

(3) *$N(S^{n+1})/N(S^n)$ and $N(T^{m+1})/N(T^m)$ have the same dimension for all sufficiently large m and n.*

(4) *$R^\infty(T)$ is a subspace of $R^\infty(S)$ with codimension less than or equal to $k^\infty(T)$.*

(5) *$\overline{N^\infty(T)}$ is a subspace of $\overline{N^\infty(S)}$ with codimension less than or equal to $k^\infty(T)$.*

(6) $k^\infty(S) \leq k^\infty(T)$.

(7) $\|S - T\| < \gamma(T) \Longrightarrow k(S) \leq k(T)$.

(8) $\overline{N^\infty(S)} \cap R^\infty(S) = \overline{N^\infty(T)} \cap R^\infty(T)$.

Theorem 4.6. ([51, Theorem 5.9, Theorem 5.8]) *If $T \in \mathcal{V}(X)$, $S \in B(X)$, $TS = ST$ and $T - S \in K(X)$, then*

(1) $S \in \mathcal{V}(X)$.

(2) $\dim(R(S^n)/R(S^{n+1})) = \dim(R(T^m)/R(T^{m+1}))$ *for all sufficiently large m and n.*

(3) $\dim(N(S^{n+1})/N(S^n)) = \dim(N(T^{m+1})/N(T^m))$ *for all sufficiently large m and n.*

(4) $\dim[R^\infty(T) + R^\infty(S)]/[R^\infty(T) \cap R^\infty(S)] < \infty$.

(5) $\dim[\overline{N^\infty(T)} + \overline{N^\infty(S)}]/[\overline{N^\infty(T)} \cap \overline{N^\infty(S)}]$.

Now, set

$$\sigma_g(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{V}_0(X)\},$$
$$\sigma_{gb}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{V}(X)\}.$$

The set $\sigma_g(T)$ and its essential version the set $\sigma_{gb}(T)$ were studied (under various names and notations) by many authors, see e.g., [56], [67], [68], [69], [70], [75], [88], [92], [100], [109], [122] and [124]. Clearly, if H is a Hilbert space, then $\sigma_g(A) = \sigma_\gamma(A)$, $A \in B(H)$. For example, in [109] $\sigma_g(T)$ and $\sigma_{gb}(T)$ were called the *generalized spectrum* of T and the *Browder's generalized spectrum* of T. These notations and terminology were used because the relation between $\sigma_g(T)$ and $\sigma_{gb}(T)$ that was exhibited in [109] resembled the relation between $\sigma_a(T)$ and the $\sigma_{ab}(T)$, or relation between $\sigma(T)$ and the $\sigma_{eb}(T)$. (In [92] $\sigma_g(A)$ and $\sigma_{gb}(A)$ were denoted by $\sigma_\gamma(A)$ and $\sigma_{\gamma e}(A)$, respectively). Now we recall some results for $\sigma_g(T)$ and $\sigma_{gb}(T)$.

Theorem 4.7. ([124]) *Let $T \in B(X)$.*

(1) *Then the functions*

$$\lambda \mapsto R^\infty(T-\lambda) \quad \text{and} \quad \lambda \mapsto \overline{N^\infty(T-\lambda)}$$

are constant on connected component of $\mathbb{C} \setminus \sigma_g(T)$;

(2) $\sigma_g(T) = \sigma_g(T^*)$*;*

(3) $\sigma_g(T)$ *is closed;*

(4) $\sigma_g(T) = \sigma(T)$, $\partial\sigma(T) \subset \sigma_g(T)$ *and* $\sigma_g(T) \neq \emptyset$.

Theorem 4.8. ([124], [92], [109]) *Let $T \in B(X)$ and f be an analytic function defined on a neighbourhood of the spectrum of T. Then*

$$\sigma_g(f(T)) = f\{\sigma_g(T)\} \quad \text{and} \quad \sigma_{gb}(f(T)) = f\{\sigma_{gb}(T)\}.$$

Theorem 4.9. ([109]) *Suppose that $T \in B(X)$. Then*

$$\sigma_{gb}(T) = \bigcap_{\substack{TK=KT \\ K\in K(X)}} \sigma_g(T+K) = \bigcap_{\substack{TK=KT \\ K\in F(X)}} \sigma_g(T+K).$$

Corollary 4.10. ([109]) *Suppose that $T \in B(X)$. Then:*

(1) $\lambda \in \sigma_g(T) \setminus \sigma_{gb}(T)$ *if and only if* λ *is an isolated point of* $\sigma_g(T)$, $0 < k(T-\lambda) < \infty$ *and* $R(T-\lambda)$ *is closed,*

(2) $\sigma_{gb}(T) \subset \sigma_{ek}(T)$,

(3) $\partial\sigma_{ek}(T) \subset \partial\sigma_{gb}(T)$ *and* $\sigma_{gb}(T)$ *is nonempty,*

(4) $\sigma_{gb}(T) = \sigma_{gb}(T^*)$.

