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ON A SEQUENCE OF LINEAR AND POSITIVE OPERATORS

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Abstract. The purpose of this paper is to study a sequence of linear and positive operators which was proposed by A. Lupaş [4]. An asymptotic formula and some quantitative estimates for the rate of convergence are given. By using a probabilistic method, this sequence is reobtained. Also two modified sequences are constructed.

1. Introduction

At the International Dortmund Meeting held in Witten (Germany, March, 1995), A. Lupaş [4] formulated the following problem.

"Starting with the identity

$$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1, \tag{1}$$

let $\alpha = nx,\ x \geq 0$, and consider the linear positive operators

$$(L_n f)(x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right), \quad x \geq 0,$$

with $f : [0, \infty) \to \mathbb{R}$. If we impose that $L_n e_1 = e_1$ we find $a = 1/2$. Therefore

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{2}$$

This L_n-operator has a form very similar with Szász-Mirakyan operators. We have $L_n h = h,\ h \in \Pi_1$ and $\lim\limits_{n\to\infty} (L_n e_2)(x) = e_2(x)$.

Find other properties of $L_n f$."

The focus of this note is to investigate these operators.

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2. Approximation Properties

Firstly, we recall the common notation

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1), \quad k \geq 1.$$

The symbol Π_n stands for the linear space of polynomials with real coefficients of degree $\leq n$. For any real $x \geq 0$ and integer $r \geq 0$ we set

$$e_r(x) := x^r, \quad \psi_{x,r}(t) := (t-x)^r \quad (t \geq 0), \quad \mu_{n,r}(x) := (L_n\psi_{x,r})(x).$$

At this point, it has been proved that

$$L_n e_r = e_r, \quad r \in \{0,1\}. \tag{3}$$

Remark 1. We can consider that $L_n,\ n \geq 1$, are defined on E where

$$E = \bigcup_{a>0} E_a$$

and E_a is the subspace of all real valued continuous functions f on $[0,\infty)$ such as $e(f;a) := \sup\limits_{x\geq 0}(\exp(-ax)|f(x)|) < \infty$. The space E_a is endowed with the norm $\|f\|_a = e(f;a)$ with respect to which it becomes a Banach lattice.

Remark 2. In our investigations we also need to consider the Banach lattice $C_B[0,\infty)$ of all real-valued bounded continuous functions on $[0,\infty)$ endowed with the sup-norm $\|\cdot\|_\infty$. The operator L_n maps $C_B[0,\infty)$ into itself, it is continuous with respect to the sup-norm and $\|L_n\| = \|L_n e_0\|_\infty = 1$.

Lemma 1. *If L_n is defined by* (2) *then, for each $x \geq 0$, the following identities are valid*

$$(L_n e_2)(x) = x^2 + \frac{2x}{n}, \tag{4}$$

$$\mu_{n,2}(x) = \frac{2x}{n}. \tag{5}$$

Proof. Taking into account the recurrence relation $(\alpha)_k = \alpha(\alpha+1)_{k-1}$, $k \geq 1$, we can write successively:

$$\begin{aligned}(L_n e_2)(x) &= 2^{-nx}x\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^k(k-1)!}\cdot\frac{k}{n}\\ &= 2^{-nx-1}x\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\cdot\frac{j}{n} + 2^{-nx-1}(x/n)\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\\ &= 2^{-nx-2}x\sum_{k=0}^{\infty}\frac{(nx+1)(nx+2)_k}{2^k k!} + \frac{x}{n} = x^2 + \frac{2x}{n}\,.\end{aligned}$$

We have also used (1) where $a = 1/2$ and $\alpha = nx + 1$ respectively $\alpha = nx + 2$. Since the operator L_n is linear, the second statement of our lemma follows as a consequence of (3) and (4).

Let φ be the function defined on $[0, \infty)$ by

$$\varphi(x) := \sqrt{2x}. \tag{6}$$

Actually, φ represents the step weight function of the Lupaş operators and it controls their rate of convergence as follows.

At this point, we fix $b > 0$ and consider the lattice homomorphism $H_b : C[0, \infty) \to C[0, b]$ defined by $H_b(f) = f|_{[0,b]}$. It is clear that $H_b(L_n e_i) \to H_b(e_i)$ uniformly on $[0, b]$, where $i \in \{0, 1, 2\}$. Hence, the well known Korovkin theorem implies the following result.

