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TRIGONOMETRIC WAVELET INTERPOLATION IN BESOV SPACES

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Dedicated to Prof. M.R. Occorsio for his 65th birthday

Abstract. In the present paper we investigate a particular kind of trigonometric interpolating wavelets, arising by an opportune discretization of the de la Vallée Poussin operator. Wavelet compression is analyzed, and estimates of the interpolating error are given in $L^p_{2\pi}$-norms and in Besov-norms.

1. Introduction

Given a continuous 2π-periodic function f, let us consider the Fourier sum of f:

$$S_m f(x) := \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + b_k \sin kx, \qquad m \in \mathbb{N}_0, \tag{1}$$

where

$$a_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, \qquad k = 0, 1, \ldots, m, \tag{2}$$

$$b_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt, \qquad k = 1, 2, \ldots, m. \tag{3}$$

It is well-known [2] that if we apply to the integrals (2) and (3), the following quadrature formula, of degree of exactness l:

$$\int_0^{2\pi} T(x)\,dx = \frac{2\pi}{l+1}\sum_{k=0}^{l} T\left(\frac{2k\pi}{l+1}\right), \tag{4}$$

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choosing $l := 2m$, we obtain the trigonometric Lagrange polynomial:

$$L_m^* f(x) := \frac{A_0}{2} + \sum_{k=1}^{m} A_k \cos kx + B_k \sin kx,$$

$$A_k := \frac{2}{2m+1} \sum_{j=0}^{2m} f\left(\frac{2j\pi}{2m+1}\right) \cos \frac{2jk\pi}{2m+1}, \qquad k = 0, 1, \ldots, m,$$

$$B_k := \frac{2}{2m+1} \sum_{j=0}^{2m} f\left(\frac{2j\pi}{2m+1}\right) \sin \frac{2jk\pi}{2m+1}, \qquad k = 1, 2, \ldots, m,$$

interpolating f at the $2m+1$ nodes $\theta_{m,k} := \dfrac{2k\pi}{2m+1}$, $k = 0, 1, \ldots, 2m$.

In this paper we apply a similar discretizing procedure to the de la Vallée Poussin operator:

$$V_m f(x) := \frac{S_m f(x) + S_{m+1} f(x) + \cdots + S_{2m-1} f(x)}{m}, \quad m \in \mathbb{N}.$$

Using the same quadrature formula (4), but with $l := 3m - 1$, we obtain another interpolating operator, denoted by $\tilde{L}_m$, which maps 2π-periodic continuous functions into polynomials of degree $2m - 1$ interpolating on $3m$ nodes, and which, like its continuous version V_m, preserves the polynomials of order m.

In the first part of this paper, we show that $\tilde{L}_m$ is a bounded map in $C^0_{2\pi}$ and in some subspaces of $L^p_{2\pi}$. Several estimates of the interpolation error are also given. In the second part of the paper, starting from an idea in [6], we consider a particular type of trigonometric interpolating scaling functions, defined as de la Vallée Poussin means of the Dirichlet kernels. Then we look the operator $\tilde{L}_m$ from a different point of view: as a projector in the sampled spaces of a given periodic *multiresolution analysis* [3].

This viewpoint allows us to use all the tools of wavelet theory, i.e. decomposition, compression and reconstruction. Thanks to the multiresolution structure, given a 2π-periodic, continuous function f, we can decompose the initial polynomial interpolating f at resolution levels lower and lower. Then we can neglect that elements of the decomposition that are unimportant for the wanted approximation of f and, finally, we can reconstruct the function at the initial resolution level. In this way we obtain a new compressed approximation of f, that is no more interpolating, but that is described by a number of decomposition coefficients less than the initial approximation, approximation degree being the same.

Studying the error of the compressed approximation, we indicate a criteria to choose the threshold for the compression. Finally we give some numerical examples, which confirm the following thesis: locally, in the regular parts of the function, the considered interpolation process has a behaviour better than the polynomial of the best approximation.

In the present paper only the trigonometric case is taken under consideration, but we advice that various results about the algebraic interpolation can be deduced generalizing this trigonometric case in several directions. For example, we can translate faithfully the trigonometric case banally, bringing about a change in variable $x := \cos t, t \in [0, \pi]$: in this way we still obtain interpolating wavelets arising by a discretization of the de la Vallée Poussin algebraic operator which is defined as mean of the Chebyshev sums (see [7]).

In [1] interpolating polynomial wavelets are constructed by means of a discretization of the Chebyshev sums, obtaining the classical Lagrange algebraic interpolation on the zeros of the Chebyshev polynomials of the 2-th kind and on the extremes ± 1. In this case we can apply the decomposition-compression-reconstruction process, but we have no more uniformly convergent sequence of interpolating polynomials. Nevertheless these are not the unique possible approaches to the algebraic case and we will return on this argument somewhere else.

The outline of the present paper is as follows. Section 2 is divided in two Subsections: in the first one, we give the notations and the definitions which are necessary in the sequel; in the second one, we discretize the de la Vallée Poussin operator introducing the interpolating operator $\tilde{L}_m$. In Section 3 we enunciate the main results about the boundedness of $\tilde{L}_m$ in the infinite norm and in Besov norms. In Section 4 we introduce the discrete version of the de la Vallée Poussin operator by means of the wavelets theory. In Section 5 we analyze the effects of the decomposition-compression-reconstruction process applied to our interpolating operator, and finally, in Section 6 we give the proofs.

2. Preliminaries

Some functional spaces. Throughout this paper we denote by $\mathcal{T}_n$ the set of all trigonometric polynomials of degree at most n and by $C_{2\pi}$ the set of all continuous 2π-periodic functions equipped with the norm

$$\|f\|_\infty := \sup_{x \in [0,2\pi)} |f(x)|.$$

As usual the notation $A \sim B$ with $A, B \geq 0$, means that there exist two positive constants C_1, C_2 such that $C_1 B \leq A \leq C_2 B$. Furthermore, throughout the paper we denote by C a positive constant which may take different values in different formulas.

For $1 \leq p < +\infty$, $L^p_{2\pi}$ is the set of 2π-periodic functions f such that

$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} < +\infty,$$

while for $p = +\infty$, with $L^\infty_{2\pi}$ we denote the set of 2π-periodic functions everywhere defined and such that $\|f\|_\infty < +\infty$.