Corollary 4.11. ([109]) *Let $T \in \mathcal{V}(X)$. Then the following statements are equivalent:*

(1) $T = V + F$, *where* $\alpha(V) = 0$, F *is finite rank and* $VF = FV$.

(2) *There exists a finite rank projection* P, $PT = TP$ *and* $\alpha(T_{|N(P)}) = 0$.

(3) *There exists* $\epsilon > 0$ *such that* $\alpha(T+\lambda) = 0$ *for* $0 < |\lambda| < \epsilon$.

(4) $a(T) < \infty$.

Let us mention that the mappings $A \to \sigma_g(A)$ and $A \to \sigma_{gb}(A)$ are not upper semi-continuous at A in general [109, Remark 4.4]. Prof. Laura Burlando (1991) has kindly informed me that:

(4.3) σ_{gb} upper semi-continuous at T implies $\sigma_{gb}(T) = \sigma_{ek}(T)$;

(4.4) σ_g upper semi-continuous at T implies $\sigma_{ek}(T) \subset \sigma_g(T)$.

Theorem 4.12. ([109]) *Let $T, T_n \in B(X)$ and $TT_n = T_nT$ for each positive integer n. Then*

$$\limsup \sigma_g(T_n) \subset \sigma_g(T) \quad \textit{and} \quad \limsup \sigma_{gb}(T_n) \subset \sigma_{gb}(T).$$

Remark 4.13. If $T \in B(X)$, then $\mathbb{C}\backslash\sigma_{gb}(T)$ is an open set in $\mathbb{C}$. Further, let U be an connected component of $\mathbb{C}\backslash\sigma_{gb}(T)$ and $G = \{\lambda \in \mathbb{C}\backslash\sigma_{gb}(T) : k(T-\lambda) \neq 0\}$. A complex number $\lambda \in G \cap U$ is called a jumping point in U. If $\lambda \in U$ is a jumping point, then by Theorem 4.4, there is an T–invariant finite dimensional subspace N_λ in X such that $T - \lambda$ is nilpotent on it. Consistent with the matrix case we define the (algebraic) multiplicity of the jumping point λ to be $\dim N_\lambda$.

Theorem 4.14. ([109]) *Let $T \in B(X)$ and let U and G be as above. Then the functions*

$$\lambda \mapsto N^\infty(T-\lambda) + R^\infty(T-\lambda) \quad \textit{and} \quad \lambda \mapsto N^\infty(T-\lambda) \cap R^\infty(T-\lambda)$$

are constant on U, while the functions

$$\lambda \mapsto R^\infty(T-\lambda) \quad \textit{and} \quad \lambda \mapsto \overline{N^\infty(T-\lambda)}$$

are constant on $U \setminus G$.

Now, suppose that the connected component U contains zero. Than the points in $G \cap U$ can be ordered in such a way that

$$|\lambda_1(T)| \leq |\lambda_2(T)| \leq \ldots < v(T),$$

where each jump appears consecutively according to its multiplicity. If there are only $p\,(= 0, 1, 2, \ldots)$ such jumps, we put $|\lambda_{p+1}(T)| = |\lambda_{p+2}(T)| = v(T)$.

