FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 14 (1999), 11–19

ON THE RATE OF APPROXIMATION ALMOST EVERYWHERE BY STEKLOV MEANS WITH RESPECT TO MULTIDIMENSIONAL INTERVALS

Alexander Stokolos

This paper is dedicated to Professor D. S. Mitrinović

Abstract. The rate of approximation a.e. by Steklov means with respect to multidimensional integrals is considered.

The classical Lebesgue theorem about differentiation of integrals states that Steklov means $S_h f(x) = h^{-1} \int_x^{x+h} f(t)\, dt$ approximate a.e. the summable function f. It is easy to estimate the rate of approximating in L^1–metric in the following way

$$\|S_h f - f\| = O\{\omega(f;h)\}\,. \tag{1}$$

Also, it is natural to consider the problem of estimation of the rate of a.e. convergence of $S_h f(x)$. In fact, we can define more precisely the order of $o_x(1)$ in the following fundamental relation, which may be considered as a.e. analogous of the statement $\|S_h f - f\| = o(1)$:

$$|I|^{-1} \int_{I \ni x} |f(y) - f(x)|\, dy = o_x(1)\,, \quad \operatorname{diam}(I) \to 0\,. \tag{2}$$

The complete solution of this problem was given by K.I. Oskolkov [1]. In certainly weaker that in [1] form, it is contained in the Theorem A below. Before the formulation of this theorem we introduce some notation.

Let $I^n = [0,1]^n$ and $\ll$ and $\gg$ designates the inequalities $\le$ and $\ge$, which are true with some constants. By the Φ we denote the class of positive

Received February 27, 1996.
1991 *Mathematics Subject Classification*. Primary 26B05; Secondary 26B35.

sub additive non decreasing on $(0,1]$ functions ϕ. Also, $\omega(f;\delta)$ and $\omega_i(f,\delta)$ denote the modulus of continuity and j–th partial modulus of continuity of $f \in L^1(I^n)$ respectively and $H_1^\omega(I^n) = \{f \in L^1(I^n) \,:\, \omega(f;\delta) = O\{\omega(\delta)\}\}$.

Let $f \in L^1(I^n)$ and $\delta > 0$. Define

$$D_\delta(f,x) = \sup |I|^{-1} \int_I |f(y) - f(x)|\, dy$$

where supremum is taking over the n–dimensional intervals $I = I_1 \times \cdots \times I_n$, $I \ni x$, $I \subset I^n$, $\operatorname{diam}(I) = \delta$.

We suppose that $\omega(\delta)/\delta \to \infty$, $\delta \to 0$ and introduce the auxiliary function $\Omega(\delta)$ which characterizes the rate of convergence to zero the quantities $\omega(\delta)$ and $\delta/\omega(\delta)$. We set

$$\delta_0 = 1\,; \quad \delta_{k+1} = \min\left\{\delta \,:\, \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}\,;\frac{\omega(\delta_k)\delta}{\omega(\delta)\delta_k}\right) \ge \frac{1}{2}\right\}\,, \tag{3}$$

and $\Omega(\delta) = 2^{1-k}$; $\delta_{k+1} \le \delta < \delta_k$.

Theorem A. *Let $\omega(\delta)/\delta \to \infty$, $\delta \to 0$, and $\psi(t)$ is positive non increasing function such that*

$$\int_0^1 \frac{dt}{t\psi(t)} < \infty\,. \tag{4}$$

Then for every $f \in H_1^\omega(I^1)$

$$D_\delta(f,x) = O_x\{\omega(\delta)\psi(\Omega(\delta))\} \quad a.e.\ on\ I^1\,.$$

If (4) false, then with the some $g \in H_1^\omega(I^1)$

$$\limsup_{\delta\to 0} \frac{D_\delta(g,x)}{\omega(\delta)\,\psi(\Omega(\delta))} = \infty \quad a.e.\ on\ I^1\,. \tag{5}$$