As usual, for each $n \in \mathbb{N}$, $1 \leq p \leq +\infty$ and $f \in L^p_{2\pi}$, $E^*_n(f)_p$ denotes the error of the best trigonometric approximation of f in the $L^p_{2\pi}$-norm, i.e.,

$$E^*_n(f)_p := \inf_{T \in \mathcal{T}_n} \|f - T\|_p. \tag{5}$$

Furthermore, we denote by $\omega_k(f,t)_p$ the ordinary modulus of continuity of f, defined as

$$\omega_k(f,t)_p := \sup_{h \leq t} \|\Delta^k_h f\|_p \,, \tag{6}$$

with

$$\Delta_h f(x) := f(x+h) - f(x), \quad \Delta^k_h f := \Delta_h \Delta^{k-1}_h f, \quad h > 0,\ k \in \mathbb{N}.$$

The Jackson inequality:

$$E^*_n(f)_p \leq C \omega_k(f; 1/n)_p, \quad k < n, \tag{7}$$

and the Salem-Stechkin inequality:

$$\omega_k \left(f; \frac{1}{n} \right)_p \leq \frac{C}{n^k} \sum_{i=0}^{n} (1+i)^{k-1} E^*_i(f)_p \,, \tag{8}$$

hold for all $n \in \mathbb{N}$, $1 \leq p \leq +\infty$, and $f \in L^p_{2\pi}$ (see [4]).

By using $\omega_k(f,t)_p$, we can define the following quasi-norm

$$\|f\|_{p,q,r} := \begin{cases} \left(\displaystyle\int_0^1 \left[\frac{\omega_k(f;t)_p}{t^r} \right]^q \frac{dt}{t} \right)^{1/q} & \text{if } \ 1 \leq q < +\infty \ , \\ \displaystyle\sup_{t>0} \frac{\omega_k(f;t)_p}{t^r} & \text{if } \ q = +\infty \ , \end{cases}$$

for $k > r$, where p, q, r are fixed real parameters such that

$$1 \le p \le +\infty; \qquad 1 \le q \le +\infty; \qquad r \ge 0.$$

Adding to the above quasi-norm the $L^p_{2\pi}$-norm, we obtain the Besov norm:

$$\|f\|_{B^p_{r,q}} := \|f\|_p + \|f\|_{p,q,r}, \tag{9}$$

and the corresponding Besov space

$$B^p_{r,q} := \Big\{ f \in L^p_{2\pi} : \|f\|_{B^p_{r,q}} < +\infty \Big\}.$$

By the inequalities (7) and (8), it can be proved that the Besov-norm (9) is equivalent to the norm defined by means of the error of the best trigonometric approximation, as follows

$$\|f\|_{E^p_{r,q}} := \|f\|_p + \begin{cases} \left(\displaystyle\sum_{k=0}^{+\infty} [(1+k)^{r-1/q} E^*_k(f)_p]^q \right)^{1/q}, & 1 \le q < +\infty, \\ \displaystyle\sup_{k \ge 0} (1+k)^r E^*_k(f)_p \,, & q = +\infty. \end{cases}$$

Thus,

$$\|f\|_{B^p_{r,q}} \sim \|f\|_{E^p_{r,q}}, \tag{10}$$

and the Besov space $B^p_{r,q}$ admits a discrete equivalent version $E^p_{r,q}$ defined as

$$E^p_{r,q} := \Big\{ f \in L^p_{2\pi} : \|f\|_{E^p_{r,q}} < +\infty \Big\}.$$

An interpolating operator. For each function $f \in L^1_{2\pi}$, let us consider the de la Vallée Poussin sum of f:

$$V_m f(x) := \frac{1}{m} \sum_{n=m}^{2m-1} S_n f(x), \quad m \in \mathbb{N},\ m \ge 2,$$

where $S_n f$ is the $(n+1)$-th partial sum of the Fourier series of f:

$$\begin{aligned} S_n f(x) : &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \quad n \in \mathbb{N}, \\ a_k : &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt, \qquad k = 0, 1, \ldots, n, \\ b_k : &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt, \qquad k = 1, 2, \ldots, n. \end{aligned}$$

The following results are well-known [8]:

i) For each $T \in \mathcal{T}_m$, $\quad V_m T = T$;

ii) $\|V_m\|_\infty := \sup_{\|f\|_\infty = 1} \|V_m f\|_\infty = \frac{4}{\pi^2} \log 2 + c_m, \quad \sup_m c_m < +\infty.$

Using the Dirichlet kernel

$$D_n(x) := 1 + 2 \sum_{k=1}^{n} \cos kx, \qquad n \in \mathbb{N},$$

we have the following integral representation

$$V_m f(x) = \frac{1}{2\pi m} \int_0^{2\pi} f(t) \sum_{n=m}^{2m-1} D_n(x-t)\, dt. \tag{11}$$

A particular type of trigonometric interpolating polynomial can be deduced from (11) by means of the following quadrature formula:

$$\int_0^{2\pi} T(x)\, dx = \frac{2\pi}{3m} \sum_{k=0}^{3m-1} T\left(\frac{2k\pi}{3m} \right), \qquad T \in \mathcal{T}_{3m-1}. \tag{12}$$

In fact, applying (12) in (11), we obtain the following trigonometric polynomial of degree $2m-1$

$$\tilde{L}_m f(x) := \frac{1}{3m} \sum_{k=0}^{3m-1} \left[\frac{1}{m} \sum_{n=m}^{2m-1} D_n(x - t_{m,k}) \right] f(t_{m,k}), \tag{13}$$

where

$$t_{m,k} := \frac{2k\pi}{3m}, \qquad k = 0, 1, \ldots, 3m-1.$$

Hence, taking into account that

$$\sum_{n=m}^{2m-1} D_n(x) = \begin{cases} 3m^2, & \text{if } \ x = 2k\pi, \ k \in \mathbb{Z}, \\ \dfrac{\sin 3mx/2 \sin mx/2}{\sin^2 x/2}, & \text{otherwise,} \end{cases}$$

it is simple to recognize that the trigonometric interpolating polynomial $\tilde{L}_m f$ interpolates the function f at the $3m$ nodes $t_{m,k}$, $k = 0, 1, \ldots, 3m-1$.