Let S denote the closed unit ball of X. Let $q(T) = \sup\{\epsilon \geq 0 : TS \supset \epsilon S\}$ be the *surjection modulus* of T. For each $r = 1, 2, \ldots$ set

$$q_r(T) = \sup\{q(T+F) : \text{rank} F < r\}.$$

Theorem 4.15. ([109]) *Let $T \in B(X), 0 \in U$, and let U, G, and $W \equiv N^\infty(T-\lambda) + R^\infty(T-\lambda)$, $\lambda \in U$ be as above. Then for each jumping point $\lambda_r(T)$, $r = 1, 2, \dots$ we have*

$$|\lambda_r(T)| = \lim_k q_r((T_{|W})^k)^{1/k}.$$

Corollary 4.16. ([109]) *If $T \in \mathcal{V}(X)$, then*

$$v_0(T) = \lim_k \gamma((T_{|W})^k)^{1/k}.$$

It was natural for me to finish paper [109] (October 1990) with the following questions (I admit I did not know the answers to these questions until I saw the papers of V. Müller and V. Kordula [68], [92]) :

Questions 4.1. *If $T \in \mathcal{V}(X)$, must*

$$\lim_k \gamma(T^k)^{1/k} = v_0(T)?$$

Question 4.2. *If $A, B \in B(X)$, and $AB = BA \in \mathcal{V}(X)$, must $A, B \in \mathcal{V}(X)$?*

Question 4.3. *If $A, B \in B(X)$, $AB = BA$ and B is a quasinilpotent operator, must*

$$\sigma_{gb}(A+B) = \sigma_{gb}(A)?$$

Question 4.4. *If $A, B \in \mathcal{V}(X)$, (or $\mathcal{V}_0(X)$) and $AB = BA$, must $AB \in \mathcal{V}(X)$ (or $\mathcal{V}_0(X)$), and possibly $k(A+B) \le k(A) + k(B)$?*

V. Kordula and V. Müller [68], among other things, have proved that the answer to Question 4.1 is positive [68, Theorem 4], and also the answer to Question 4.3 is positive [68, Theorem 6 (2)]; V. Müller [92], among other things, has proved that the answer to Question 4.2 is positive [92, Theorem 3.5], and the answer to both parts of Question 4.4 is negative [92, Example 2.2]. V. Müller set the following question ([92, Problem 3.11])

Question 4.5. *If $T \in \mathcal{V}_0(X)$ and A is a finite-dimensional operator, is then $T + A \in \mathcal{V}(X)$?*

V. Kordula [67] and P. W. Poon [100], among other things, have proved that the answer to Question 4.5 is positive.

Theorem 4.17. ([68]) *Let $T \in \mathcal{V}_0(X)$. Then*

$$dist\{0, \sigma_g(T)\} = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

Theorem 4.18. ([68]) *Let $T \in \mathcal{V}(X)$. Then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and*

$$\begin{aligned}\lim_{n\to\infty} \gamma(T^n)^{1/n} &= \max\{r : T - \lambda \in \mathcal{V}_0(X) \text{ for } 0 < |\lambda| < r\}\\ &= dist\,\{0, \sigma_g(T) \setminus \{0\}\}.\end{aligned}$$

Theorem 4.19. ([70]) *Let T be an operator on a Banach space X. Then the following conditions are equivalent:*

(1) $T \in \mathcal{V}(X)$,

(2) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Fredholm and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Fredholm,*

(3) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,*

(4) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is surjective and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,*

(5) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is bounded below.*

It is well-known that if $T \in \mathcal{V}(X)$ and K is a compact operator commuting with T then $T + K \in \mathcal{V}(X)$. Recently, we get a sharper result [70].

Theorem 4.20. ([70]) *Let $T, S \in B(X)$, $TS = ST$ and let $T \in \mathcal{V}(X)$. Let $\hat{T} = T|_{R^\infty(T)}$ and let $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ be the induced operator by T. Then*

$$r_e(S) < \min(r_-(\hat{T}), r_+(\tilde{T})) \quad \textit{implies} \quad T + S \in \mathcal{V}(X).$$

Corollary 4.21. ([70]) *Let $T \in \mathcal{V}(X)$, $S \in B(X)$, $TS = ST$ and S is a Riesz operator (i.e., $r_e(S) = 0$). Then $T + S \in \mathcal{V}(X)$.*

Corollary 4.22. ([70]) *Let $T \in B(X)$. Then*

$$\sigma_{gb}(T) = \bigcap \sigma_g(T+S)$$

where the intersection is taken over all Riesz operators in X commuting with T.

Remark 4.23. In Section 3 and above in this section (see also Section 5) we see that the reduced minimum modulus of $T \in B(X)$, $\gamma(T)$, plays important role in perturbation theory of linear operators. Also the behavior of the sequence $\{\gamma(T^n)^{1/n}\}$, is extremely important, and we find convenient to include here some Zemanek's results.