Theorem 1. *If L_n is defined by* (2) *then one has*

$$\lim_{n\to\infty} L_n f = f \ \textit{uniformly on } [0, b],$$

for any $b > 0$.

Next, we are interested in some quantitative estimates for the rate of the convergence.

We shall give estimates concerning the pointwise convergence in terms of the usual first and second moduli of smoothness of a function g, which are defined by

$$\omega_k(g; \delta) := \sup\{|\Delta_h^k g(x)| : \ |h| \le \delta, \ x, x + kh \in I\}, \quad \delta > 0,$$

where $k \in \{1, 2\}$, $g : I \to \mathbb{R}$ is a bounded real function, $\Delta_h^1 g(x) = g(x+h) - g(x)$ and $\Delta_h^2 g(x) = \Delta_h^1 g(x + h) - \Delta_h^1 g(x)$.

Theorem 2. *Let L_n be defined by* (2) *and $b > 0$. One has*

$$|(L_n f)(x) - f(x)| \le \left(1 + \sqrt{2b}\right) \omega_1 \left(f; \frac{1}{\sqrt{n}}\right), \quad x \in [0, b]. \tag{7}$$

If f has a continuous derivative on $[0, b]$ then

$$|(L_n f)(x) - f(x)| \le \frac{2b + \sqrt{2b}}{\sqrt{n}} \omega_1 \left(f'; \frac{1}{\sqrt{n}}\right), \quad x \in [0, b]. \tag{8}$$

Proof. For every x, y belonging to $[0, b]$ and $\delta > 0$ we obviously have

$$|f(x) - f(y)| \leq (1 + \delta^{-1}|x - y|)\omega_1(f; \delta). \tag{9}$$

By using the preceding inequality as well as the Schwarz inequality we obtain

$$|(L_n f)(x) - f(x)| \leq (1 + \delta(L_n|\psi_{x,1}|)(x))\omega_1(f; \delta) \leq (1 + \delta\mu_{n,2}^{1/2}(x))\omega_1(f; \delta).$$

Choosing $\delta = 1/\sqrt{n}$ and taking identity (5) into account we arrive at (7).

In what follows we assume that f possesses a continuous derivative on $[0, b]$. For every x, y belonging to $[0, b]$ by the mean value theorem we get $f(x) - f(y) = (x - y)f'(x) + (x - y)(f'(\xi) - f'(x))$, where ξ lies between x and y. Using (9) for f' and following an argument similar to the one used in the preceding proof, we obtain

$$\begin{aligned}|(L_n f)(x) - f(x)| &\leq \{(L_n|\psi_{x,1}|)(x) + \frac{1}{\delta}(L_n|\psi_{x,2}|)(x)\}\omega_1(f'; \delta) \\ &\leq \mu_{n,2}^{1/2}(x)(1 + \delta^{-1}\mu_{n,2}^{1/2}(x))\omega_1(f'; \delta)\,.\end{aligned}$$

Since $x \in [0, b]$ and $\delta = 1/\sqrt{n}$, lemma 1 implies (8).

In the following we are going to prove another estimate by involving the second order modulus of smoothness. In fact, our estimate will be based upon a more general theorem which is due to Gonska ([3], theorem 4.1, page 331). So, if we consider the identities (3), (4) and the above mentioned result of Gonska, we can state

Theorem 3. *Let L_n be defined by* (2) *and $b > 0$. The following inequality*

$$|(L_n f)(x) - f(x)| \leq \left(3 + 2b \max\left(1, \frac{b}{n}\right)\right)\omega_2\left(f; \frac{1}{\sqrt{n}}\right), \quad x \in [0, b],$$

holds.

In the final part of this section we establish a Voronovskaja-type formula.

Theorem 4. *Let $f \in C[0, \infty)$ be twice differentiable at some point $x > 0$ and let us assume that $f(t) = \mathbf{O}(t^2)$ as $t \to \infty$. If the operators L_n are defined by* (2) *then*

$$\lim_{n \to \infty} n((L_n f)(x) - f(x)) = \frac{\varphi^2(x)}{2} f''(x), \tag{10}$$

holds, where φ *is defined by* (6).