An explicit expression of the polynomial $\tilde{L}_m f$ is the following

$$\tilde{L}_m f(x) = \frac{\tilde{A}_0}{2} + \sum_{k=1}^{2m-1} \lambda_{k,m} (\tilde{A}_k \cos kx + \tilde{B}_k \sin kx),$$

where

$$\lambda_{k,m} := \begin{cases} 1, & \text{if } \; k \le m, \\ \dfrac{2m-k}{m}, & \text{if } \; k > m, \end{cases}$$

for $k = 1, 2, \ldots, 2m-1$, and

$$\begin{aligned} \tilde{A}_k : &= \frac{2}{3m} \sum_{j=0}^{3m-1} f(t_{m,j}) \cos k t_{m,j}, \quad k = 0, 1, \ldots, 2m-1, \\ \tilde{B}_k : &= \frac{2}{3m} \sum_{j=0}^{3m-1} f(t_{m,j}) \sin k t_{m,j}, \quad k = 1, 2, \ldots, 2m-1, \\ t_{m,j} : &= \frac{2\pi j}{3m}, \qquad j = 0, 1, \ldots, 3m-1. \end{aligned}$$

We note that, by the well-made choice of the quadrature formula, the discrete operator $\tilde{L}_m$, like its continuous version V_m, satisfies the following invariance property:

$$(\forall T \in \mathcal{T}_m) \qquad \tilde{L}_m T = T.$$

3. On the Boundedness of the Operator $\tilde{\mathbf{L}}_\mathbf{m}$

First of all, the following theorem holds.

Theorem 1. *For each $m \in \mathbb{N}$, $1 \le p < +\infty$, and $f \in C_{2\pi}$, it results*

$$A \left(\frac{1}{3m} \sum_{k=0}^{3m-1} |f(t_{m,k})|^p \right)^{1/p} \le \|\tilde{L}_m f\|_p \le B \left(\frac{1}{3m} \sum_{k=0}^{3m-1} |f(t_{m,k})|^p \right)^{1/p}, \tag{14}$$

where $t_{m,k} = 2k\pi/(3m)$, $k = 0, 1, \ldots, 3m-1$, and

$$A = \frac{3}{3+4\pi}, \qquad B = 3 + 4\pi$$

are suitable values of the constants A and B.

The same result holds for $p = +\infty$, i.e.,

$$\frac{3}{3+4\pi} \max_k \left| f\left(\frac{2k\pi}{3m}\right) \right| \le \|\tilde{L}_m f\|_\infty \le (3+4\pi) \max_k \left| f\left(\frac{2k\pi}{3m}\right) \right|. \tag{15}$$

This theorem can be found in [6], in a lightly different version. For convenience of the reader, in Section 6 we give a simple proof of it.

Obviously, by Theorem 1 we immediately deduce the following result:

Corollary 1. *Let $m \in \mathbb{N}$, $1 \le p \le +\infty$, and $f \in C_{2\pi}$. Then*

$$\|f - \tilde{L}_m f\|_p \le 4(1+\pi) E_m^*(f)_\infty \tag{16}$$

holds.

We observe that for $1 \le p < +\infty$, (16) is not a homogeneous estimate by the presence of the infinite norm on the right hand side. A better estimate is given by the following statement.

Theorem 2. *For each p such that $1 \le p \le +\infty$ and for every function $f \in C_{2\pi}$ satisfying*

$$\int_0^1 \frac{\omega_k(f;t)_p}{t^{1+1/p}} dt < +\infty,$$

we have

$$\|f - \tilde{L}_m f\|_p \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\omega_k(f;t)_p}{t^{1+1/p}} dt, \tag{17}$$

where C is a constant independent of f and m.

By Theorem 2, we obtain the following corollary.

Corollary 2. *Let $1 \le p, q \le +\infty$ and $r > 1/p$ arbitrarily fixed. Then for each function $f \in C_{2\pi} \cap B^p_{r,q}$, we have*

$$\|f - \tilde{L}_m f\|_p \le \frac{C}{m^r} \|f\|_{B^p_{r,q}}, \tag{18}$$

C being a constant independent of f and m.

Finally, if we consider $\tilde{L}_m$ as a map from the Besov space $B^p_{r,q}$ into itself, we can state the following uniform boundedness result.

Theorem 3. *For each $m \in \mathbb{N}$, $1 \le p, q \le +\infty$, $s \ge r \ge 0$, $s > 1/p$, and $f \in C_{2\pi} \cap B^p_{s,q}$, the following estimate*

$$\|f - \tilde{L}_m f\|_{B^p_{r,q}} \le \frac{C}{m^{s-r}} \|f\|_{B^p_{s,q}} \tag{19}$$

holds, C being a constant independent of f and m. Consequently, for all $1 \le p, q \le +\infty$ and $r \in \mathbb{R}^+$ with $r > 1/p$,

$$\sup_m \|\tilde{L}_m\|_{\tilde{B}^p_{r,q} \to \tilde{B}^p_{r,q}} < +\infty \tag{20}$$

holds, where $\tilde{B}^p_{r,q}$ is the space $B^p_{r,q} \cap C_{2\pi}$, equipped with the Besov norm $\| \cdot \|_{B^p_{r,q}}$.

4. Interpolating Scaling Functions and Wavelets

Let us look at the interpolating operator $\tilde{L}_m$ from a different point of view. For this end, from now on, let us consider m as a function of $j \in \mathbb{N}_0$ and more precisely, let us assume:

$$m = m_j := 2^{j+1}, \qquad j \in \mathbb{N}_0.$$

For each $j \in \mathbb{N}_0$, let for $k = 0, 1, \ldots, 3m_j - 1$

$$\varphi_j(x) := \frac{1}{3m_j^2} \sum_{n=m_j}^{2m_j-1} D_n(x), \quad \varphi_{jk}(x) := \varphi_j(x - t_{jk}), \quad t_{jk} := \frac{2k\pi}{3m_j}.$$

It has been proved [6] that the functions φ_j are scaling functions generating a periodic Multiresolution Analysis (shortly MRA) of $L^2_{2\pi}$, i.e., the subspaces of $L^2_{2\pi}$

$$V_j := \operatorname{span}\{\varphi_{jk} : \ k = 0, 1, \ldots, 3m_j - 1\}, \qquad j \in \mathbb{N}_0,$$

satisfy all properties defining a periodic MRA of $L^2_{2\pi}$ (see [3]):

(P1) For each $j \in \mathbb{N}_0$, $V_j \subset V_{j+1}$;

(P2) $\operatorname{clos}_{L^2_{2\pi}} \bigcup_{j=0}^{+\infty} V_j = L^2_{2\pi}$;