If $T \in B(X)$ is semi-Fredholm operator then there is an $\epsilon > 0$ such that both $\dim N(T-\lambda)$ and $\mathrm{codim} R(T-\lambda)$ are constant on $0 < |\lambda| < \epsilon$. We can define

$$\delta_+(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \text{ and } \alpha(T-\lambda) = \text{constant for } 0 < |\lambda| < \epsilon\},$$

$$\delta_-(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \text{ and } \beta(T-\lambda) = \text{constant for } 0 < |\lambda| < \epsilon\}.$$

Let us remark that $r_+(T) \geq \delta_+(T)$ and $r_-(T) \geq \delta_-(T)$.

Theorem 4.24. ([140, Theorem 1]) *Suppose that $T \in \Phi_\pm(X)$. Then the limit $\lim \gamma(T^n)^{1/n}$ exists and*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \delta_\pm(T).$$

Recall that an operator $T \in B(X)$ is *bounded below* if and only if $R(T)$ is closed and $N(T) = \{0\}$, i.e., if and only if $\mu(T) > 0$; an operator $T \in B(X)$ is *surjective* if and only if $R(T) = X$, i.e., if and only if $q(T) > 0$.

For $T \in B(X)$ set

$$\mu_r(T) = \sup\{\mu(T+F) : \text{ rank } F < r\},$$

and $g_r(T) = \max\{\mu_r(T), q_r(T)\}$. If T is semi-Fredholm either $\alpha(T+\lambda)$ or $\beta(T+\lambda)$ (the nullity or the defect) is constant ($= n$) for λ in the semi-Fredholm domain of T except at a discrete set of jump points which may be ordered by their moduli

$$|\lambda_1(T)| \leq |\lambda_2(T)| \leq \ldots \max\{\delta_+(T), \delta_-(T)\},$$

where each jump appears consecutively according to its multiplicity.

Theorem 4.25. ([113, Theorem 1.1]) *Let T be a semi-Fredholm operator. Then for each jumping point $\lambda_r(T)$, $r = 1, 2, \dots$ we have*

$$|\lambda_r(T)| = \lim_{k\to\infty} g_{kn+r}(T^k)^{1/k},$$

where $g_r(T) = \max(\mu_r(T), q_r(T))$.

For more details, connected with semi-Fredholm radius and related topics, the reader is referred to [36], [47], [48], [68], [91], [108], [113], [121], [131], [132], [138], [140], [142] and [144].

5. $\mathcal{S}(X), \mathcal{S}_e(X)$ and Corresponding Spectra

Let X be an infinite-dimensional complex Banach space. An operator $S \in B(X)$ is a *generalized inverse (pseudo inverse)* of T if $TST = T$. We then say that T is *relatively regular*. It is easy to see that if $TST = T$, then the operator $S_1 = STS$ satisfies the equations $TS_1T = T$ and $S_1TS_1 = S_1$. It is well known that T is relatively regular if and only if $N(T)$ and $R(T)$ are closed, complemented subspaces of X. In this case TS is a projection onto $R(T)$ and $I - ST$ is a projection onto $N(T)$; $T \in B(X)$ is called an operator of *Saphar* type (*Saphar* operator), ([23], [126], [127]) or *hyper-regular* ([56], [57]) or *regular* ([92]), if T is relatively regular and $N(T) \subset R^\infty(T)$. This class of operators has been studied by P. Saphar ([115], [116]). If π is the natural homomorphism of $B(X)$ onto the Calkin algebra $C(X)$, set

$$\begin{aligned}
\mathcal{S}(X) &= \{A \in B(X) : A \quad \text{is Saphar operator}\},\\
\mathcal{S}_e(X) &= \{A \in B(X) : A \quad \text{is relatively regular and} \quad k(A) < \infty\},\\
\Phi_l(X) &= \pi^{-1}(C(X)_l^{-1}),\\
\Phi_r(X) &= \pi^{-1}(C(X)_r^{-1}).
\end{aligned}$$