Proof. In order to prove this identity we use Taylor's expansion

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left(\frac{1}{2} f''(x) + \varepsilon\left(\frac{k}{n} - x\right)\right),$$

where ε is bounded and $\lim_{t\to 0} \varepsilon(t) = 0$. By applying the operator L_n to the above relation we obtain

$$(L_n f)(x) - f(x) = \mu_{n,1}(x) f'(x) + \frac{1}{2}\mu_{n,2}(x) f''(x) + (L_n s_x)(x), \tag{11}$$

where $s_x(t) = \psi_{x,1}^2(t)\varepsilon(t-x)$ and clearly $\mu_{n,1}(x) = 0$. Recalling the Cauchy-Schwarz inequality and (5) we can infer

$$(L_n s_x)(x) \le (L_n \varepsilon^2 \psi_{x,1}^2)(x)(L_n \psi_{x,1}^2)(x) \le \|\varepsilon^2\|_\infty \mu_{n,2}^2(x) = 4\|\varepsilon^2\|_\infty \frac{x^2}{n^2}.$$

Thus, $\lim_{n\to\infty} n(L_n s_x)(x) = 0$ holds, and therefore we conclude that (11) and (5) lead us to the asymptotic formula (10).

3. A Probabilistic Investigation

It is known that by using some concepts of the probability theory have been obtained several classical positive and linear operators. Pioneers in this field to be mentioned here are W. Feller [2] and D.D. Stancu [6].

Let $(X_{j,x})_{j\ge 1}$ be a sequence of independent random variables identically distributed

$$P(X_{j,x} = k) = 2^{-x-k}\frac{(x)_k}{k!}, \quad k \ge 0, \tag{12}$$

where x is a positive real parameter. Denoting by θ the common characteristic function of these random variables, the identity (1) implies

$$\theta(t) = \sum_{k=0}^{\infty} e^{itk} P(X_{j,x} = k) = (2 - e^{it})^{-x}.$$

If we set $Y_{n,x} := \frac{1}{n}\sum_{j=1}^{n} X_j,\ n \ge 1$, then the characteristic function of $Y_{n,x}$ will be $\phi_n(t) = \theta^n(t/n)$ which corresponds to the following distribution $P(Y_{n,x} = k/n) = l_{n,k}(x)$ where

$$l_{n,k}(x) := 2^{-nx}\frac{(nx)_k}{2^k k!}, \quad k \ge 0. \tag{13}$$

Furthermore, for every $n \geq 1$ and every $f \in E$ we consider the function $L_n f : [0, \infty) \to \mathbb{R}$ defined by $(L_n f)(x) := M(f \circ Y_{n,x})$ where $M(Z)$ represents the mathematical expectation of Z. This way we obtain the Lupaş operators.

As a matter of fact, all those approximation processes $(P_n)_{n\geq 1}$ of probabilistic type which are associated with a random scheme

$$Z_{n,x} = \frac{1}{n} \sum_{k=1}^{n} X_{k,x} \quad (n \geq 1,\ x \in I,\ X_{k,x} \text{ i.i.d.})$$

satisfy the formula $\lim\limits_{n\to\infty} n((P_n f)(x) - f(x)) = \frac{\sigma^2(x)}{2} f''(x)$ for every $f \in C_B^2(I)$, see [1, page 368]. Here $\sigma^2(x) = \mathrm{Var}(X_{k,x})$ represents the variance of $X_{k,x}$. For the variables $X_{n,k}$ defined by (12), after a few calculations, we obtain $M(X_{k,x}) = x,\ M(X_{k,x}^2) = x^2 + 2x,\ \mathrm{Var}(X_{k,x}) = 2x$. So, in the particular case when $f \in C_B^2[0, \infty)$ we come across (10).

The next step is to present some properties of $l_{n,k}(x)$.