(P3) $\forall j \in \mathbb{N}_0$, $f(x) \in V_j \Longrightarrow f\left(x - \frac{2\pi}{3m_j}\right) \in V_j$;

(P4) The set

$$\left\{\varphi_{jk}(x) := \varphi_j\left(x - \frac{2k\pi}{3m_j}\right) : k = 0, 1, \dots 3m_j - 1\right\}$$

constitutes a Riesz basis of V_j, i.e., it is a basis of V_j and moreover, there exist two constants $A, B > 0$ such that

$$A\left[\frac{1}{3m_j}\sum_{k=0}^{3m_j-1}|a_k|^2\right]^{1/2} \leq \left\|\sum_{k=0}^{3m_j-1} a_k\varphi_{jk}\right\|_2 \leq B\left[\frac{1}{3m_j}\sum_{k=0}^{3m_j-1}|a_k|^2\right]^{1/2} \tag{21}$$

holds for each $j \in \mathbb{N}_0$ and each $\{a_k\}_{k=0}^{3m_j-1} \in \mathbb{C}^{3m_j}$.

Furthermore, it has been shown that the following additional properties hold:

(I) $\mathcal{T}_{m_j} \subset V_j \subset \mathcal{T}_{2m_j-1}$, for each $j \in \mathbb{N}_0$;

(II) $\varphi_{jk}(t_{jh}) = \delta_{hk}$, for each $j \in \mathbb{N}_0$ and each $h, k = 0, 1, \dots, 3m_j - 1$;

(III) There exist two constants $C_1, C_2 > 0$ such that for each $j \in \mathbb{N}_0$, every $\{a_k\}_{k=0}^{3m_j-1} \in \mathbb{C}^{3m_j}$, and each p $(1 \leq p \leq +\infty)$, it results

$$C_1\left[\frac{1}{3m_j}\sum_{k=0}^{3m_j-1}|a_k|^p\right]^{1/p} \leq \left\|\sum_{k=0}^{3m_j-1} a_k\varphi_{jk}\right\|_p \leq C_2\left[\frac{1}{3m_j}\sum_{k=0}^{3m_j-1}|a_k|^p\right]^{1/p}. \tag{22}$$

For each $j \in \mathbb{N}_0$, the function

$$\psi_j(x) := 2\varphi_{j+1}\left(x - \frac{\pi}{3m_j}\right) - \varphi_j\left(x - \frac{\pi}{3m_j}\right) \tag{23}$$

is the wavelet of level j (see [6]). It is the unique function whose shifts

$$\psi_{jk}(x) := \psi_j(x - t_{jk}), \qquad k = 0, 1, \dots, 3m_j - 1,$$

firstly, constitute a Riesz basis of the wavelet space W_j, defined by the conditions

$$V_{j+1} = V_j \oplus W_j, \qquad V_j \perp W_j, \qquad j \in \mathbb{N}_0, \tag{24}$$

and secondly, they satisfy the additional interpolation property:

$$\psi_{jk}\left(\frac{(2h+1)\pi}{3m_j}\right) = \delta_{hk}.$$

The Riesz stability condition (22) holds for wavelet functions too, i.e. by (23), (22) we can deduce that there exist two constants, namely C_1' and C_2', such that $\forall j \in \mathbb{N}_0$, $\forall \{a_k\}_{k=0}^{3m_j-1} \in \mathbb{C}^{3m_j}$ and $\forall p$ $(1 \le p \le +\infty)$, we have

$$(25)\quad C_1' \left[\frac{1}{3m_j} \sum_{k=0}^{3m_j-1} |a_k|^p \right]^{1/p} \le \left\| \sum_{k=0}^{3m_j-1} a_k \psi_{jk} \right\|_p \le C_2' \left[\frac{1}{3m_j} \sum_{k=0}^{3m_j-1} |a_k|^p \right]^{1/p}.$$

Taking into account the interpolation property of the scaling functions φ_j (property II), for each level $j \in \mathbb{N}_0$, we can define the following interpolating operator:

$$\tilde{L}_j f(x) := \sum_{k=0}^{3m_j-1} f(t_{jk}) \varphi_{jk}(x); \qquad \tilde{L}_j f(t_{jk}) = f(t_{jk}), \quad k = 0, 1, \ldots, 3m_j - 1.$$

This is nothing else than the interpolating operator $\tilde{L}_m$ in (13) calculated with $m = m_j = 2^{j+1}$. But since obviously

$$\tilde{L}_j \varphi_{jk} = \varphi_{jk}, \qquad j \in \mathbb{N}_0, \quad k = 0, 1, \ldots, 3m_j - 1,$$

we also have for each $j \in \mathbb{N}_0$ that

$$(\forall f \in V_j) \qquad \tilde{L}_j f = f.$$

Hence we can interpret the discrete approximation $\tilde{L}_{m_j} = \tilde{L}_j$ of the de la Vallée Poussin operator V_{m_j} as a projection on the sampled space V_j too. This fact permits us to apply the typical strategies of wavelet theory, i.e. the decomposition, compression and reconstruction, as described in the next section.

5. Compression

Fixed a function $f \in L_{2\pi}^\infty$ and a resolution level $J \in \mathbb{N}$, let us consider the approximation f_J of f at the level J, i.e. the interpolating polynomial:

$$(26)\qquad f_J(x) := \tilde{L}_J f(x) = \sum_{k=0}^{3m_J-1} a_{Jk} \varphi_{Jk}(x),$$

$$(27)\qquad a_{Jk} := f\left(\frac{2k\pi}{3m_J}\right), \qquad k = 0, 1, \ldots, 3m_J - 1,$$

where, as usual, $m_J = 2^{J+1}$. By virtue of (24), we have

$$f_J \in V_J = V_0 \oplus W_0 \oplus W_1 \oplus \ldots . \oplus W_{J-1}.$$

Hence f_J can be decomposed as follows

$$f_J(x) = \sum_{k=0}^{3m_0-1} a_{0k}\varphi_{0k}(x) + \sum_{j=0}^{J-1}\sum_{k=0}^{3m_j-1} b_{jk}\psi_{jk}(x), \tag{28}$$

where the decomposition coefficients $\{a_{0k}\}$ and $\{b_{jk}\}$ in (28), can be calculated easily, starting from the initial coefficients $\{a_{Jk}\}$ in (27), by means of a *decomposition algorithm* that works in a very fast way, by using the FFT algorithm [6].

