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DUAL SPACE OF A QUATERNION HILBERT SPACE

Aleksandar Torgašev

Abstract. In this paper we prove some facts known from the real and complex Hilbert spaces, for the left quaternion Hilbert spaces (called Wachs spaces). Among other things we prove that every Wachs space is reflexive. Difficulties arising in proofs are due to the noncommutativity of the ring of quaternions Q. Several constructions have no analogy with the commutative case, and are specific for quaternions.

Let $Q = \{\alpha = a + bi + cj + dk \,|\, a, b, c, d \in \mathbb{R}\}$ be the noncommutative division ring of real quaternions. $\overline{\alpha} = a-bi-cj-dk$ will denote the conjugate of α and $|\alpha| = \sqrt{a^2 + b^2 + c^2 + d^2}$ the absolute value of α. $R = \{\alpha | b = c = d = 0\}$ can be identified with the real field and $C = \{\alpha | c = d = 0\}$ with the complex field. If $\alpha = a + bi + cj + dk \in Q$, then $a = \mathrm{Re}(\alpha)$ is called the real part of α. Every quaternion α satisfies the identity

$$\alpha = \mathrm{Re}(\alpha) + i\,\mathrm{Re}(-i\alpha) + j\,\mathrm{Re}(-j\alpha) + k\,\mathrm{Re}(-k\alpha).$$

Next, let H be an arbitrary left quaternion Hilbert space, which is sometimes called a left Wachs space. The quaternion scalar product $(x, y) \mapsto \langle x, y\rangle$ has all usual properties of a complex scalar product, but in view of the noncommutativity of the ring Q, it holds $\langle x, \alpha y\rangle = \langle x, y\rangle\overline{\alpha}$ for arbitrary vectors $x, y \in H$ and $\alpha \in Q$.

So far, quaternion Hilbert and Banach spaces have been considered several times in the literature. See, for instance [1], [4], [5], [6], [7], and the other papers quoted in [1 ; Ch.13].

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If H is a left quaternion Hilbert space, denote by H' the corresponding left dual space of H, i.e. the set of all bounded left linear functionals on H with the usual norm

$$||f|| = \sup\{|f(x)| : ||x|| = 1\}.$$

It is known that H' is a real vector space. But, in order to transform H' into a Wachs space, it is convenient to introduce also a structure of a right quaternion vector space in H.

Let $\{e_\nu | \nu \in \Lambda\}$ be a fixed left orthonormal basis in H. If $x \in H$, and if

$$x = \sum_{\nu \in \Lambda} \widehat{x}_\nu e_\nu \,,$$

where $\widehat{x}_\nu = \langle x, e_\nu \rangle$ $(\nu \in \Lambda)$ are the Fourier coefficients of x, we shall define

$$x\lambda = \sum_{\nu \in \Lambda} (\widehat{x}_\nu \lambda) e_\nu, \tag{1}$$

for any quaternion λ.

Then H becomes also a right vector space over Q, with the right scalar multiplication so defined. In particular, we have

$$rx = xr \qquad (x \in H, r \in R)\,, \tag{2}$$

$$||x\lambda|| = |\lambda|\, ||x|| \qquad (x \in H, \lambda \in Q)\,, \tag{3}$$

and

$$\langle x\lambda, y \rangle = \langle x, y\overline{\lambda} \rangle \qquad (x, y \in H, \lambda \in Q). \tag{4}$$

Relation (4) is obviously true since for any two vectors x, y in H we have the Parseval's equality

$$\langle x, y \rangle = \sum_{\nu \in \Lambda} \widehat{x}_\nu \overline{\widehat{y}}_\nu.$$

Now, for any functional $f \in H'$ and any $\lambda \in Q$, we define λf and $f\lambda$ as follows:

$$(\lambda f)(x) = f(x\lambda) \quad , \quad (f\lambda)(x) = f(x)\lambda \qquad (x \in H).$$