Now, let us point our attention to the multidimensional case. The multidimensional Steklov means $S_{h_1,\dots,h_n}f$ approximate summable function f only by L^1–metric with similar to (1) estimation. However, they approximate a.e. only functions from $L(\log + L)^{n-1}(I^n)$ by the well–known Jessen–Marcinkiewicz–Zygmund theorem. Therefore, the embedding $H_1^\omega(I^n) \subset L(\log + L)^{n-1}(I^n)$, is sufficient for the availability of (2) for all $f \in H_1^\omega(I^n)$. The corresponding condition of embedding has the following form:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\left(\log\frac{1}{\delta}\right)^{n-2} d\delta < \infty\,. \tag{6}$$

We [2] managed to establish, that (6) is a necessary condition for (2) also. By the other words, *the condition of strong differentiation of integrals of all functions from $H_1^\omega(I^n)$ coincide with the condition of embedding $H_1^\omega(I^n) \subset L(\log +L)^{n-1}(I^n)$.*

So, we can consider the problem of estimation of the rate of strong differentiation.

Theorem 1. *Let $n \geq 2$ and $\omega(\delta)$ modulus of continuity which satisfying* (6) *and function $\psi \in \Phi$ such that*

$$\int_0^1 \frac{\omega(\delta)}{\delta\psi(\delta)} \left(\log \frac{1}{\delta}\right)^{n-2} d\delta < \infty \,.$$

Then for every $f \in H_1^\omega(I^n)$

$$D_\delta(f,x) = O_x\{\psi(\delta)\} \quad \textit{a.e. on } I^n \,. \tag{7}$$

The method of investigation of strong differentiation due to A. Zygmund consists of majorization of multidimensional operators by the compositions of partial one–dimensional operators. Then the problem of finitenesses a.e. is reducing to the problem of summability and embedding to the some Orlicz classes low–dimensional operators. We shall also follow this schedule and introduce the maximal function $\mathcal{N}_\phi^s f(x)$ connected with local smoothness

$$\mathcal{N}_\varphi^s f(x) = \sup_{I, x \in I} (|I|_\varphi(\operatorname{diam}(I)))^{-1} \int_I |f(y) - f(x)|\, dy \,.$$

Theorem 2. *Let $\omega(\delta)$ is modulus of continuity which satisfies* (6) *and function $\phi \in \phi$ such that*

$$\int_0^1 \frac{\omega(\delta)}{\delta\, \phi(\delta)} \left(\log \frac{1}{\delta}\right)^{n-1} d\delta < \infty \,.$$

Then $\mathcal{N}_\phi^s f(x) \in L^1(I^n)$ for every $f \in H_1^\omega(I^n)$.

Proof. With $\tau \equiv (\tau(1), \dots, \tau(n))$ we denote some rearrangements of $(1, \dots, n)$. Define n–dimensional simplexes $E_r \subset I^n$

$$E_r \equiv \{(x_1, \dots, x_n) : 0 \leq x_{\tau(1)} \leq 1,\ x_{\tau(1)} \leq x_{\tau(2)} \leq 1\,, \dots, x_{\tau(n-1)} \leq x_{\tau(n)} \leq 1\} \,.$$

For example in two–dimensional case there are two different rearrangements $\tau : (1,2) \to (1,2)$ and $\tau : (1,2) \to (2,1)$. Then

$$E_{1,2} = \{(x_1, x_2) \,:\, 0 \le x_1 \le 1\,,\ x_1 \le x_2 \le 1\}\,,$$
$$E_{2,1} = \{(x_1, x_2) \,:\, 0 \le x_2 \le 1\,,\ x_2 \le x_1 \le 1\}\,,$$

and $I^2 = E_{1,2} \cup E_{2,1}$.

It is obvious (by induction), that for every $x \in I^n$

$$1 \le \sum_\tau \chi_\tau(x) \le n! \tag{8}$$

where sum is taken over all rearrangements τ and χ_τ denote the characteristic function of E_τ.