Once decomposed the initial approximation f_J of f, we can compress it, i.e. fixed an opportune threshold $\varepsilon > 0$, we can put equal to zero those coefficients in (28) whose modulus is less than ε. In this way we obtain other decomposition coefficients:

$$\tilde{a}_{0k} := \begin{cases} 0, & \text{if } |a_{0k}| < \varepsilon, \\ a_{0k}, & \text{otherwise,} \end{cases} \tag{29}$$

for $k = 0, 1, \ldots, m_0$, and

$$\tilde{b}_{jk} := \begin{cases} 0, & \text{if } |b_{jk}| < \varepsilon, \\ b_{jk}, & \text{otherwise,} \end{cases} \tag{30}$$

for $k = 0, 1, \ldots, m_j$, $j = 0, \ldots, J-1$, by means of which we can reconstruct a new approximation of f. Namely,

$$\begin{aligned} \tilde{f}_J^{(\varepsilon)}(x) &:= \sum_{k=0}^{3m_0-1} \tilde{a}_{0k}\varphi_{0k}(x) + \sum_{j=0}^{J-1}\sum_{k=0}^{3m_j-1} \tilde{b}_{jk}\psi_{jk}(x) \\ &= \sum_{k=0}^{3m_J-1} \tilde{a}_{Jk}\varphi_{Jk}(x). \end{aligned} \tag{31}$$

We explicitly note that also to reconstruct $\tilde{f}_J^{(\varepsilon)}(x)$ i.e. to calculate the coefficients $\{\tilde{a}_{Jk}\}_{k=0}^{3m_J-1}$, we can utilize a *reconstruction algorithm* that works recursively in a very simple and fast way. For more details we refer the reader to [6].

Here we are interested to solve the following questions. How can we choose the threshold ε in such a way as the compressed approximation $\tilde{f}_J^{(\varepsilon)}$ results comparable with the initial one f_J? Furthermore, how many can we compress? That is, how many coefficients in (28) can we put equal to zero? In Section 3 (Corollary 1) we have estimated the interpolation error by means of the error of the best uniform approximation:

$$\|f - f_J\|_p \le C E^*_{m_J}(f)_\infty, \qquad 1 \le p \le +\infty.$$

Nevertheless locally, in the "regular parts" of f, the error $|f(x) - f_J(x)|$ is smaller than $E^*_{m_J}(f)_\infty$. Consequently, taken

$$\varepsilon \sim E^*_{m_J}(f)_\infty,$$

if the function is everywhere "regular", we do not expect great results by the compression, but on the contrary, if the function f has some isolated points of "singularity", we expect to neglect a lot of coefficients in (28).

At the moment we are not yet able to prove pointwise estimate for the error $|f(x) - f_J(x)|$ confirming these facts, but several numerical tests give value to this idea. In the meantime, we state the following result:

Theorem 4. *For each $J \in \mathbb{N}_0$, $\varepsilon > 0$ and $f \in C_{2\pi}$, let f_J and $\tilde{f}_J^{(\varepsilon)}$ be given by* (26) *and* (31)*, respectively. Then, for all p, $1 \le p \le +\infty$, it results*

$$\|f_J - \tilde{f}_J^{(\varepsilon)}\|_p \le C J \varepsilon, \tag{32}$$

C being a constant independent of f, J, and ε.

Then, taking into account that

$$\|f - \tilde{f}_J^{(\varepsilon)}\|_p \le \|f - f_J\|_p + \|f_J - \tilde{f}_J^{(\varepsilon)}\|_p,$$

by Theorem 4, easily we can deduce how to fix the value of the threshold ε in such a way as the approximation degree does not change. In particular, choosing the threshold

$$\overline{\varepsilon} \sim \frac{E^*_{m_J}(f)_\infty}{J}, \tag{33}$$

by Theorem 4 and Corollary 1, we obtain the following estimate

$$\|f - \tilde{f}_J^{(\overline{\varepsilon})}\|_p \le C E^*_{m_J}(f)_\infty, \qquad 1 \le p \le +\infty, \tag{34}$$

where C is a constant independent of f and J.

The following tables show some numerical results of the compression, for several resolution level J. Two cases are analyzed. In the first table, we have tested the function $f(x) = |\sin x|$ (for which $E^*_{m_J}(f)_\infty = O(1/m_J)$), while in the second table the $f(x) = |\sin x|^{1/2}$ is considered (in this case $E^*_{m_J}(f)_\infty = O(1/\sqrt{m_J})$). In the first case we have chosen $\varepsilon = (Jm_J)^{-1}$, while in the second case we have taken $\varepsilon = (Jm_J^{1/2})^{-1}$. In these tables we have denoted by N the number of non vanishing coefficients in (31), by N_0 the number of vanishing coefficients in (31) and by P_0 the percentage of vanishing coefficients in (31).

$f(x) = |\sin x|$.				
	nodes	N	N_0	P_0
$J = 3$	48	28	20	41.667%
$J = 5$	192	102	90	46.875%
$J = 7$	768	312	456	59.375%
$J = 8$	1536	586	950	61.849%

$f(x) = |\sin x|^{1/2}$.				
	nodes	N	N_0	P_0
$J = 3$	48	6	42	87.500%
$J = 5$	192	22	170	88.541%
$J = 7$	768	46	722	94.014%
$J = 8$	1536	64	1472	95.833%

These results can be explained with the following heuristic arguments.

We have seen that the sequence $\|f - f_j\|_\infty \to 0$ like the error of the best trigonometric approximation $E^*_{m_j}(f)_\infty$. But if the function f is everywhere regular unless in some isolated point, then $E^*_{m_j}(f)_\infty$ does not take into account the regular parts of f which, instead, should be sufficiently described by a lower resolution level. Then the more "*heterogeneous*" the function to approximate is, the more successful the compression has. On the other side, for very regular function we can obtain a good error already with low resolution levels. Then it is obvious that for such functions we cannot expect great results about the compression (in fact, for example, if $f(x) = e^{\sin x}$ then we eliminate just two decomposition coefficients for all $J = 3, 4, 5, 6, 7, 8$).

