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THE REGULATION INDEPENDENT OF THE POTENTIAL SYMMETRICAL TO THE CENTER $[\tau, \pi]$ FOR STURM–LIOUVILLE OPERATOR WITH A CONSTANT DELAY

M. Pikula and T. Marjanović

This paper is dedicated to Professor D. S. Mitrinović

Abstract. Let $L = L(\tau, q)$ be operator defined with $-y''(x) + q(x)y(x-\tau) = \lambda y(x)$, $\lambda = z^2$; $y(x-\tau) \equiv 0$, $x \le \tau$, $\pi/3 \le \tau < \pi/2$; $y(\pi) = 0$. The aim of this work is to prove the existence and unicity of the operator L, if the range of proper values is given. Potential q is a complex function in $L_1[0,\pi]$, and $q(x) = q(\pi - x)$. If q is an analytic function, the problem is solved in [6] for $\tau \in (0,\pi)$. With $q \in L_1[0,\pi]$, $\tau \in [\pi/2, \pi]$ the corresponding problem is solved in [7]. For an arbitrary $\tau \in (0,\pi)$ and "small" potential q the problem is solved in [8]. In this paper, the same method of characteristic functions like in [4], [5], and [6], is used.

1. Asymptotics of Characteristic Values

Let $L = L(\tau, q)$ be operator defined with

$$-y''(x) + q(x)y(x-\tau) = \lambda y(x),\ \ \lambda = z^2, \tag{1}$$

$$y(x-\tau) \equiv 0, \quad x \le \tau, \quad \pi/3 \le \tau < \pi/2, \tag{2}$$

$$y(\pi) = 0. \tag{3}$$

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The relations (1) and (2) are in fact the integral equation

$$(4) \qquad y(x,z,\tau) = \sin zx + \frac{1}{z}\int_\tau^x q(t)\sin z(x-t)\sin z(t-\tau)dt$$
$$+\frac{1}{z^2}\int_{2\tau}^x q(t_1)\sin z(x-t_1)\int_\tau^{t_1-\tau} q(t_2)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau)dt_2dt_1.$$

From (4) we get the characteristic function $F(z)$ of the operator L in the form

$$(5) \quad F(z) = \sin \pi z + \frac{1}{z}\int_\tau^\pi q(t)\sin z(\pi-t)\sin z(t-\tau)dt$$
$$+\frac{1}{z^2}\int_{2\tau}^x q(t_1)\sin z(\pi-t_1)\int_\tau^{t_1-\tau} q(t_2)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau)dt_2dt_1.$$

Theorem 1. *If* $q \in L_1[0,\pi]$ *then zeros* z_n *of the function* (5) *have the following asymptotics*

$$(6) \qquad \pm z_n = n + \frac{\cos n\tau}{2\pi n}\int_\tau^\pi q(t)dt + O\Big(\frac{\cos n\tau}{n}\Big),\ \ n=1,2,\dots\ .$$

Proof. Take $z_n = n + c_1(n)/n + c_2(n)/n^2 + o(1/n^2)$. Putting this expression into the equation $F(z)=0$ and grouping expression by degrees, we get

$$c_1(n) = \frac{\cos n\tau}{2\pi}\int_\tau^\pi q(t)dt - \frac{1}{2\pi}\int_\tau^\pi q(t)\cos 2n\Big(t-\frac{\tau}{2}\Big)dt$$

and

$$c_2(n) = \frac{1}{4\pi}\int_{2\tau}^\pi q(t_1)\int_\tau^{t_1-\tau} q(t_2)dt_2dt_1.$$

Because $q \in L_1[0,\pi]$, we have $a_{2n} = \int_\tau^\pi q(t)\cos 2n(t-\tau/2)dt \to 0$ $(n\to+\infty)$.

From that and from the fact that $F(z)$ is an odd function, the proof follows directly. □

Remark 1. If function q is of the bounded variation then

$$\int_\tau^\pi q(t)\cos 2n\big(t-\frac{\tau}{2}\big)dt = O\Big(\frac{1}{n}\Big),$$

and

$$z_n = n + \frac{\cos n\tau}{2n\pi} \int_{\tau}^{\pi} q(t)dt + \frac{1}{n^2} \left(\frac{1}{4\pi} \int_{2\tau}^{\pi} q(t_1) \int_{\tau}^{t_1-\tau} q(t_2)dt_2dt_1 + O(1) \right) + o\Big(\frac{1}{n^2}\Big). \tag{7}$$

Since $\lambda_n = z_n^2$, then from (6) we get asymptotics of the eigenvalues of the operator L in the form

$$\lambda_n = n^2 + \frac{1}{\pi} \int_{\tau}^{\pi} q(t) \cos n\tau dt + O(\alpha_n),\ \alpha_n \to 0\ (n \to +\infty), \tag{8}$$

and from (7) in the form

$$\lambda_n = n^2 + \frac{1}{\pi} \int_{\tau}^{\pi} q(t) \cos n\tau\ dt + O\Big(\frac{1}{n}\Big). \tag{9}$$