It is well-known that $\Phi_l(X)$ and $\Phi_r(X)$ are open semigroups in $B(X)$ ([24], [55]). Further, $T \in \Phi_l(X)$ if and only if $T \in \Phi_+(X)$ and there exists a bounded projection of X onto $R(T)$; $T \in \Phi_r(X))$ if and only if $T \in \Phi_-(X)$ and there exists a bounded projection of X onto $N(T)$. Hence, $\Phi_r(X) \cup \Phi_l(X) \subset \mathcal{S}_e(X)$; $\mathcal{S}(X)$ and $\mathcal{S}_e(X)$ are neither semigroups nor open or closed subset of $B(X)$. From the papers of M. A. Goldman [42] and C. Schmoeger [125] we get

$$\begin{aligned}
\operatorname{int}(\mathcal{S}_e(X)) &= \Phi_r(X) \cup \Phi_l(X),\\
\operatorname{int}(\mathcal{S}(X)) &= \{A \in B(X) : A \,\text{is left or right invertible in}\, B(X)\}.
\end{aligned}$$

Theorem 5.1. ([10], [23], [56], [57], [79], [92], [124], [126]) *$T \in \mathcal{S}(X)$ if and only if there is a neighbourhood $U \subset \mathbb{C}$ of 0 and a holomorphic function $F : U \mapsto B(X)$ such that*

$$(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda, \quad \textit{and} \quad F(\lambda)(T-\lambda)F(\lambda) = F(\lambda), \quad \textit{for all} \quad \lambda \in U.$$

Let us remark that for F it is possible to take

$$F(\lambda) = \sum_{i=0}^{\infty} S^{i+1}\lambda^i, \quad \lambda \in U,$$

where $S \in B(X)$ is a generalized inverse of T, i.e., $TST = T$ and $TST = T$, and $U \equiv \{\lambda \in \mathbb{C} : |\lambda| < \|S\|^{-1}\}$. Further

$$F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \text{for all } \lambda, \mu \in U,$$

i.e., $F(\lambda)$ satisfies the resolvent identity on U.

Theorem 5.2. ([92], [124], [126]) *Let $T \in B(X)$. Denote by $G = \{\lambda \in \mathbb{C} : T - \lambda \in \mathcal{S}(X)\}$. Then G is an open set and there exists an analytic function $F : G \mapsto B(X)$ such that*

$$(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda \quad \textit{and} \quad F(\lambda)(T-\lambda)F(\lambda) = F(\lambda), \quad \textit{for all} \ \lambda \in G.$$

Set

$$\sigma_s(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{S}(X)\},$$
$$\sigma_{se}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{S}_e(X)\};$$

$\sigma_s(T)$ and its essential version the set $\sigma_{se}(T)$ were studied (under various names and notations) by many authors, see e.g., [69], [92], [124], [126]. Clearly, if H is a Hilbert space, then $\sigma_s(A) = \sigma_g(A) = \sigma_\gamma(A)$ and $\sigma_{se}(A) = \sigma_{gb}(A), A \in B(H)$.

Theorem 5.3. ([124, Proposition 1]) *Let $T \in B(X)$ Then:*

(1) *$\sigma_s(T)$ is closed,*

(2) *$\sigma_g(T) \subset \sigma_s(T) \subset \sigma(T)$ and $\sigma_s(T) \neq \emptyset$,*

(3) *$\sigma_s(T^*) \subset \sigma_s(T)$, and in general $\sigma_s(T^*) \neq \sigma_s(T)$.*

To the best of my knowledge the next problem is still open

Question 5.1. ([92, Remark 4.2], [127, Question 3]) *Let $T \in B(X)$. Does there exist a holomorphic function $F : \mathbb{C} \setminus \sigma_s(T) \mapsto B(X)$ such that*

$$(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda, \quad \textit{for all} \quad \lambda \in \mathbb{C} \setminus \sigma_s(T),$$

and

$$F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \textit{for all} \quad \lambda, \mu \in \mathbb{C} \setminus \sigma_s(T)?$$

Theorem 5.4. ([69, Corollary 2.14], [124, Theorem 3]) *Let $T \in B(X)$ and f be an analytic function defined on a neighbourhood of the spectrum of T. Then*

$$\sigma_s(f(T)) = f\{\sigma_s(T)\} \quad \textit{and} \quad \sigma_{se}(f(T)) = f\{\sigma_{se}(T)\}.$$

Theorem 5.5. ([69, Theorem 2.13]) *Let $T, T_n \in B(X)$ and $TT_n = T_nT$ for each positive integer n. Then*

$$\limsup \sigma_s(T_n) \subset \sigma_s(T) \quad \textit{and} \quad \limsup \sigma_{se}(T_n) \subset \sigma_{se}(T).$$

