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# ASYMPTOTICALLY PERIODIC SOLUTION OF SOME LINEAR DIFFERENCE EQUATIONS

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**Abstract.** For the linear difference equation

$$(E) \qquad a_n^r x_{n+r} + \cdots + a_n^1 x_{n+1} + a_n^0 x_n = d_n, \quad n \in \mathbb{N},$$

sufficient conditions for the existence of an asymptotically periodic solutions are given.

In the paper by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_0$ we denote the set of positive integers, real numbers, and nonnegative real numbers respectively.

**Definition.** The sequence $v : \mathbb{N} \to \mathbb{R}$ is periodic ($\sigma$-periodic) if $v_{n+\sigma} = v_n$ for all $n \in \mathbb{N}$. The sequence $v : \mathbb{N} \to \mathbb{R}$ is asymptotically periodic (asymptotically $\sigma$-periodic) if there exist two sequences $u, w : \mathbb{N} \to \mathbb{R}$ such that $u$ is periodic ($\sigma$-periodic), $\lim\limits_{n\to\infty} w_n = 0$, and $v_n = u_n + w_n$ for all $n \in \mathbb{N}$.

We study the equation $(E)$ when one of the coefficient $a^i$ is periodic or constant and the rest asymptotically approach zero. In the first theorem we give negative answer to the question: does the equation $(E)$ possess periodic solution if all coefficients approach zero. Therefore looking for equations with periodic solutions we should turn our attention to these equations for which at least one coefficient $a^i$ satisfies $\lim\limits_{n\to\infty} a_n^i \neq 0$ (or this limit does not exist). The second theorem gives answer why considering equations (with only one $a^i$ satisfying above mentioned conditions) our thesis can not have the form "all solutions are periodic".

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**Theorem 1.** *Let* $\lim_{n\to\infty} a_n^i = 0$ *for all* $i \in \{0, 1, \dots, r\}$, *and the sequence* $\{d_n\}_{n=1}^{\infty}$ *possesses bounded away from zero some infinite subsequence* $\{d_{n_k}\}_{k=1}^{\infty}$ *then* $(E)$ *has no asymptotically periodic solution.*

*Proof.* Let $\{d_{n_k}\}_{k=1}^{\infty}$ be this subsequence of $\{d_n\}_{n=1}^{\infty}$ such that $|d_{n_k}| > \delta$ for some positive $\delta$ and all $k \in \mathbb{N}$. Suppose that $(E)$ possesses asymptotically periodic solution $x = \{x_n\}_{n=1}^{\infty}$. This yields that there exists some constant $D$ such that $|x_n| < D$ for all $n \in \mathbb{N}$. Furthermore from $\lim_{n\to\infty} a_n^i = 0$ it follows that for arbitrary $\varepsilon > 0$ we can find integer $n(\varepsilon)$ such that

$$\sum_{i=0}^{r} |a_n^i| < \varepsilon \quad \text{for all} \quad n \geq n(\varepsilon). \tag{1}$$

Let us take $\varepsilon = \delta/D$ and any $n \geq n(\varepsilon)$. Then from $(E)$ and (1) we get

$$\begin{aligned}
|d_n| &= |a_n^r x_{n+r} + \dots + a_n^1 x_{n+1} + a_n^0 x_n| \\
&< |a_n^r||x_{n+r}| + \dots + |a_n^1||x_{n+1}| + |a_n^0||x_n| \\
&< D(|a_n^r| + \dots + |a_n^1| + |a_n^0|) < D\varepsilon = \delta \quad \text{for all} \quad n \geq n(\varepsilon).
\end{aligned}$$

On the other hand there exists $n_k \geq n(\varepsilon)$ such that $|d_{n_k}| > \delta$. The obtained contradiction completes the proof.

**Remark 1.** Theorem 1 yields that if $\{d_n\}_{n=1}^{\infty}$ is periodic (asymptotically periodic) sequence different (asymptotically) from zero then a necessary condition for $(E)$ to possess periodic (asymptotically periodic) solution is: for at least one sequence of coefficients $\{a_n^i\}_{n=1}^{\infty}$ there should be $\lim_{n\to\infty} a_n^i \neq 0$.