Now, let $x \in I = I_1 \times \cdots \times I_n$, and integer m_j such that $2^{-m_j-1} < |I_j| \le 2^{-m_j}$. Denote by $v(I) = (2^{-m_1}, \dots, 2^{-m_n})$, and taking into consideration (8) we have for all x and y in I^n

$$\begin{aligned} |f(y) - f(x)| \ll \sum_\tau & |f(y_1, \dots, x_{\tau(1)}, \dots, y_n) - f(y_1, \dots, y_n)|\, \chi_\tau(v(I)) \\ + \sum_\tau & |f(y_1, \dots, x_{\tau(1)}, \dots, y_n) - f(x_1, \dots, x_n)|\, \chi_\tau(v(I))\,. \end{aligned}$$

Let

$$A(j) = \{k \,:\, 1 \le k \le n\,;\ k \ne j\}\,, \quad I(j) = \prod_{k \in A(j)} I_k\,, \quad dy(j) = \prod_{k \in A(j)} dy_k\,.$$

Then

$$\begin{aligned} |I|^{-1} \int_I |f(y) - f(x)|\, dy \le \sum_r & |I|^{-1} \\ & \times \int_I |f(y_1, \dots, y_n) - f(y_1, \dots, x_{\tau(1)}, \dots, y_n)|\, dy \\ & \times \chi_\tau(v(I)) + n! \sum_{j=1}^n |I(j)|^{-1} \\ & \times \int_{I(j)} |f(x_1, \dots, x_n) - f(y_1, \dots, x_j, \dots, y_n)|\, dy(j)\,. \end{aligned}$$

If we denote by

$$\mathcal{R}_\psi^\tau f(x) = \sup_{I,x\in I,v(I)\in E_\tau} \left(|I|\psi_{\tau(1)}\left(|I|_{\tau(1)}\right)\right)^{-1} \times \int_I |f(y_1,\dots,y_n) - f(y_1,\dots,x_{\tau(1)},\dots,Y_n)|\,dy\,;$$

$$\mathcal{N}_\psi^{s,j} f(x) = \sup_{I,x\in I} (|I(j)|\psi(I(j)))^{-1} \times \int_{I_{(j)}} |f(x_1,\dots,x_n) - f(y_1,\dots,x_j,\dots,y_n)|\,dy(j)\,,$$

as $1/\psi(\mathrm{diam}\,(I)) \le 1/\psi(|I_j|)$ for all $j = 1,\dots,n$, then

$$\mathcal{N}_\psi^s f(x) \le \sum_\tau \mathcal{R}_\psi^\tau(f,x) + n!\sum_{j=1}^n \mathcal{N}_\psi^{s,j} f(x)\,.$$

The second sum, in essence, contains $(n-1)$–dimensional $\mathcal{N}_\varphi$–functions, therefore, it is sufficient to establish summability $\mathcal{R}_\psi^\tau f(x)$.

Let

$$E_\tau^* = \{m = (m_1,\dots,m_n) \,:\, \left(2^{-m_1},\dots,2^{-m_n}\right) \in E_\tau\}\,.$$

Then

$$\begin{aligned}\mathcal{R}_\psi^\tau(f,x) &\ll \sum_{m\in E_\tau^*} 2^{m_1}\int_{-2^{-m_n}}^{2^{-m_1}} dt_1 \cdots 2^{m_n}\int_{-2^{-m_n}}^{2^{-m_n}} |f(x_1+t_1,\dots,x_n+t_n)\\ &- f(x_1+t_1,\dots,x_{\tau(1)},\dots,x_n+t_n)|\,dt_n \left[\psi_{\tau(1)}(2^{-m_{\tau(1)}})\right]^{-1}\\ &\ll \sum_{m\in E_\tau^*}\sum_{|\epsilon_1|=1,\dots,|\epsilon_n|=1} 2^{m_1}\int_0^{2^{-m_1}} dt_1\cdots 2^{m_n}\\ &\times \int_0^{2^{-m_n}} |f(x_1+\epsilon_1\cdot t_1,\dots,x_n+\epsilon_n\cdot t_n)\\ &- f(x_1+\epsilon_1\cdot t_1,\dots,x_{\tau(1)},\dots,x_n+\epsilon_n\cdot t_n)|\,dt_n\left[\psi_{\tau(1)}(2^{-m_{\tau(1)}})\right]^{-1}.\end{aligned}$$