Let us prove an explicit connection between delay τ and given eigenvalues. Since

$$\begin{aligned} \beta_n &= \frac{\lambda_{n+2} - \lambda_{n-2} - (n+2)^2 + (n-2)^2}{\lambda_{n+1} - \lambda_{n-1} - (n+1)^2 - (n-1)^2} \\ &= 2\cos\tau + \begin{cases} O(\alpha_n), & \alpha_n \to 0,\ n \to +\infty,\ q \in L_1[0,\pi], \\ O(1/n), & n \to +\infty,\ q \text{ is a bounded variation function}, \end{cases} \end{aligned}$$

then

$$\cos\tau = \frac{1}{2} \lim_{n\to+\infty} \beta_n = \frac{1}{2}\beta \qquad (-2 < \beta < 2). \tag{10}$$

Notice that if $0 < \beta \le \sqrt{3}$, then $\tau \in [\pi/3, \pi/2)$.

2. Relation Between Potential and Characteristic Values

Let λ_n be eigenvalues of the operator L. Then the characteristic function (5) can be done in the form

$$F(z) = Az \prod_{n=1}^{+\infty} \Big(1 - \frac{z^2}{\lambda_n}\Big),$$

where A is a undetermined constant. Equating those two forms of the same function, we get

$$A = \pi \prod_{n=1}^{+\infty} \frac{\lambda_n}{n^2}.$$

Thus,

$$F(z) = \pi \prod_{n=1}^{+\infty} \frac{\lambda_n}{n^2} \cdot z \prod_{n=1}^{+\infty} \Big(1 - \frac{z^2}{\lambda_n}\Big). \tag{11}$$

Using asymptotics (8) and taking $z = -i\sqrt{\mu}$, we obtain

$$\lim_{\mu \to +\infty} \frac{F(-i\sqrt{\mu}) - \sin(-\pi i\sqrt{\mu})}{\cosh(\pi - \tau)\sqrt{\mu}} \cdot 2\sqrt{\mu} = \int_\tau^\pi q(t)dt = J_1. \tag{12}$$

Putting $S(z, t_1, t_2, \tau) = \sin z(\pi - t_1) \sin z(t_1 - \tau - t_2) \sin zt_2$, for $z = m$, $m \in \mathbb{N}$, we can write

$$\begin{aligned} S(m, t_1, t_2, \tau) &= (-1)^{m+1} S_m(t_1, t_2, \tau) \\ &= \frac{(-1)^{m+1}}{4} \big\{ \sin 2m(t_1 - t_2) - \sin 2m\tau \\ &\qquad - \sin 2m(t_1 - \tau) + \sin 2mt_2 \big\}. \end{aligned} \tag{13}$$

In order to find a relation between the potential q and eigenvalues λ_n of the operator L, we start from

$$\begin{aligned} &\pi \prod_{n=1}^{+\infty} \frac{\lambda_n}{n^2} \prod_{n=1}^{+\infty} \Big(1 - \frac{z^2}{\lambda_n}\Big) \\ &\equiv \sin \pi z - \frac{J_1}{2z} \cos z(\pi - \tau) + \frac{1}{2z} \int_\tau^\pi q(t) \cos z(\pi - 2t + \tau) dt \\ &+ \frac{1}{z^2} \int_{2\tau}^\pi q(t_1) \int_\tau^{t_1 - \tau} q(t_2) S(z, t_1, t_2, \tau) dt_2 dt_1, \end{aligned} \tag{14}$$

for $z \in \mathbb{C}$. The identity (14) gives

$$
\begin{aligned}
(15) \qquad & F(m) + \frac{(-1)^m J_1}{2m}\cos m\tau \\
&= \frac{(-1)^m}{2m}\int_{\tau}^{\pi} q(t)\cos 2m\Big(t-\frac{\tau}{2}\Big)\,dt \\
&+ \frac{(-1)^{m+1}}{4m^2}\Bigg\{\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2m(t_1-t_2)\,dt_2\,dt_1 \\
&- \sin 2m\tau \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\,dt_1\,dt_2 \\
&- \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2m(t_1-\tau)\,dt_2\,dt_1 \\
&+ \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2mt_2\,dt_2\,dt_1\Bigg\},\ \ m=1,2,\dots\ .
\end{aligned}
$$