**Example 1.** The Theorem 1 (Remark 1) does not hold if we have homogeneous equation

$$a_n^r x_{n+r} + \dots + a_n^1 x_{n+1} + a_n^0 x_n = 0. \tag{$E1$}$$

Notice that if

$$\sum_{i=0}^{r} a_n^i = 0 \quad \text{for all} \quad n \in \mathbb{N}, \tag{2}$$

then, for arbitrary constant $C$, homogeneous equation $(E1)$ possesses 2-periodic solutions $\{C(-1)^n\}_{n=1}^{\infty}$ . We can easily find the equation for which condition (2) is fulfilled and $\lim_{n\to\infty} a_n^i = 0$ for all $i \in \{0, 1, \dots, r\}$.

In the next theorem we present one necessary condition for all solutions of the equation $(E1)$ be $\sigma$-periodic.

**Theorem 2.** *Let $a^i : \mathbb{N} \to \mathbb{R}$, $i \in \{0, 1, \dots, r\}$ with $a_n^r \neq 0$ for all $n \in \mathbb{N}$. The necessary condition for all solutions of $(E1)$ to be $\sigma$-periodic is: the sequences $\{a_n^i/a_n^r\}_{n=1}^{\infty}$, $i \in \{0, 1, \dots, r-1\}$ are $\sigma$-periodic.*

*Proof.* Let $\{u_{i,n}\}_{n=1}^{\infty}$, $i \in \{1, \dots, r\}$ be linearly independent solutions of $(E1)$. Every solution $\{u_n\}_{n=1}^{\infty}$ of $(E1)$ can be written in the form

$$u_n = C_1 u_{1,n} + C_2 u_{2,n} + \dots + C_r u_{r,n}, \quad n \in \mathbb{N}, \tag{3}$$

for some constants $C_1, C_2, \dots, C_r$. From (3) it follows that $\sigma$-periodicity of the solutions $\{u_{i,n}\}_{n=1}^{\infty}$, $i \in \{1, \dots, r\}$ yields $\sigma$-periodicity of any other solution of $(E1)$. Therefore necessary conditions for $\{u_{i,n}\}_{n=1}^{\infty}$ to be $\sigma$-periodic are necessary conditions for all solutions to be $\sigma$-periodic. Suppose that $\{u_{i,n}\}_{n=1}^{\infty}$ form $r$ independent $\sigma$-periodic solutions of $(E1)$ and denote suitable Casorati matrix by

$$W_n = \begin{bmatrix} u_{1,n} & u_{2,n} & \cdots & u_{r,n} \\ u_{1,n+1} & u_{2,n+1} & & u_{r,n+1} \\ \vdots & & & \\ u_{1,n+r-1} & u_{2,n+r-1} & & u_{r,n+r-1} \end{bmatrix}$$

and Casoratian

$$w_n = \det(W_n), \quad n \in \mathbb{N}.$$

Notice (see e.g. [4]) that $w_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore $\sigma$-periodicity of $u_i$ yields $\sigma$-periodicity of the sequence $\{w_n\}_{n=1}^{\infty}$. Because $w$ is never vanishing, $\sigma$-periodic solution of the equation

$$w_{n+1} = (-1)^r \frac{a_n^0}{a_n^r} w_n, \quad n \in \mathbb{N}, \tag{4}$$

therefore (see [1]) $\left\{a_n^0/a_n^r\right\}_{n=1}^{\infty}$ is $\sigma$-periodic.