6. Proofs

Proof of Theorem 1. The proof is based on the following trigonometric inequality (see [8, p. 228]):

$$\left(\frac{1}{3m}\sum_{k=0}^{3m-1}\left|T\left(\frac{2k\pi}{3m}\right)\right|^p\right)^{1/p} \le \left(1+\frac{2\pi N}{3m}\right)\|T\|_p, \tag{35}$$

which holds for all $T \in \mathcal{T}_N$, $N \in \mathbb{N}$, and $1 \le p \le +\infty$ (for $p = +\infty$ the left hand side denoting $\max_k \left|T\left(\frac{2k\pi}{3m}\right)\right|$). In fact, taking into account the interpolation property of the operator $\tilde{L}_m$ and applying the inequality (35) for $T = \tilde{L}_m f \in \mathcal{T}_{2m-1}$, we have:

$$\begin{aligned}\left(\frac{1}{3m}\sum_{k=0}^{3m-1}|f(t_{m,k})|^p\right)^{1/p} &= \left(\frac{1}{3m}\sum_{k=0}^{3m-1}\left|\tilde{L}_m f(t_{m,k})\right|^p\right)^{1/p}\\ &\le \left(1+\frac{2\pi(2m-1)}{3m}\right)\|\tilde{L}_m f\|_p\\ &\le \left(1+\frac{4\pi}{3}\right)\|\tilde{L}_m f\|_p.\end{aligned}$$

Hence , for $A = 3/(3+4\pi)$, we obtain the first inequality in (14) and (15).

In order to prove the second inequality in (14) and (15), let us start by considering the case $1 \le p < +\infty$. In this case there exists a unique function $g \in L^q_{2\pi}$ with $1/p + 1/q = 1$ and $\|g\|_q = 1$, such that

$$\|\tilde{L}_m f\|_p = \frac{1}{2\pi}\int_0^{2\pi}\tilde{L}_m f(x)g(x)\,dx.$$

Then, by using the Hölder inequality and (35) with $T = V_m f \in \mathcal{T}_{2m-1}$, we have for $1 < p < +\infty$ (the case $p = 1$ being analogous)

$$\begin{aligned}\|\tilde{L}_m f\|_p &= \frac{1}{2\pi}\int_0^{2\pi}\tilde{L}_m f(x)g(x)\,dx = \frac{1}{3m}\sum_{k=0}^{3m-1} f(t_{m,k})V_m g(t_{m,k})\\ &\le \left(\frac{1}{3m}\sum_{k=0}^{3m-1}|V_m g(t_{m,k})|^q\right)^{1/q}\left(\frac{1}{3m}\sum_{k=0}^{3m-1}|f(t_{m,k})|^p\right)^{1/p}\end{aligned}$$

$$\begin{aligned}
&\le \left(1+\frac{2\pi(2m-1)}{3m}\right)\|V_m g\|_q \left(\frac{1}{3m}\sum_{k=0}^{3m-1}|f(t_{m,k})|^p\right)^{1/p} \\
&\le \left(1+\frac{4\pi}{3}\right)\|V_m\|_q \left(\frac{1}{3m}\sum_{k=0}^{3m-1}|f(t_{m,k})|^p\right)^{1/p} \\
&\le \frac{3+4\pi}{3}\|V_m\|_\infty \left(\frac{1}{3m}\sum_{k=0}^{3m-1}|f(t_{m,k})|^p\right)^{1/p}.
\end{aligned}$$

In the case $p=+\infty$, the same result can be proved by proceeding in the following way. For each $x\in[0,2\pi)$, using the inequality (35) with $p=1$ and $T(t)=\sum_{n=m}^{2m-1} D_n(x-t)$, we have

$$\begin{aligned}
|\tilde{L}_m f(x)| &\le \frac{1}{3m^2}\sum_{k=0}^{3m-1}|f(t_{m,k})|\left|\sum_{n=m}^{2m-1}D_n(x-t_{m,k})\right| \\
&\le \frac{\max_k|f(t_{m,k})|}{3m^2}\sum_{k=0}^{3m-1}\left|\sum_{n=m}^{2m-1}D_n(x-t_{m,k})\right| \\
&\le \max_k|f(t_{m,k})|\left(1+\frac{2\pi(2m-1)}{3m}\right)\left(\frac{1}{2\pi m}\int_0^{2\pi}\left|\sum_{n=m}^{2m-1}D_n(x-t)\right|dt\right) \\
&\le \max_k|f(t_{m,k})|\left(1+\frac{4\pi}{3}\right)\left(\frac{1}{2\pi m}\int_0^{2\pi}\left|\sum_{n=m}^{2m-1}D_n(t)\right|dt\right),
\end{aligned}$$

and taking the supremum with respect to $x\in[0,2\pi)$, we obtain:

$$\|\tilde{L}_m f\|_\infty \le \frac{3+4\pi}{3}\left(\frac{1}{2\pi m}\int_0^{2\pi}\left|\sum_{n=m}^{2m-1}D_n(t)\right|dt\right)\max_k|f(t_{m,k})|.$$

But we recall that

$$\begin{aligned}
\|V_m\|_\infty &= \frac{1}{2\pi m}\int_0^{2\pi}\left|\sum_{n=m}^{2m-1}D_n(t)\right|dt \\
&\le \frac{1}{2\pi m}\left[\int_0^{2\pi}\left|\sum_{n=0}^{2m-1}D_n(t)\right|dt+\int_0^{2\pi}\left|\sum_{n=0}^{m-1}D_n(t)\right|dt\right]
\end{aligned}$$

$$= \frac{1}{2\pi m}\left[\int_0^{2\pi}\left(\frac{\sin\frac{2mt}{2}}{\sin\frac{t}{2}}\right)^2 dt + \int_0^{2\pi}\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^2 dt\right],$$

where we used the identity

$$\sum_{k=0}^{m-1} D_k(t) = \left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^2 \qquad (m \in \mathbb{N}).$$

Then, since

$$\frac{1}{2\pi m}\int_0^{2\pi}\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^2 dt = 1 \qquad (m \in \mathbb{N}),$$

we can conclude

$$\|V_m\|_\infty \le 2 + 1 = 3 \qquad (m \in \mathbb{N}).$$

Hence, in both the cases $1 \le p < +\infty$ and $p = +\infty$, we obtain

$$\|\tilde{L}_m f\|_p \le \frac{3+4\pi}{3}\|V_m\|_\infty \begin{cases} \left[\frac{1}{3m}\sum_{k=0}^{3m-1} |f(t_{m,k})|^p\right]^{1/p}, & 1 \le p < +\infty, \\ \max_k |f(t_{m,k})|, & p = +\infty, \end{cases}$$

$$\le (3+4\pi) \begin{cases} \left[\frac{1}{3m}\sum_{k=0}^{3m-1} |f(t_{m,k})|^p\right]^{1/p}, & 1 \le p < +\infty, \\ \max_k |f(t_{m,k})|, & p = +\infty. \end{cases} \qquad \square$$

Proof of the Corollary 1. It is sufficient to consider the case $p = +\infty$.