Let $T \in \mathcal{S}(X)$. Then $T^n \in \mathcal{S}(X)$ for each $n \in \mathbb{N}$, and set ([127])

$$\operatorname{dist}\{0, \sigma_s(T)\} = d(T),$$

$$\delta_n(T) = \sup\{r(A)^{-1} : A \in B(X),\, T^nAT^n = T^n\}, \quad \delta(T) = \sup_{n \geq 1} \delta_n(T)^{1/n}.$$

Theorem 5.6. ([127, Proposition 6]) *Suppose that $T \in \mathcal{S}(X)$ and put $G = \{\lambda \in \mathbb{C} : |\lambda| < d(T)\}$. Then $G \subset \mathbb{C} \setminus \sigma_s(T)$ and there is a holomorphic function $F : G \mapsto B(X)$ such that*

$$(T-\lambda)F(\lambda)(T-\lambda) = T-\lambda \quad \textit{for all} \quad \lambda \in G.$$

The function F has the following properties:

(1) $(T-\lambda)^{n+1}F^{(n)}(\lambda)(T-\lambda)^{n+1} = n!(T-\lambda)^{n+1}$ *for all $\lambda \in G$ and all $n \in \mathbb{N}$.*

(2) *If $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ for $|\lambda| < d(T)$, then $d(T) = (\limsup \|A_n\|^{1/n})^{-1}$ and*

$$T^{n+1}A_nT^{n+1} = T^{n+1} \quad \textit{for each } n = 0, 1, 2, \ldots$$

Theorem 5.7. ([127, Theorem 3]) *Suppose that $T \in \mathcal{S}(X)$. Then*

(1) $\delta(T) = \lim_{n\to\infty} \delta_n(T)^{1/n} = d(T)$.

(2) *If* $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ *is a holomorphic pseudo-inverse of* $T - \lambda$ *for* $|\lambda| < d(T)$, *then* $d(T) = (\lim_{n\to\infty} \|A_n\|^{1/n})^{-1} = \lim_{n\to\infty} (r(A_n)^{1/n})^{-1}$.

Question 5.2. ([127, Question 1]) *If* $T \in \mathcal{S}(X)$, *must* $\lim_k \gamma(T^k)^{1/k} = d(T)$?

According to the Question 5.1 we can prove the following theorem.

Theorem 5.8. *Suppose that* $T \in \mathcal{S}(X)$, *dist*$\{0, \sigma_s(T)\} = d(T)$ *and* $\mathbf{G} = \{\lambda \in C : |\lambda| < d(T)\}$. *Then for any compact subset* $\mathbb{K}$ *of* G *there exists an analytic function* $F : U \mapsto B(X)$, *were* $U \subset G$, *is a neighbourhood of* $\mathbb{K}$, *such that*

(1) $(T - \lambda)F(\lambda)(T - \lambda) = T - \lambda$ *for all* $\lambda \in \mathbb{K}$,

(2) $F(\lambda)(T - \lambda)F(\lambda) = F(\lambda)$ *for all* $\lambda \in \mathbb{K}$, *and*

(3) $F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu)$ *for all* $\lambda, \mu \in \mathbb{K}$, *if and only if*

$$d(T) = \sup\{r(A)^{-1} : A \in B(X), TAT = T\}.$$

Question 5.3. *If* $T \in \mathcal{S}(X)$, *must* $R^\infty(T)$ *be a complemented subspace of* X?

Set

$$\begin{aligned} \mathcal{S}_l(X) = \{A \in B(X) :\ & R(A) = \overline{R(A)},\ N(A) \text{ is complemented} \\ & \text{subspaces of } X \text{ and } k(A) = 0\}, \\ \mathcal{S}_r(X) = \{A \in B(X) :\ & R(A) \text{ is closed complemented} \\ & \text{subspaces of } X \text{ and } k(A) = 0\}. \end{aligned}$$

Now we include some results of Goldman and Kračkovskii ([43], [44]). These results were proved for closed linear operators, but to simplify notations we shall consider bounded operators.