Because $w_n \neq 0$ for all $n \in \mathbb{N}$ then for each $n \in \mathbb{N}$ there exist minors $w_{(i,\mathbf{j}(n)),n}$ of order $i$ $(i = 1, \dots, r-1)$,

$$w_{(i,\mathbf{j}(n)),n} = \det \begin{bmatrix} u_{j_1(n),n} & u_{j_2(n),n} & \cdots & u_{j_i(n),n} \\ u_{j_1(n),n+1} & u_{j_2(n),n+1} & & u_{j_i(n),n+1} \\ \vdots & & & \\ u_{j_1(n),n+i-1} & u_{j_2(n),n+i-1} & & u_{j_i(n),n+i-1} \end{bmatrix}$$

of the Casorati matrix such that $w_{(i,\mathbf{j}(n)),n} \neq 0$. Here

$$\mathbf{j}(n) = \big(j_1(n), j_2(n), \dots, j_i(n)\big) \ , \quad 1 \le j_1(n) < j_2(n) < \cdots < j_i(n) \le r \ ,$$

and $j_k$ can possibly depend both on $i$ and on $n$. Of course each of $\binom{r}{i}$ possible minors $w_{(i,\mathbf{j}(n)),n}$ can be different from zero or some of them, but at least one such a minor exists.

Let us observe that $w_{(i,\mathbf{j}(n)),m+\sigma} = w_{(i,\mathbf{j}(n)),m}$ for all $m \in \mathbb{N}$. Furthermore if we take $\mathbf{j}(n+\sigma) = \mathbf{j}(n)$, then $w_{(i,\mathbf{j}(n+\sigma)),m+\sigma} = w_{(i,\mathbf{j}(n)),m}$.

Choose $\mathbf{j} = \mathbf{j}(2) = (j_1, j_2, \dots, j_{r-1})$ such that minor $w_{(r-1,\mathbf{j}),2} \neq 0$. Then using

$$u_{i,n+r} = -\frac{a_n^{r-1}}{a_n^r} u_{i,n+r-1} - \cdots - \frac{a_n^1}{a_n^r} u_{i,n+1} - \frac{a_n^0}{a_n^r} u_{i,n} \tag{5}$$

for $i = j_1, \dots, j_{r-1}$, we get the equation

$$w_{(r-1,\mathbf{j}),n+1} = (-1)^{r-1} \frac{a_{n-1}^1}{a_{n-1}^r} w_{(r-1,\mathbf{j}),n} + \phi(r-1,\mathbf{j},n), \quad n > 1, \tag{6}$$

where $\{\phi(r-1,\mathbf{j},n\}$ is $\sigma$-periodic (by $\sigma$-periodicity of $\big\{a_n^0/a_n^r\big\}_{n=1}^{\infty}$) sequence defined by

$$\begin{aligned} &\phi(r-1,\mathbf{j},n) \\ &= \det \begin{bmatrix} u_{j_1(n),n+1} & u_{j_2(n),n+1} & \cdots & u_{j_{r-1}(n),n+1} \\ \vdots & & & \\ u_{j_1(n),n+r-2} & u_{j_2(n),n+r-2} & & u_{j_{r-1}(n),n+r-2} \\ -\frac{a_{n-1}^0}{a_{n-1}^r} u_{j_1(n),n-1} & -\frac{a_{n-1}^0}{a_{n-1}^r} u_{j_2(n),n-1} & & -\frac{a_{n-1}^0}{a_{n-1}^r} u_{j_{r-1}(n),n-1} \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} w_{(r-1,\mathbf{j}),n+k\sigma+1} &= w_{(r-1,\mathbf{j}),n+1}, \\ w_{(r-1,\mathbf{j}),2+k\sigma} &= w_{(r-1,\mathbf{j}),2} \neq 0, \\ w_{(r-1,\mathbf{j}),3+k\sigma} &= w_{(r-1,\mathbf{j}),3} \end{aligned}$$

for all $n, k \in \mathbb{N}$, then we get from (6)

$$\begin{aligned} \frac{a_{1+k\sigma}^1}{a_{1+k\sigma}^r} &= (-1)^{r-1} \big(w_{(r-1,\mathbf{j}),k\sigma+3} - \phi(r-1,\mathbf{j},2+k\sigma)\big) \big/ \big(w_{(r-1,\mathbf{j}),k\sigma+2}\big) \\ &= (-1)^{r-1} \big(w_{(r-1,\mathbf{j}),3} - \phi(r-1,\mathbf{j},2)\big) \big/ \big(w_{(r-1,\mathbf{j}),2}\big) = \frac{a_1^1}{a_1^r}, \end{aligned}$$