Let $f \in C_{2\pi}$ and $m \in \mathbb{N}$. Since obviously $\max_k |f(t_{m,k})| \le \|f\|_\infty$, by (15) we obtain

$$\|\tilde{L}_m f\|_\infty \le (3+4\pi)\|f\|_\infty. \tag{36}$$

Then the inequality (16) follows trivially by (36), recalling that $\tilde{L}_m$ is a *quasi-projector* on $\mathcal{T}_m$, i.e.,

$$(\forall T \in \mathcal{T}_m) \quad \tilde{L}_m T = T. \tag{37}$$

In fact, for each $T \in \mathcal{T}_m$ we have

$$\begin{aligned}\|\tilde{L}_m f - f\|_\infty &\le \|\tilde{L}_m f - T\|_\infty + \|T - f\|_\infty \\ &= \|\tilde{L}_m (f - T)\|_\infty + \|T - f\|_\infty \\ &\le (4 + 4\pi)\|f - T\|_\infty,\end{aligned}$$

which gives (16) by taking the infimum with respect to $T \in \mathcal{T}_m$. $\square$

Proof of Theorem 2. Let be $m \in \mathbb{N}$, $1 \le p \le +\infty$ and $f \in C_{2\pi}$ such that

$$\int_0^1 \frac{\omega_k(f,t)_p}{t^{1+1/p}}\, dt < +\infty.$$

To prove (17) we use the following inequalities (see [5, p. 18 and p. 169])

$$\tilde{E}^*_m(f)_p \le C\tau_k\left(f; \frac{1}{m}\right)_p \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\omega_k(f,t)_p}{t^{1+1/p}}\, dt, \tag{38}$$

where C is a constant independent of f and m, $\tilde{E}^*_n(f)_p$ is the *error of the best one-sided trigonometric approximation* of f in the $L^p_{2\pi}$-norm, namely

$$\tilde{E}^*_n(f)_p := \inf\Big\{\|T^+ - T^-\|_p :\ T^\pm \in \mathcal{T}_n,\ T^- \le f \le T^+\Big\},$$

and $\tau_k(f,t)_p$ denotes the τ-modulus of continuity, defined as

$$\tau_k(f,t)_p := \|\omega_k(f,\cdot,t)\|_p\,,$$

with

$$\omega_k(f,x,t) := \sup\Big\{|\Delta^k_h f(s)| : s, s+kh \in [0,2\pi] \cap [x - kt/2, x + kt/2]\Big\}.$$

More precisely, by (37) and Theorem 1, for all $T^\pm \in \mathcal{T}_m$ such that $T^- \le f \le T^+$, we have

$$\begin{aligned}\|f - \tilde{L}_m f\|_p &\le \|f - T^-\|_p + \|\tilde{L}_m f - T^-\|_p \\ &\le \|T^+ - T^-\|_p + \|\tilde{L}_m (f - T^-)\|_p \\ &\le \|T^+ - T^-\|_p + C\left(\frac{1}{3m}\sum_{k=0}^{3m-1} |(f - T^-)(t_{m,k})|^p\right)^{1/p} \\ &\le \|T^+ - T^-\|_p + C\left(\frac{1}{3m}\sum_{k=0}^{3m-1} |(T^+ - T^-)(t_{m,k})|^p\right)^{1/p} \\ &\le \|T^+ - T^-\|_p + C\|\tilde{L}_m (T^+ - T^-)\|_p \\ &= C\|T^+ - T^-\|_p.\end{aligned}$$

Thus, taking the infimum with respect to $T^{\pm} \in \mathcal{T}_m$ such that $T^- \le f \le T^+$ and using (38), we get

$$\|f - \tilde{L}_m f\|_p \le C\tilde{E}^*_m(f)_p \le C\tau_k\left(f; \frac{1}{m}\right)_p \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\omega_k(f,t)_p}{t^{1+1/p}}\, dt. \quad \square$$

Proof of Corollary 2. Let be $1 \le p, q \le +\infty$, $r > 1/p$, $f \in C_{2\pi} \cap B^p_{r,q}$, and $m \in \mathbb{N}$, arbitrarily fixed. Let us examine the case $1 \le q < +\infty$ (the case $q = +\infty$ being the same). If $q' = (q-1)/q$ is the conjugate exponent of q, then by (17) and Hölder inequality, we obtain

$$\begin{aligned}
\|f - \tilde{L}_m f\|_p &\le \frac{C}{m^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\omega_k(f;t)_p}{t^{1+\frac{1}{p}}} dt = \frac{C}{m^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\omega_k(f;t)_p}{t^{r+\frac{1}{q}}} [t^{-\frac{1}{q'}+r-\frac{1}{p}}] dt \\
&\le \frac{C}{m^{\frac{1}{p}}} \left(\int_0^{\frac{1}{m}} t^{[-\frac{1}{q'}+r-\frac{1}{p}]q'} dt \right)^{\frac{1}{q'}} \left(\int_0^{\frac{1}{m}} \left[\frac{\omega_k(f;t)_p}{t^{r+\frac{1}{q}}} \right]^q dt \right)^{\frac{1}{q}} \\
&= \frac{C}{m^{\frac{1}{p}}} \frac{1}{m^{r-\frac{1}{p}}} \|f\|_{p,q,r} \le \frac{C}{m^r} \|f\|_{B^p_{r,q}}. \quad \square
\end{aligned}$$

Proof of Theorem 3. Let be $m \in \mathbb{N}$, $1 \le p, q \le +\infty$, $s \ge r \ge 0$ with $s > 1/p$ and $f \in C_{2\pi} \cap B^p_{s,q}$, arbitrarily fixed.