Now

$$\|\mathcal{R}_\psi^\tau f\|_1 \ll \sum_{m\in E_\tau^*}\sum_{|\epsilon_1|=1,\dots,|\epsilon_n|=1} 2^{m_{\tau(1)}}\left[\psi_{\tau(1)}(2^{-m_{\tau(1)}})\right]^{-1} \times \int_0^{2^{-m_{\tau(1)}}} \|\Delta^{\tau(1)}_{\epsilon_{\tau(1)}t} f\|_1\,dt\,.$$

Let us estimate the arbitrary addendum. Without loss of generality we may assume that $\epsilon_1 = \cdots = \epsilon_n = 1$; $\tau(1) = 1, \ldots, \tau(n) = n$. Then

$$\sum_{m \in E_\tau^*} 2^{m_1} \left[\psi\left(2^{-m_1}\right)\right]^{-1} \int_0^{2^{-m_1}} \|\Delta_t^1 f\|_1 \, dt$$
$$\ll \sum_{m_1=0}^{\infty} \left\{ \sum_{m_2=1}^{m_1} \cdots \sum_{m_n=1}^{m_{n-1}} 1 \right\} 2^{m_1} \left[\psi\left(2^{-m_1}\right)\right]^{-1} \int_0^{2^{-m_1}} \|\Delta_t^1 f\|_1 \, dt \, .$$

Obviously,

$$\sum_{m_2=1}^{m_1} \cdots \sum_{m_n=1}^{m_{n-1}} 1 \ll (m_1)^{n-2} \, .$$

Taking into consideration monotonicity of $\psi(t)$, we get

$$\|\mathcal{R}_\psi^\tau f\|_1 \ll \sum_{m=0}^{\infty} m^{n-2} 2^m \left[\psi(2^{-m})\right]^{-1} \int_0^{2^{-m}} \|\Delta_t^1 f\|_1 \, dt$$
$$\ll \sum_{m=0}^{\infty} 2^m \int_0^{2^{-m}} \frac{\|\Delta_t^1 f\|_1}{\psi(t)} \left(\log \frac{1}{t}\right)^{n-2} dt \, .$$

Thus, applying the Abel's transform, we get

$$\|\mathcal{R}_\psi^\tau f\|_1 \ll \int_0^1 \frac{\omega_1(t)}{t\psi_1(t)} \left(\log \frac{1}{t}\right)^{n-2} dt < \infty \, .$$

The proof of theorem 2 is complete. $\square$

Now, Theorem 1 follows from theorem 2 by the following simple observation

$$\begin{aligned} |I|^{-1} \int_{I \ni x} |f(y) - f(x)| \, dy &\le |I|^{-1} \int_I |f(y_1, \ldots, y_n) - f(y_1, x_2, \ldots, x_n)| \, dy \\ &\quad + |I|^{-1} \int_{I_1} |f(x_1, \ldots, x_n) - f(y_1, x_2, \ldots, x_n)| \, dy_1 \\ &\le \mathcal{M}^1 \left(\mathcal{N}_\phi^{s,1} f\right)(x) \, \phi(\mathrm{diam}\,(I)) + \mathcal{N}_\phi^1 f(x) \, \phi(|I_1|) \end{aligned}$$

where $\mathcal{M}^1$ denotes partial Hardy–Littlewood maximal function at x_1–coordinate, which is finite a.e. if $\mathcal{N}_\phi^{s,1} f \in L^1$.