for all $k \in \mathbb{N}$. Similary choose $\mathbf{j} = \mathbf{j}(\alpha) = (j_1, j_2, \dots, j_{r-1})$ for any fixed $\alpha \in \{2, 3, \dots, \sigma+1\}$ such that minor $w_{(r-1,\mathbf{j}),\alpha} \neq 0$. Proceeding in the same way we get

$$\begin{aligned}\frac{a^1_{\alpha-1+k\sigma}}{a^r_{\alpha-1+k\sigma}} &= (-1)^{r-1}\big(w_{(r-1,\mathbf{j}),k\sigma+\alpha+1} - \phi(r-1,\mathbf{j},\alpha+k\sigma)\big)/\big(w_{(r-1,\mathbf{j}),k\sigma+\alpha}\big)\\ &= (-1)^{r-1}\big(w_{(r-1,\mathbf{j}),\alpha+1} - \phi(r-1,\mathbf{j},\alpha)\big)/\big(w_{(r-1,\mathbf{j}),\alpha}\big)\\ &= \frac{a^1_{\alpha-1}}{a^r_{\alpha-1}}.\end{aligned}$$

This yields that the sequence $\{a^1_n/a^r_n\}_{n=1}^{\infty}$ is $\sigma$-periodic.

We have proved that $\{a^0_n/a^r_n\}_{n=1}^{\infty}$ and $\{a^1_n/a^r_n\}_{n=1}^{\infty}$ are $\sigma$-periodic sequences.

Suppose that all

$$\{a^k_n/a^r_n\}_{n=1}^{\infty}, \quad k = 0, 1, \dots, r-i-1 \quad \text{are } \sigma\text{-periodic.} \tag{7}$$

Take the minor

$$w_{(i,\mathbf{j}),n} = \det\begin{bmatrix} u_{j_1,n} & u_{j_2,n} & \cdots & u_{j_i,n}\\ u_{j_1,n+1} & u_{j_2,n+1} & & u_{j_i,n+1}\\ \vdots & & & \\ u_{j_1,n+i-1} & u_{j_2,n+i-1} & & u_{j_i,n+i-1}\end{bmatrix} \tag{8}$$

such that $w_{(i,\mathbf{j}),\alpha} \neq 0$ for some fixed $\alpha \in \{r-i+1, \dots, r-i+\sigma\}$. Because of (5) we obtain

$$\begin{aligned} u_{j_k,n+i} = &-\frac{a^{r-1}_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n+i-1} - \cdots -\\ &-\frac{a^{r-i+1}_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n+1} - \frac{a^{r-i}_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n} - \frac{a^{r-i-1}_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n-1} - \cdots -\\ &-\frac{a^1_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n+i-r+1} - \frac{a^0_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n+i-r}, \quad k = 1, \dots, i.\end{aligned} \tag{9}$$

Let us substitute the terms of the last row of $w_{(i,\mathbf{j}),n+1}$ (in (8)) by formulae obtained from (9), we get

$$w_{(i,\mathbf{j}),n+1} = (-1)^i \frac{a^{r-i}_{n+i-r}}{a^r_{n+i-r}} w_{(i,\mathbf{j}),n} + \phi(i,\mathbf{j},n), \tag{10}$$

where $\{\phi(i,\mathbf{j},n)\}$ is, by (7), $\sigma$-periodic sequence defined as follows

$$\phi(i,\mathbf{j},n)=\det\begin{bmatrix} w_{(r-1,\mathbf{j}),1}1 & w_{(r-1,\mathbf{j}),2}1 & \cdots & w_{(r-1,\mathbf{j}),i}1 \\ \vdots & & & \\ w_{(r-1,\mathbf{j}),1}i-1 & w_{(r-1,\mathbf{j}),2}i-1 & & w_{(r-1,\mathbf{j}),i}i-1 \\ \phi(i,\mathbf{j},n)_{i,1} & \phi(i,\mathbf{j},n)_{i,2} & & \phi(i,\mathbf{j},n)_{i,i} \end{bmatrix}$$

with

$$\phi(i,\mathbf{j},n)_{i,k}=-\sum_{s=0}^{r-i-1}\frac{a^s_{n+i-r}}{a^r_{n+i-r}}u_{j_k,n+i-r+s}\,, \qquad k=1,\dots,i.$$