First of all we observe that to demonstrate the theorem, it is enough to prove the following estimate

$$\|f - \tilde{L}_m\|_{E^p_{r,q}} \le \frac{C}{m^{s-r}} \|f\|_{E^p_{s,q}}. \tag{39}$$

In fact on the one hand, (39) implies directly (19) by virtue of the norm-equivalence (10), and, on the other hand, (20) trivially follows from (19) by taking $s = r$. Limit ourself to the case $q \ne +\infty$, the case $q = +\infty$ being analogous. We note that $\forall T \in \mathcal{T}_{m-1}$, we have

$$E^*_k(f-T)_p \begin{cases} \le \|f - T\|_p & \text{if } k < m, \\ = E^*_k(f)_p & \text{if } k \ge m. \end{cases}$$

Hence for $T = \tilde{L}_m f \in \mathcal{T}_{2m-1}$, we obtain

$$\|f - \tilde{L}_m f\|_{E^p_{r,q}} = \|f - \tilde{L}_m f\|_p + \left(\sum_{k=0}^{+\infty} \left[(1+k)^{r-\frac{1}{q}} E^*_k(f - \tilde{L}_m f)_p \right]^q \right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\le \|f-\tilde L_m f\|_p \\
&\quad + C\left(\sum_{k<2m}\left[(1+k)^{r-\frac1q}E_k^*(f-\tilde L_m f)_p\right]^q\right)^{\frac1q} \\
&\quad + C\left(\sum_{k\ge 2m}\left[(1+k)^{r-\frac1q}E_k^*(f-\tilde L_m f)_p\right]^q\right)^{\frac1q} \\
&\le \|f-\tilde L_m f\|_p + C\|f-\tilde L_m f\|_p\left(\sum_{k<2m}(1+k)^{rq-1}\right)^{\frac1q} \\
&\quad + C\left(\sum_{k\ge 2m}\left[(1+k)^{r-\frac1q}E_k^*(f)_p\right]^q\right)^{\frac1q} \\
&\le C\left[m^r\|f-\tilde L_m f\|_p + \left(\sum_{k\ge 2m}\left[(1+k)^{r-\frac1q}E_k^*(f)_p\right]^q\right)^{\frac1q}\right].
\end{aligned}
$$

Then (39) follows immediately by Corollary 2 (recall that $s>1/p$) and by the equivalence $\|f\|_{B^p_{r,q}}\sim\|f\|_{E^p_{r,q}}$. In fact, we have

$$
\begin{aligned}
\|f-\tilde L_m f\|_{E^p_{r,q}} &\le C\left[m^r\|f-\tilde L_m f\|_p + \left(\sum_{k\ge 2m}(1+k)^{qr-1}E_k^*(f)_p^q\right)^{\frac1q}\right] \\
&\le C\left[m^r\frac{\|f\|_{B^p_{s,q}}}{m^s} + \left(\frac{1}{m^{q(s-r)}}\sum_{k\ge 2m}(1+k)^{qs-1}E_k^*(f)_p^q\right)^{\frac1q}\right] \\
&\le C\left[\frac{\|f\|_{B^p_{s,q}}}{m^{s-r}} + \frac{1}{m^{s-r}}\|f\|_{E^p_{s,q}}\right] \le C\,\frac{\|f\|_{E^p_{s,q}}}{m^{s-r}}. \quad \square
\end{aligned}
$$

Proof of Theorem 4. Let be $1\le p\le+\infty$, $f\in C_{2\pi}$, $J\in\mathbb{N}_0$ and $\varepsilon>0$ arbitrarily fixed. Let us write the initial approximation of f at the level J (26), in the decomposition form

$$
f_J(x)=\sum_{k=0}^{3m_0-1}a_{0k}\varphi_{0k}(x)+\sum_{j=0}^{J-1}\sum_{k=0}^{3m_j-1}b_{jk}\psi_{jk}(x). \tag{40}
$$

Then let us consider the corresponding compressed approximation of f

$$
\tilde f_J^{(\varepsilon)}(x)=\sum_{k=0}^{3m_0-1}\tilde a_{0k}\varphi_{0k}(x)+\sum_{j=0}^{J-1}\sum_{k=0}^{3m_j-1}\tilde b_{jk}\psi_{jk}(x), \tag{41}
$$

with the coefficients $\{\tilde{a}_{0,k}\}$ and $\{\tilde{b}_{j,k}\}$, respectively given by (29) and (30).

If we set

$$\alpha_{0,k} := a_{0,k} - \tilde{a}_{0,k}; \quad k = 0, 1, \ldots, 3m_0,$$

$$\beta_{j,k} := b_{j,k} - \tilde{b}_{j,k}; \quad k = 0, 1, \ldots, 3m_j; \quad j = 0, 1, \ldots, J-1,$$

by (29) and (30), obviously we deduce that

$$\begin{aligned} &|\alpha_{0,k}| < \varepsilon, \qquad k = 0, 1, \ldots, 3m_0, \\ &|\beta_{j,k}| < \varepsilon, \qquad k = 0, 1, \ldots, 3m_j; \quad j = 0, 1, \ldots, J-1. \end{aligned}$$

Consequently, for all $p : 1 \le p \le +\infty$, it results

$$\|\{\alpha_{0,k}\}\|_{l^p} \le \varepsilon, \quad \|\{\beta_{j,k}\}\|_{l^p} \le \varepsilon, \quad j = 0, 1, \ldots, J-1, \tag{42}$$

where we have denoted by $\|\cdot\|_{\ell^p}$ the vectorial norm

$$\|\{x_k\}\|_{l^p} := \begin{cases} \left(\dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} |x_k|^p \right)^{1/p} & 1 \le p < +\infty, \\ \max\limits_k |x_k| & p = +\infty, \end{cases}$$

where $\{x_k\}_{k=0}^{N-1} \in \mathbb{C}^N$. Thus, subtracting (41) from (40) and taking the p-norm, by (22), (25) and (42), we get

$$\begin{aligned} \|f_J - \tilde{f}_J^{(\varepsilon)}\|_p &\le \left\| \sum_{k=0}^{3m_0-1} \alpha_{0,k} \varphi_{0k} \right\|_p + \sum_{j=0}^{J-1} \left\| \sum_{k=0}^{3m_j-1} \beta_{j,k} \psi_{jk} \right\|_p \\ &\le C_2 \, \|\{\alpha_{0,k}\}\|_{\ell^p} + C_2' \sum_{j=0}^{J-1} \|\{\beta_{j,k}\}\|_{\ell^p} \\ &\le \varepsilon(C_2 + C_2' J) \le CJ\varepsilon. \quad \square \end{aligned}$$

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