Let us discuss the sharpness of (7). It is not difficult to prove, that well known Bari–Zygmund condition on $\omega(\delta)$ is equivalent with the following condition: $\delta_k/\delta_{k+1} = O(1)$, where sequence δ_k was generated by (2). So, in this case $\Omega(\delta)$ has the power grow near zero. On the other hand, $\psi(t)$ has the logarithmic behavior, hence the order of $(\psi(\Omega(\delta))$ and $\psi(\delta)$ is identical (this unformal consideration may be rewritten as formal statement). Thus for these modulus, like $\omega(\delta) = \delta^\alpha(\log(1/\delta))^\beta$, $0 < \alpha < 1$, the sharpness of (7) follows from (5).

However, for the logarithmic–type modulus like $\omega(\delta) = (\log(1/\delta))^\alpha$, $\alpha < -1$ with some condition of regularity the estimation (7) is also sharp. This condition is decreasing of $\psi(t)/\omega(t)$ and $\omega(t)t^{-1/2}$. The latter implies that $\omega(\delta_k) = 2^{-k}$.

Further, for the simplicity of notations, we consider two–dimensional case. Then

$$\int_0^1 \frac{\omega(\delta)}{\delta\psi(\delta)}\,d\delta \le \sum_{k=1}^\infty \int_{\delta_k}^{\delta_{k-1}} \frac{\omega(\delta)}{\delta\psi(\delta)}\,d\delta \le \sum_{k=1}^\infty \frac{\omega(\delta_{k-1})}{\psi(\delta_{k-1})} \log\frac{\delta_{k-1}}{\delta_k}\,.$$

Hence

$$\sum_{k=1}^\infty \frac{\omega(\delta_{k-1})}{\psi(\delta_{k-1})} \log\frac{\delta_{k-1}}{\delta_k} = \infty\,.$$

If $\sum \omega(\delta_{k-1})/\psi(\delta_{k-1})$ is divergence, then necessary estimation follows from one–dimensional case. So, let us assume, that

$$\sum_{k=1}^\infty \frac{\omega(\delta_k)}{\psi(\delta_k)} < \infty\,.$$

Denote

$$Q_k[0,\delta_k]^2\,; \quad E_k = \{(x_1,x_2)\,:\,\delta_k \le x_1 \le \delta_{k-1}\,;\ 0 \le x_2\cdot x_1 \le \delta_k\delta_{k-1}\}\,,$$

$$\lambda_k = \psi(\delta_{k-1})\,\frac{\delta_{k-1}}{\delta_k}\,, \quad m_k = \left[\frac{\omega(\delta_{k-1})}{\psi(\delta_{k-1})\delta_{k-1}\delta_k}\right] + 1\,.$$

Then $|E_k| = \delta_k\delta_{k-1}\log(\delta_{k-1}/\delta_k)$ and $m_k|E_k| \ge (\omega(\delta_{k-1})/\psi(\delta_{k-1}))\times$ $\times\log(\delta_{k-1}/\delta_k)$, so

$$\sum_{k=1}^\infty m_k|E_k| = \infty\,.$$

Now by the well known Calderon lemma we can define translations τ_i^k, $i = 1, \ldots, m_k$; $k \geq 0$ such that $|\limsup \tau_i^k(E_k)| = 1$.

Let χ_i^k denote the characteristic function of $\tau_i^k(E_k)$ and define

$$f_k^i = \lambda_k \chi_i^k\,; \quad f = \sum_{k=1}^{\infty} \sum_{i=1}^{m_k} f_k^i\,.$$

Then $\lambda_k m_k \delta_k^2 \leq 2\omega(\delta_{k-1}) = 4\omega(\delta_k)$ and for $\delta_{j-1} < h \leq \delta_j$ and $i = 1, 2$ we have

$$\omega_i(f, h) \ll h \sum_{k=1}^{j} \frac{\lambda_k m_k \delta_k^2}{\delta_k} + \sum_{k=j+1}^{\infty} \lambda_k m_k \delta_k^2 \ll h \sum_{k=1}^{j} \frac{\omega(\delta_k)}{\delta_k} + \sum_{k=j+1}^{\infty} \omega(\delta_k)\,.$$

The latter sum is less than $C\omega(h)$ (see [1]), so $f \in H_1^{\omega}(I^2)$.