Hence from (10) applying $\sigma$-periodicity of $w_{(i,\mathbf{j}),n+1}, w_{(i,\mathbf{j}),n}, \phi(i,\mathbf{j},n)$ we get for $n=\alpha+k\sigma$ and $k=1,2,\dots$

$$\begin{aligned}\frac{a^{r-i}_{\alpha+k\sigma+i-r}}{a^r_{\alpha+k\sigma+i-r}}&=(-1)^i\big(w_{(i,\mathbf{j}),\alpha+k\sigma+1}-\phi(i,\mathbf{j},\alpha+k\sigma)\big)/\big(w_{(i,\mathbf{j}),\alpha+k\sigma}\big)\\ &=(-1)^i\big(w_{(i,\mathbf{j}),\alpha+1}-\phi(i,\mathbf{j},\alpha)\big)/\big(w_{(i,\mathbf{j}),\alpha}\big)=\frac{a^{r-i}_{\alpha+i-r}}{a^r_{\alpha+i-r}}.\end{aligned}$$

However for $\alpha\in\{r-i+1,\dots,r-i+\sigma\}$ this yields $\frac{a^{r-i}_{s+k\sigma}}{a^r_{s+k\sigma}}=\frac{a^{r-i}_s}{a^r_s}$ for all $s\in\{1,\dots,\sigma\}$, $k\in\mathbb{N}$.

That is $\{a^{r-i}_n/a^r_n\}_{n=1}^{\infty}$ is $\sigma$-periodic sequence. Following this way we get step by step $\sigma$-periodicity of all $\{a^{r-i}_n/a^r_n\}_{n=1}^{\infty}$ for $i=r,r-1,\dots,2$. For $i=1$ the conditions imposed on the minors $w_{(i,\mathbf{j}),n}$ reduce to the fact that for each $n\in\mathbb{N}$ there exists a solution $u_{j_1}, j_1\in\{1,\dots,r\}$, such that $u_{j_1,n}\neq 0$. Suitable equations (6), (10) reduce to

$$u_{j_1,n+r}=-\frac{a^{r-1}_n}{a^r_n}u_{j_1,n+r-1}+\phi(1,j_1,n),$$

where $\phi(1,j_1,n)=-\frac{a^{r-2}_n}{a^r_n}u_{j_1,n+r-2}-\cdots-\frac{a^0_n}{a^r_n}u_{j_1,n}$ is $\sigma$-periodic sequence becuse of previous obtained results and $\sigma$-periodicity of $u_{j_1}$. This yields in a similar manner $\sigma$-periodicity of $\{a^{r-1}_n/a^r_n\}_{n=1}^{\infty}$.

**Example 2.** Periodicity of the sequences $\{a_n^i/a_n^r\}_{n=1}^{\infty}$ are necessary but not sufficient conditions for all solutions of $(E1)$ to be periodic. Consider the equation

$$x_{n+2} + (1-(-1)^n)\, x_{n+1} + x_n = 0, \quad n \in \mathbb{N}.$$

It is evident that sequences of the coefficients are 2-periodic. However the general solution of this equation can be presented as follows

$$\begin{cases} x_{2n+1} &= (-1)^n C_1 + (-1)^n 2nC_2, \quad n \in \mathbb{N}, \\ x_{2n} &= (-1)^{n-1} C_2, \quad n \in \mathbb{N}. \end{cases}$$

Of course in generally this solution is not periodic.