It is easy to see that then the dual space H' becomes a two-side quaternion Banach space, with the scalar multiplication so defined. In particular, we have

$$rf = fr \qquad (f \in H', r \in R),$$

and

$$||\lambda f|| = ||f\lambda|| = |\lambda|||f|| \qquad (f \in H', \lambda \in Q).$$

Next, we recall that for any functional $f \in H'$, exactly as in the real and the complex case, we have the Riesz representation of f in the form

$$f(x) = \langle x, y \rangle \qquad (x \in H), \tag{5}$$

for a vector $y \in H$, and then $||f|| = ||y||$. Let $J\colon H' \mapsto H$ be the canonical mapping from the space H' into the space H defined by relation (5), i.e. by

$$f(x) = \langle x, Jf \rangle \qquad (x \in H).$$

The mapping J is additive, isometric, bijective, and it also satisfies next two relations:

$$J(f\alpha) = \overline{\alpha} J(f)\,,$$

$$J(\alpha f) = (Jf)\overline{\alpha} \qquad (f \in H', \alpha \in Q).$$

Next, we shall define a mapping $K\colon H \mapsto H$ as follows. If $x \in H$ and if $x = \sum_{\nu \in \Lambda} \widehat{x}_\nu e_\nu$, let

$$Kx = \sum_{\nu \in \Lambda} \overline{\widehat{x}}_\nu e_\nu\,.$$

It is easy to see that K is an additive mapping defined on the space H, it holds true

$$||Kx|| = ||x|| \qquad (x \in H),$$

and also

$$K(\alpha x) = K(x)\overline{\alpha} \quad , \quad K(x\alpha) = \overline{\alpha} K(x)\,,$$

for every $x \in H$ and every $\alpha \in Q$.

Proposition 1. *The left dual space H' of a Wachs space H, is also a two-side Wachs space, if we introduce the inner product in H' by*

$$\langle f, g \rangle = \langle KJf, KJg \rangle \qquad (f, g \in H'). \tag{6}$$

The inner product (6) *is in accordance with the norm of the space* H'.

Proof. Since K and J are isometric mappings, we have that

$$\langle f, f\rangle = \langle KJf, KJf\rangle = ||KJf||^2 = ||Jf||^2 = ||f||^2 \qquad (f \in H').$$

Hence, expression (6) is in accordance with the norm of the space H'.

Relation $\langle f, g\rangle = \overline{\langle g, f\rangle}$ is obvious, and the functional $\langle f, g\rangle$ is evidently additive in the first argument. Further we have

$$\begin{aligned}\langle \alpha f, g\rangle &= \langle (KJ)(\alpha f), KJg\rangle = \langle K((Jf)\overline{\alpha}), KJg\rangle \\ &= \langle \alpha(KJf), KJg\rangle = \alpha\langle KJf, KJg\rangle \\ &= \alpha\langle f, g\rangle \qquad (\alpha \in Q),\end{aligned}$$

so that $\langle f, g\rangle$ is linear in the first argument, and generally sesquilinear.

It remains only to prove relation (4) for an inner product.

For arbitrary functionals $f, g \in H'$ and arbitrary quaternion α we have

$$\begin{aligned}\langle f\alpha, g\rangle &= \langle (KJ)(f\alpha), KJg\rangle = \langle K(\overline{\alpha}(Jf)), KJg\rangle \\ &= \langle (KJf)\alpha, KJg\rangle = \langle KJf, (KJg)\overline{\alpha}\rangle \\ &= \langle KJf, K(\alpha(Jg))\rangle = \langle KJf, (KJ)(g\overline{\alpha})\rangle \\ &= \langle f, g\overline{\alpha}\rangle.\end{aligned}$$

Therefore, H' is a two-side Wachs space with inner product (6). $\square$

Proposition 2. *The left dual space* H' *of a Wachs space* H *is congruent to the space* H.