Theorem 5. *If $n \geq 1,\ k \geq 0,\ x \in (0, \infty)$ and $l_{n,k}(x)$ is defined by* (13) *then the following relations hold true:*

i) $l_{n,k+1}(x) = \frac{nx+k}{2(k+1)} l_{n,k}(x)$,

ii) $l'_{n,k}(x) = n l_{n,k}(x) \left(\sum\limits_{i=0}^{k-1} (nx+i)^{-1} - \log 2 \right)$ $(k \neq 0)$,

iii) $\int_0^\infty l_{n,k}(x) = (n 2^k k!)^{-1} \sum\limits_{i=0}^{k} (-1)^{k-i} s_{k,i} i! (\log 2)^{-i-1}$,

iv) $l_{n,k}(x) < 4(2x^2 + 3x + 2)/\sqrt{nx}$.

Here $s_{k,i}$ represents the Stirling numbers of the first kind.

Proof. The first two identities can be obtained by an easy computation, so we omit them.

For the third estimate we recall $(x)_k = \sum\limits_{i=0}^{k} (-1)^{k-i} s_{k,i} x^i$, where $s_{k,i}$ are the Stirling numbers of the first kind. Also, we need the identity

$$\int_0^\infty \frac{x^i}{2^x} dx = \frac{i!}{(\log 2)^{i+1}}, \quad i \geq 0.$$

By using these relations the result follows.

In order to prove the last inequality we resort to a probabilistic way. By virtue of Berry-Essen theorem [5, page 286] there is an absolute constant

$0 < C_1 < 1.33$ such that

$$\sup_x \left| P\left(\sum_{k=1}^{n} X_{k,x} - n\mu < \sigma\sqrt{n}x\right) - (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}dy \right| < \frac{C_1}{\sqrt{n}}\left(\frac{\rho}{\sigma}\right)^3 \tag{14}$$

where $\mu = M(X_{k,x})$, $\sigma^2 = \mathrm{Var}(X_{k,x})$, $\rho^3 = M(|X_{k,x} - \mu|^3)$. For our variables $X_{k,x}$ we already know that $\mu = x$, $\sigma^2 = 2x$ and $M(X_{k,x}^2) = x^2 + 2x$. Also, by using a standard computation method, we obtain $M(X_{k,x}^3) = x(x^2 + 6x + 6)$. It follows that

$$\rho^3 \leq M(X_{k,x}^3) + 3xM(X_{k,x}^2) + 3x^2 M(X_{k,x}) + x^3 = 2x(4x^2 + 6x + 3).$$

So, we have

$$\begin{aligned} l_{n,k}(x) &= P\left(k - 1 < \sum_{k=1}^{n} X_{k,x} \leq k\right) \\ &= P\left(\frac{k-1-nx}{\sqrt{2nx}} < \frac{\sum_{k=1}^{n} X_{k,x} - nx}{\sqrt{2nx}} \leq \frac{k-nx}{\sqrt{2nx}}\right). \end{aligned}$$

By using (14) we obtain

$$\begin{aligned} l_{n,k}(x) &< \frac{1}{\sqrt{2\pi}}\int_{(k-1-nx)/\sqrt{2nx}}^{(k-nx)/\sqrt{2nx}} \exp(-t^2/2)dt + \frac{2C_1}{\sqrt{2nx}}(4x^2 + 6x + 3) \\ &< \frac{1}{2\sqrt{\pi nx}} + \frac{2}{\sqrt{nx}}(4x^2 + 6x + 3), \end{aligned}$$

which implies the claimed result.

4. Extensions

In order to obtain an approximation process in spaces of integrable functions, we introduce two integral modifications of these operators, Kantorovich-type operators

$$(K_n f)(x) = n\sum_{k=0}^{\infty} l_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt,$$

respectively Durrmeyer-type operators

$$(D_n f)(x) = \sum_{k=0}^{\infty} c_{n,k} l_{n,k}(x) \int_0^{\infty} l_{n,k}(u) f(u) du.$$

The coefficients $c_{n,k}$ are defined as follows $c_{n,k}^{-1} = \int_0^{\infty} l_{n,k}(x)dx$. In fact, this guarantees the relation $D_n e_0 = e_0$. Also, we easily obtain

$$(K_n e_0)(x) = 1, \quad (K_n e_1)(x) = x + \frac{1}{n}, \quad (K_n e_2)(x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}.$$

As regards these integral operators we raise the problem to investigate their convergence in L_p-spaces.

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