**Remark 2.** Looking for other conditions let us observe that from (4), to get $\sigma$-periodicity of $\{w_n\}$, we should have furthermore $\prod_{j=1}^{\sigma} \frac{a_j^0}{a_j^r} = 1$, however it is not sufficient as it is satisfied for the equation considered in Example 2. Notice that if all solutions of nonhomogeneous equation $(E)$ are $\sigma$-periodic then all solutions of suitable homogeneous equations are $\sigma$-periodic also. Therefore necessary condition given in the Theorem 2 for the equation $(E1)$ is also necessary condition for the equation $(E)$.

**Theorem 3.** *Let $a^i : \mathbb{N} \to \mathbb{R}$, $i \in \{1, \dots, r\}$ with $\lim_{n\to\infty} a_n^i = 0$, $d : \mathbb{N} \to \mathbb{R}$ be $\sigma$-periodic. Then there exists asymptotically $\sigma$-periodic solution of the equation*

$$(E2) \qquad a_n^r x_{n+r} + \cdots + a_n^1 x_{n+1} + x_n = d_n, \quad n \in \mathbb{N}.$$

*Proof.* Take any $\varepsilon > 0$. Let us denote

$$\begin{aligned} C &= \max_{1\le i\le\sigma} |d_i|, \\ C_1 &= C + \varepsilon, \\ \alpha_n &= C_1 r \sup_{m\ge n} \big( \max_{1\le i\le r} |a_m^i| \big), \quad n \in \mathbb{N}, \\ I &= \big[ -C_1, C_1 \big], \\ I_n &= \big[ d_n - \alpha_n, d_n + \alpha_n \big]. \end{aligned}$$

Let us observe that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is non-increasing and

$$(11) \qquad \lim_{n\to\infty} \alpha_n = 0,$$

therefore

$$\text{diam} \quad I_n = 2\alpha_n \to 0 \quad \text{with} \quad n \to +\infty. \tag{12}$$

By $\ell_\infty$ we denote the Banach space of bounded sequences $x = \{x_n\}_{n=1}^{\infty}$ with the norm $||x|| = \sup_{n\geq 1} |x_n|$.

From (11) it follows that there exists $n_1$ such that $\alpha_n < \varepsilon$ for all $n \geq n_1$. Consequently $I_n \subset I$ for all $n \geq n_1$.

Let $\mathbf{T} \subset \ell_\infty$ be such a set that $x = \{x_n\}_{n=1}^{\infty} \in \mathbf{T}$ if

$$\begin{cases} x_n = d_n & \text{for} \quad n = 1, 2, \ldots, n_1 - 1 \\ x_n \in I_n & \text{for} \quad n \geq n_1. \end{cases}$$

It is easy to check that $\mathbf{T}$ is a closed, convex, and compact subset of $\ell_\infty$. By (12), for arbitrary $\varepsilon_1 > 0$ we can set up a finite $\varepsilon_1$-net for the set $\mathbf{T}$. Hence by Hausdorff's theorem $\mathbf{T}$ is compact.

Define now an operator $\mathcal{A}$. Let $y = \{y_n\}_{n=1}^{\infty} \in \mathbf{T}$ and $\mathcal{A}\, y = \eta = \{\eta_n\}_{n=1}^{\infty}$ if

$$\eta_n = \begin{cases} d_n & \text{for} \quad n = 1, 2, \ldots, n_1 - 1, \\ d_n - a_n^r y_{n+r} - \cdots - a_n^1 y_{n+1} & \text{for} \quad n \geq n_1. \end{cases}$$

Let us observe that

$$\begin{aligned} |\eta_n - d_n| &= |a_n^r y_{n+r} + \cdots + a_n^1 y_{n+1}| \leq (\max_{1\leq i\leq r} |a_n^i|)(|y_{n+r}| + \cdots + |y_{n+1}|) \\ &\leq C_1 r (\max_{1\leq i\leq r} |a_n^i|) \leq \alpha_n \quad \text{for} \quad n \geq n_1. \end{aligned}$$