Proof. Consider the mapping $U\colon H' \mapsto H$ defined by $U = KJ$. It is easily seen that U is two-linear, that is we have

$$U(\alpha f) = \alpha U(f), \quad U(f\alpha) = U(f)\alpha \qquad (f \in H', \alpha \in Q).$$

Besides, U is a bijective mapping, and relation (6) gives

$$\langle Uf, Ug\rangle = \langle f, g\rangle \qquad (f, g \in H').$$

Hence, U is an isomorphism of the Wachs spaces H' and H. $\square$

In the sequel, we want to show that any Wachs space is reflexive.

If X is an arbitrary two-side quaternion normed space, denote by X' the space of all bounded left linear functionals on X, and let $X'' = (X')'$. In [6], it is shown that both X' and X'' are two-side quaternion normed spaces, and X can be isometrically imbedded into X'' by the mapping

$$x \mapsto F_x, \tag{7}$$

where $F_x \in X''$ is for any $g \in X'$ defined by

$$F_x(g) = \mathrm{Re}(g(x)) - i\,\mathrm{Re}(g(xi)) - j\,\mathrm{Re}(g(xj)) - k\,\mathrm{Re}(g(xk)). \tag{8}$$

Note that $F_x(g) \neq g(x)$, so that we have not a direct analogy with the commutative cases.

The canonical mapping (7) is isometric and two-linear ([6]), so that X becomes a two-side subspace of the space X'' under this mapping. The space X is called reflexive if $X = X''$ by the mapping (7).

Next, we need a technical result on the mapping K.

Lemma 1. *If H is a Wachs space, then for any two vectors $x, y \in H$, the next identity holds:*

$$\langle Kx, Ky\rangle = Re(\langle x, y\rangle) + i\,Re(\langle xi, y\rangle) + j\,Re(\langle xj, y\rangle) + k\,Re(\langle xk, y\rangle). \tag{9}$$

Proof. We obviously have that

$$\begin{aligned}
\langle Kx, Ky\rangle &= \mathrm{Re}(\langle Kx, Ky\rangle) + i\,\mathrm{Re}(-i\langle Kx, Ky\rangle)\\
&\quad + j\,\mathrm{Re}(-j\langle Kx, Ky\rangle) + k\,\mathrm{Re}(-k\langle Kx, Ky\rangle)\\
&= \mathrm{Re}(\langle Kx, Ky\rangle) + i\,\mathrm{Re}(\langle -iKx, Ky\rangle)\\
&\quad + j\,\mathrm{Re}(\langle -jKx, Ky\rangle) + k\,\mathrm{Re}(\langle -kKx, Ky\rangle)\\
&= \mathrm{Re}(\langle Kx, Ky\rangle) + i\,\mathrm{Re}(\langle K(xi), Ky\rangle)\\
&\quad + j\,\mathrm{Re}(\langle K(xj), Ky\rangle) + k\,\mathrm{Re}(\langle K(xk), Ky\rangle).
\end{aligned}$$

Now, observe that for any two vectors $u, v \in H$ we have

$$\mathrm{Re}(\langle Ku, Kv\rangle) = \mathrm{Re}(\langle u, v\rangle). \tag{10}$$

Really, if

$$u = \sum_{\nu\in\Lambda} \widehat{u}_\nu e_\nu\,, \quad v = \sum_{\nu\in\Lambda} \widehat{v}_\nu e_\nu\,,$$

then

$$\langle u, v\rangle = \sum_{\nu\in\Lambda} \widehat{u}_\nu \overline{\widehat{v}}_\nu \, ,$$

and

$$Ku = \sum_{\nu\in\Lambda} \overline{\widehat{u}}_\nu e_\nu \, , \quad Kv = \sum_{\nu\in\Lambda} \overline{\widehat{v}}_\nu e_\nu \, ,$$

whence

$$\langle Ku, Kv\rangle = \sum_{\nu\in\Lambda} \overline{\widehat{u}}_\nu \widehat{v}_\nu \, , \quad \overline{\langle Ku, Kv\rangle} = \sum_{\nu\in\Lambda} \overline{\widehat{v}}_\nu \widehat{u}_\nu \, .$$