Theorem 5.9. ([43, Theorem 1]) *Suppose that* $A \in \mathcal{S}_l(X)$, $B \in B(X)$ *and* $AB = BA$. *Let* $P_A \in B(X)$ *be a projection onto* $N(A)$. *Then there is an* $\epsilon > 0$ *such that if* $\|B\| < \epsilon$, *then:*

(1) P_A *is a homeomorphism from* $N(A + B)$ *onto* $N(A)$;

(2) $A + B \in \mathcal{S}_l(X)$;

(3) *there is a projection* $P_{A+B} \in B(X)$ *onto* $N(A+B)$ *such that* $R(I - P_A) = R(I - P_{A+B})$ *(hence* $N(A)$ *and* $N(A+B)$ *have a joint complement);*

(4) *there is a homeomorphism from* $N(A+B)$ *onto* $N(A)$*;*

(5) *the gap between* $R(A)$ *and* $R(A+B)$ *is a small value of the same order, as* ϵ*;*

(6) $R^\infty(A) = R^\infty(A+B)$*;*

(7) $\overline{N^\infty(A)} = \overline{N^\infty(A+B)}$.

Theorem 5.10. ([44, Theorem 2]) *Suppose that* $A \in \mathcal{S}_r(X)$*,* $B \in B(X)$ *and* $AB = BA$*. Let* $Q_A \in B(X)$ *be a projection onto* $R(A)$*. Then for sufficiently small* B *:*

(1) $R(A+B)$ *is closed subspace of* X*, and* Q_A *is a homeomorphism from* $R(A+B)$ *onto* $R(A)$*;*

(2) $A + B \in \mathcal{S}_r(X)$*;*

(3) *there is a projection* $Q_{A+B} \in B(X)$ *onto* $R(A+B)$ *such that* $R(I - Q_A) = R(I - Q_{A+B})$ *(hence* $R(A)$ *and* $R(A+B)$ *have a joint complement);*

(4) *there is a homeomorphism from* $N(A+B)$ *onto* $N(A)$*;*

(5) *the gap between* $R(A)$ *and* $R(A+B)$ *is arbitrary small value;*

(6) $R^\infty(A) = R^\infty(A+B)$*;*

(7) $\overline{N^\infty(A)} = \overline{N^\infty(A+B)}$.

The following sets $\sigma_{sl}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{S}_l(X)\}$ and $\sigma_{sr}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{S}_r(X)\}$ have not been investigated for the Banach space operators, as we know, and in our opinion, deserve further attention.

Remark 5.11. Some of above results are connected with the global existence of finite-meromorphic relative inverses [10]. Meromorphic generalized resolvents were studied by several authors (see, e.g., [6], [7], [8], [9], [10], [23], [37], [38], [130], and references therein).

Remark 5.12. Let us point out that recently, V. Kordula and V. Müller [69] have studied an axiomatic theory for spectra which do not fit into the axiomatic theory of Żelazko [143].

Definition 5.13. ([69]) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset $\mathcal{R}$ of $\mathcal{A}$ is called a *regularity* if:

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in \mathcal{R} \Leftrightarrow a^n \in \mathcal{R}$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$.

A regularity $\mathcal{R} \subset \mathcal{A}$ defines a mapping $\tilde{\sigma}_{\mathcal{R}}$ from $\mathcal{A}$ into subsets of $\mathbb{C}$ by

$$\tilde{\sigma}_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{R}\} \qquad (a \in \mathcal{A}).$$

This mapping is the *spectrum corresponding to the regularity* $\mathcal{R}$.

V. Kordula and V. Müller [69] have investigated in details the spectrum corresponding to the regularity $\mathcal{R}$; in particular they have studied several examples of regularities of $B(X)$ and corresponding spectra (e.g. $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\mathcal{B}(X)$, $\mathcal{V}_0(X)$, $\mathcal{V}(X)$, $\mathcal{S}(X)$, $\mathcal{S}_e(X), \dots$). Let us remark that $\mathcal{B}_+(X)$ and $\mathcal{B}_+(X)$ are regularities of $B(X)$ (not mentioned in [69], but included in [87]) whith property **(P1)** from [69], as it follows.

(P1) $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$ for all commuting elements $a, b \in \mathcal{A}$.

Hence, the the spectrum of $T \in B(X)$ corresponding to the regularity $\mathcal{B}_+(X)$ is $\sigma_{ab}(T)$ the Browder´s essential approximate point spectrum of T, while the spectrum of $T \in B(X)$ corresponding to the regularity $\mathcal{B}_-(X)$ is $\sigma_{ab}(T)$ the Browder´s essential defect spectrum of T.

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University of Niš
Faculty of Philosophy
Department of Mathematics
Ćirila i Metodija 2
18000 Niš, Yugoslavia