Now, we introduce

$$
(16) \qquad A_{2m} = (-1)^m\frac{4m}{\pi}\left[\pi\prod_{n=1}^{+\infty}\frac{\lambda_n}{n^2}\prod_{n=1}^{+\infty}(1-\frac{m^2}{\lambda_n}) + \frac{(-1)^m J_1}{2m}\cos m\tau\right],
$$

$$
(17) \qquad a_{2m} = \frac{2}{\pi}\int_{\tau/2}^{\pi-\tau/2} q\Big(t+\frac{\tau}{2}\Big)\cos 2mtdt,
$$

$$
(18) \qquad
\begin{aligned}
\sigma_{1,m}(q) &= \frac{1}{\pi m}\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2m(t_1-t_2)dt_2dt_1,\\
\sigma_{2,m}(q) &= -\frac{\sin 2m\tau}{\pi m}\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)dt_2dt_1,\\
\sigma_{3,m}(q) &= -\frac{1}{\pi m}\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2m(t_1-\tau)dt_2dt_1,\\
\sigma_{4,m}(q) &= \frac{1}{\pi m}\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin 2mt_2dt_2dt_1.
\end{aligned}
$$

Then (15) becomes

$$
(19) \qquad a_{2m} = A_{2m} + \sum_{k=1}^{4}\sigma_{k,m}(q) = A_{2m} + X_{2m}(q).
$$

Using an integration by parts, exchanging the order of the integration, and putting

$$Q(t_1,t_2,\tau)=\frac{\pi-\tau-t_1}{t_1-\tau}\,q\big(\tau+t_1+\frac{\pi-\tau-t_1}{t_1-\tau}(t_2-\tau)\big),$$

the functional $X_{2m}(q)$ can be transformed in the form

$$\begin{aligned}X_{2m}(q)=\frac{2}{\pi}\int_\tau^{\pi-\tau}\Bigg\{\int_\tau^{t_1}\Big\{Q(t_1,t_2,\tau)\int_\tau^{\tau+(\pi-\tau-t_1)(t_2-\tau)/(t_1-\tau)}&q(t_3)\,dt_3\\ +\ q(t_2+\tau)\int_\tau^{t_2}q(t_3)\,dt_3-q(t_2)\int_{\tau+t_2}^{\pi}q(t_3)dt_3\Big\}dt_2\Bigg\}&\cos 2mt_1\,dt_1,\end{aligned}$$

i.e.,

$$X_{2m}(q)=\frac{2}{\pi}\int_\tau^{\pi-\tau}\left[\int_\tau^{t_1}K(t_1,t_2,q(t_1,t_2),\tau)\,dt_2\right]\cos 2mt_1\,dt_1. \tag{20}$$

Therefore (19) becomes

$$a_{2m}=A_{2m}+\frac{2}{\pi}\int_\tau^{\pi-\tau}\left[\int_\tau^{t_1}K(t_1,t_2,q(t_1,t_2),\tau)dt_2\right]\cos 2mt_1dt_1. \tag{21}$$

Now, we put

$$q_1(t)=\begin{cases}q(t+\tau/2), & t\in(\tau/2,\pi-\tau/2),\\ 0, & t\in(0,\tau/2)\cup(\pi-\tau/2,\pi),\end{cases}$$

and

$$H(t_1)=\begin{cases}\int_\tau^{t_1}K\big(t_1,t_2,q(t_1,t_2),\tau\big)dt_2, & t_1\in(\tau,\pi-\tau),\\ 0, & t_1\in(0,\tau)\cup(\pi-\tau,\pi).\end{cases}$$

If $q(x)=q(\pi-x)$, $x\in(0,\pi)$, then $q_1(x)=q_1(\pi-x)$. Then a_{2m} are Fourier coefficients of the function $q_1(x)$, $x\in(0,\pi)$.

It is easy to show that $A_{2m}\to 0$, if $m\to+\infty$. It means that A_{2m} present Fourier cosinus coefficients of a certain function $f\in L_1[0,\pi]$. Besides the functionals $X_{2m}(q)$ represent Fourier coefficients of the function $H(t_1)$.

If we multiply (21) with $\cos 2mx$ and summing in m, we get

$$q\Big(x+\frac{\tau}{2}\Big)=f(x)+\int_{\tau}^{x}K\big(x,t,q(t)\big)dt,\ x\in(\tau,\pi-\tau),$$

and

$$q\Big(x+\frac{\tau}{2}\Big)=f(x),\ x\in\Big(\frac{\tau}{2},\tau\Big)\cup\Big(\pi-\tau,\pi-\frac{\tau}{2}\Big),$$

i.e.,

$$q(x)=f\Big(x-\frac{\tau}{2}\Big),\ x\in\Big(\tau,\frac{3}{2},\tau\Big)\cup\Big(\pi-\frac{\tau}{2},\pi\Big), \tag{22}$$

and

$$q(x)=f\Big(x-\frac{\tau}{2}\Big)+\int_{\tau}^{x-\pi/2}K\Big(x-\frac{\pi}{2},t,q(t)\Big)dt,\ x\in\Big(\frac{3}{2}\tau,\pi-\frac{\tau}{2}\Big). \tag{23}$$

Remark 2. The interval A_n in (23) for $\tau=\pi/2$ disappears, and for $\tau=\pi/3$ it becomes $(\pi/2,5\pi/6)$.