But, as is known, one has $\mathrm{Re}(\alpha\beta) = \mathrm{Re}(\beta\alpha)$ for arbitrary quaternions α and β. Hence

$$\mathrm{Re}(\langle Ku, Kv\rangle) = \mathrm{Re}(\overline{\langle Ku, Kv\rangle}) = \mathrm{Re}(\langle u, v\rangle).$$

From relation (10) we obviously have that

$$\mathrm{Re}(\langle Kx, Ky\rangle) = \mathrm{Re}(\langle x, y\rangle)\, , \quad \mathrm{Re}(\langle K(xi), Ky\rangle) = \mathrm{Re}(\langle xi, y\rangle)\, ,$$
$$\mathrm{Re}(\langle K(xj), Ky\rangle) = \mathrm{Re}(\langle xj, y\rangle)\, , \quad \mathrm{Re}(\langle K(xk), Ky\rangle) = \mathrm{Re}(\langle xk, y\rangle)\, ,$$

whence we find relation (9).

Note also that relation (9) can be written in the equivalent form

$$\langle Ky, Kx\rangle = \mathrm{Re}(\langle x, y\rangle) - i\,\mathrm{Re}(\langle xi, y\rangle) - j\,\mathrm{Re}(\langle xj, y\rangle) - k\,\mathrm{Re}(\langle xk, y\rangle) \tag{11}$$

for arbitrary $x, y \in H$. □

Proposition 3. *Every Wachs space H is reflexive.*

Proof. Take any functional $F \in H''$. We have to prove that there is a vector $x_0 \in X$ such that $F = F_{x_0}$, that is

$$F(g) = F_{x_0}(g) \qquad (g \in H'),$$

or equivalently

$$F(g) = \mathrm{Re}(g(x_0)) - i\,\mathrm{Re}(g(x_0 i)) - j\,\mathrm{Re}(g(x_0 j)) - k\,\mathrm{Re}(g(x_0 k))$$

for any $g \in H'$.

Since H' is also a Wachs space, by the Riesz representation theorem applied to the functional $F \in H''$, we conclude that there is a functional $h_0 \in H'$ such that

$$F(g) = \langle g, h_0\rangle$$

for all $g \in H'$. In other words we have

$$F(g) = \langle KJg, KJh_0 \rangle = \langle KJg, Kx_0 \rangle,$$

for all $g \in H'$, where we put $x_0 = Jh_0 \in H$.

Now, denoting $Jg = y \in H$, and applying formula (11), we find

$$\begin{aligned} F(g) &= \langle Ky, Kx_0 \rangle \\ &= \mathrm{Re}(\langle x_0, y \rangle) - i\,\mathrm{Re}(\langle x_0 i, y \rangle) - j\,\mathrm{Re}(\langle x_0 j, y \rangle) - k\,\mathrm{Re}(\langle x_0 k, y \rangle) \\ &= \mathrm{Re}(\langle x_0, Jg \rangle) - i\,\mathrm{Re}(\langle x_0 i, Jg \rangle) - j\,\mathrm{Re}(\langle x_0 j, Jg \rangle) - k\,\mathrm{Re}(\langle x_0 k, Jg \rangle). \end{aligned}$$

Since $\langle x, Jg \rangle = g(x)$ for any $x \in H$, we obviously get

$$\begin{aligned} F(g) &= \mathrm{Re}(g(x_0)) - i\,\mathrm{Re}(g(x_0 i)) - j\,\mathrm{Re}(g(x_0 j)) - k\,\mathrm{Re}(g(x_0 k)) \\ &= F_{x_0}(g), \end{aligned}$$

for any $g \in H'$.

This completes the proof. $\square$

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University of Belgrade
Mathematical Faculty
Studentski Trg 16a, 11000 Beograd
Yugoslavia