Further

$$\sum_{k=1}^{\infty} m_k |Q_k| \ll \sum_{k=1}^{\infty} \frac{\omega(\delta_{k-1})}{\psi(\delta_{k-1})} \frac{\delta_k}{\delta_{k-1}} \ll \sum_{k=1}^{\infty} \frac{\omega(\delta_k)}{\psi(\delta_k)} < \infty\,.$$

Hence $|\limsup \tau_i^k(Q_k)| = 0$.

If $F = \lambda_k \chi_{Q_k}$ then for $x \in E_k \setminus Q_k$ there exists rectangle $I \ni x$ (see Fig. 1) such that $(F(x) = 0)$

$$|I|^{-1} \int_I |F(y) - F(x)|\, dy = \frac{\lambda_k \delta_k^2}{\delta_k \delta_{k-1}} = \psi(\delta_{k-1}) \gg \psi(\operatorname{diam}(I))\,. \tag{9}$$

Let $Q_i^k = \tau_i^k(Q_k)$ and x belong to finite numbers of Q_i^k. Let $x \in I$ and $\operatorname{diam}(I)$ is so small, that either $I \subset Q_i^k$ or $I \cap Q_i^k = \emptyset$ for $k \leq n$, and for $k = n$ there is relation like (9). Then

$$\begin{aligned} |I|^{-1} \int_I |f(y) - f(x)|\, dy \geq \Bigg| & \sum_{k=1}^{n-1} \sum_{i=1}^{m_k} |I|^{-1} \int_I \left[f_k^i(y) - f_k^i(x) \right]\, dy \\ & + \sum_{i=1}^{m_n} |I|^{-1} \int_I \left[f_n^i(y) - f_n^i(x) \right]\, dy \\ & + \sum_{k=n+1}^{\infty} \sum_{i=1}^{m_k} |I|^{-1} \int_I \left[f_k^i(y) - f_k^i(x) \right]\, dy \Bigg| \\ = {} & |\Sigma_1 + \Sigma_2 + \Sigma_3|\,. \end{aligned}$$

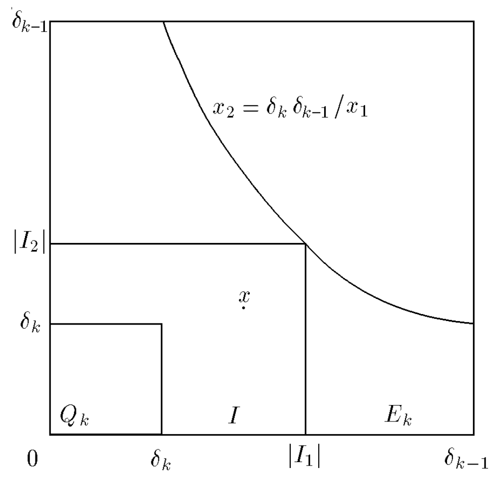

FIG. 1

By the assumption about mutual disposition of I and $\operatorname{supp}(f_k^i)$ for small k, $\Sigma_1 = 0$. By (9) $\Sigma_2 \geq \psi(\operatorname{diam}(I))$, and as $\Sigma_3 \geq 0$, then $\Sigma_1 + \Sigma_2 + \Sigma_3 \geq \psi(\operatorname{diam}(I))$.

Therefore almost everywhere on I^2 $\displaystyle\limsup_{\delta\to 0} \frac{D_\delta(f,x)}{\psi(\delta)} > 0$.

REFERENCES

1. K. I. OSKOLKOV: *Approximation properties of integrable functions on sets of full measure.* Mat. Sb. (N.S.) **103 (145)** (1977), 563–589 (Russian).
2. A. M. STOKOLOS: *On the strong differentiation of integrals of functions in Hölder classes.* Mat. Zametki **55** (1994), 84–104 (Russian).

Odessa University, IMEM
Petra Velikogo 2
270000 Odessa
Ukraine