Since (23) is a nonlinear integral equation of Volterra type, and because the kernel $K\big(x-\tau/2,t,q(t)\big)$ satisfies the Lipschitz condition by a q, then it has only one solution which can be obtained by the method of successive approximations.

Thus, one sequence of eigenvalues of the operator L defines that operator.

3. The Solution of the Inverse Problem

Theorem 2. *In order that a sequence of numbers λ_n, $n\in\mathbb{N}$, $|\lambda_n|<|\lambda_{n+1}|$ be a sequence of eigenvalues of the operator L type* (1), (2), (3) *it is necessary and sufficient that:*

1° *The sequence λ_n has asimptotics*

$$\lambda_n=n^2+a_0\cos n\tau+O(\alpha_n),\ \alpha_n\to 0,\ n\to+\infty;$$

2°

$$\beta=\lim_{n\to+\infty}\frac{\lambda_{n+2}-\lambda_{n-2}-(n+2)^2+(n-2)^2}{\lambda_{n+1}-\lambda_{n-1}-(n+1)^2+(n-1)^2}=2\cos\tau,\ \beta\in(0,\sqrt{3}).$$

Proof. The necessity of conditions 1° and 2° is evident from the direct problem. Let us prove that 1° and 2° are sufficient conditions. We construct the function

$$\Phi(z) = \pi \prod_{n=1}^{+\infty} \frac{\lambda_n}{n^2} \prod_{n=1}^{+\infty} \Big(1 - \frac{z^2}{\lambda_n}\Big)$$

and consider identity (14). Take $\tilde{\tau} = \arccos \beta/2$ and

$$\int_{\tau}^{\pi} q(t)dt = \lim_{\mu \to +\infty} \frac{\Phi(-i\sqrt{\mu}) - \sin(-\pi i\sqrt{\mu})}{\cosh(\pi - \tau)\sqrt{\mu}} \cdot 2\sqrt{\mu}.$$

The presented identity is transformed into the system of equations (22) and (23), where K is defined by (20). Let $\tilde{q}(x)$ be the unique solution of the system (22), (23), $x \in [\tau, \pi]$. According to the symmetry $q(x) = g(\pi - x)$, the function $\tilde{q} \in L_1[0, \pi]$ is defined. With such defined $\tilde{\tau}$ and $\tilde{q}$ we construct the operator $\tilde{L} = L(\tilde{\tau}, \tilde{q})$. Let us make the characteristic function $\tilde{F}(z)$ of the operator $\tilde{L}$ and let us find its zeros $\pm\tilde{z}_n$. Let $\tilde{\lambda}_n$ be eigenvalues of the operator $\tilde{L}$.

From the way of defining the function $\tilde{q}$ and number $\tilde{\tau}$ it is evident that $\Phi(z) \equiv \tilde{F}(z)$, which means that $\lambda_n = \tilde{\lambda}_n$. $\square$

REFERENCES

1. D. S. Mitrinović and J. D. Kečkić: *Mathematics II.* Građevinska knjiga, Belgrade, 1989 (Serbian).
2. M. Pikula and T. Marjanović: *The construction of the small potential for an equation of Sturm–Liouville type with constant delay.* Proc. II Math. Conf. Priština (Priština, 1996), (Ed. Lj. D. Kočinac), Univ. Priština, Priština, 1997, 135-141.
3. S. B. Norkin: *Differential equations of the second order with retarded argument.* Nauka, Moscow, 1965 (Russian).
4. L. E. El'sgol'ts and S. B. Norkin: *Introduction to the Theory of Differential Equations with Deviating Argument.* Nauka, Moscow, 1971 (Russian).
5. V. A. Sadovnichiĭ: *Operator Theory.* Moskov. Gos. Univ., Moscow, 1979 (Russian).
6. M. Pikula: *Regularized traces of higher-order differential operators with retarded argument.* Diff. Equations **21** (1985), 664-668 [Translation from Diff. Uravn. **21** (1985), 986-991 (Russian)].

7. M. Pikula: *Determination of a Sturm-Liouville-type differential operator with retarded argument from two spectra.* Mat. Vesnik **43** (1991), 159–171 (Russian).

8. M. Pikula: *A solution of the inverse problem for equations with constant delay.* Congress of Yugoslav Mathematicians (May 22–27, 1996), Petrovac, Yugoslavia.

The Teacher Training Faculty
31000 Užice
Yugoslavia

Higher Technical School
32000 Čačak
Yugoslavia