Hence $\mathcal{A}(\mathbf{T}) \subset \mathbf{T}$. Furthermore for any $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty}$ two elements of the set $\mathbf{T}$ we have

$$\begin{aligned} ||\mathcal{A}x - \mathcal{A}y|| &= \sup_{n\in N} |(\mathcal{A}x)_n - (\mathcal{A}y)_n| \\ &= \sup_{n\geq n_1} |(d_n - a_n^r x_{n+r} - \cdots - a_n^1 x_{n+1}) \\ &\quad - (d_n - a_n^r y_{n+r} - \cdots - a_n^1 y_{n+1})| \\ &\leq \sup_{n\geq n_1} \left\{ (\max_{1\leq i\leq r} |a_n^i|)(|x_{n+r} - y_{n+r}| + \cdots + |x_n - y_n|) \right\} \\ &\leq \frac{\alpha_{n_1}}{C_1 r} \sup_{n\geq n_1} (|x_{n+r} - y_{n+r}| + \cdots + |x_n - y_n|) \leq \frac{\alpha_{n_1}}{C_1 r} r||x - y||. \end{aligned}$$

From this we get continuity of the operator $\mathcal{A}$. Hence by Schauder's fixed point theorem there exists in $\mathbf{T}$ a solution of the operator equation $x = \mathcal{A}x$. Let $z = \{z_n\}_{n=1}^{\infty}$ be this fixed point of $\mathcal{A}$. Then

$$z = \{d_1, \dots, d_{n_1-1}, z_{n_1}, \dots, z_n, \dots\}$$

and by definition of $\mathcal{A}$ we have

$$z_n = d_n - a_n^r z_{n+r} - \cdots - a_n^1 z_{n+1} \quad \text{for} \quad n \geq n_1. \tag{13}$$

That is $z = \{z_n\}_{n=1}^{\infty}$ is the solution of $(E2)$ for $n \geq n_1$. Furthermore this solution, by (12), possesses asymptotic property:

$$z_n = d_n + o(1). \tag{14}$$

Hence $\sigma$-periodicity of the sequence $d$ yields asymptotic $\sigma$-periodicity of the sequence $z$. The obtained sequence does not satisfy $(E2)$ for all $n \in \mathbb{N}$, however using (13) we can build solution of $(E2)$ back, step by step up to $z_1$.

**Remark 3.** The same result we present in Theorem 3 can be obtained in one of the following cases.

i) $a_n^0 \equiv a \neq 0$ (constant) in $(E)$. In this case it suffices to divide the equation $(E)$ by $a$, and check that conditions of Theorem 3 are satisfied.

ii) $\{d_n\}_{n=1}^{\infty}$ is asymptotically $\sigma$-periodic sequence.

iii) $\left\{a_n^0\right\}_{n=1}^{\infty}$ is $\sigma$-periodic with $a_n^0 \neq 0$ for all $n \in \mathbb{N}$. As in the case (i) dividing $(E1)$ by $a_n^0$ we come back to the case considered in the Theorem 1, because then $\left\{d_n/a_n^0\right\}_{n=1}^{\infty}$ is $\sigma$-periodic and $\lim_{n\to\infty} (a_n^i/a_n^0) = 0$.

iv) $a_n^0 \neq 0$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n^0 = 0$ and there exists $k \in \{1, \dots, r\}$ such that $a_n^k \equiv a \neq 0$, and for other $i \in \{1, \dots, r\}$, $i \neq k$, $\lim_{n\to\infty} a_n^i = 0$.

In the paper [7] we have considered equation $(E1)$ in relation to the problem of existence of asymptotically periodic solutions when $\left\{a_n^0\right\}$ is asymptotically periodic, and $\left\{a_n^1\right\}$ asymptotically approaches 1. In [5] the method similar to this presented in proof of Theorem 3 has been applied to obtain existence of asymptotically periodic perturbation for second order nonlinear difference equations with asymptotically periodic solutions. Also similar method has been used in [6] to obtain particular case of periodic sequences i.e. constant approaching solutions. For other treatment of the problem of periodicity see e.g. [2,3].

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