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Ser. Math. Inform. 14 (1999), 1–10

ON SOME FINITE SUMS WITH FACTORIALS

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Abstract. The summation formula

$$\sum_{i=0}^{n-1} \varepsilon^i i!(i^k + u_k) = v_k + \varepsilon^{n-1} n! A_{k-1}(n)$$

($\varepsilon = \pm 1$; $k = 1, 2, ...$; $u_k, v_k \in \mathbb{Z}$; A_{k-1} is a polynomial) is derived and its various aspects are considered. In particular, divisibility with respect to n is investigated. Infinitely many equivalents to Kurepa's hypothesis on the left factorial are found.

1. Introduction

The subject of the present paper is an investigation of finite sums of the form

$$\sum_{i=0}^{n-1} \varepsilon^i i! P_k(i)\,, \tag{1}$$

where $\varepsilon = \pm 1$, and

$$P_k(i) = C_k i^k + \cdots + C_1 i + C_0 \tag{2}$$

is a polynomial with $k, i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and coefficients $C_0, C_1, \ldots, C_k \in \mathbb{Z}$.

We mainly consider the following three problems of (1): a) summation formula, b) divisibility by $n!$ and c) connection with the Kurepa hypothesis (KH) on the left factorial. All these problems depend on the form of the polynomial $P_k(i)$ and have something in common with it.

In Sec. 2 we find a few ways to determine $P_k(i)$ which give simple and useful summation formulae. Sec. 3 contains divisibility properties. The results concerning KH on the left factorial are given in Sec. 4. Infinitely many equivalents to KH are found.

Received December 20, 1997.

1991 *Mathematics Subject Classification*. Primary 11A05.

2. Summation Formulae

Lemma 1. *Let* $\varepsilon = \pm 1$ *and*

$$A_{k-1}(n) = a_{k-1}n^{k-1} + \cdots + a_1 n + a_0\,, \quad k \in \mathbb{N}\,, \quad n \in \mathbb{N}_0\,, \tag{3}$$

is a polynomial. One can find coefficients $a_{k-1} = 1$ *and* $a_{k-2}, \ldots, a_0 \in \mathbb{Z}$ *such that identity*

$$(n+1)A_{k-1}(n+1) - \varepsilon A_{k-1}(n) = n^k + A_{k-1}(1) - \varepsilon A_{k-1}(0) \tag{4}$$

holds for all $n \in \mathbb{N}_0$.

Proof. Formula (4) has the form

$$(n+1)A_{k-1}(n+1) - \varepsilon A_{k-1}(n) = n^k + u_k\,. \tag{5}$$

Replacing $A_{k-1}(n)$ by (3) and demanding (5) to be an identity, the following system of linear equations must be satisfied:

$$\begin{aligned}
\binom{k}{0} a_{k-1} &= 1\,,\\
\left[\binom{k}{1} - \varepsilon\right] a_{k-1} + a_{k-2} &= 0\,,\\
\binom{k}{2} a_{k-1} + \left[\binom{k-1}{1} - \varepsilon\right] a_{k-2} + a_{k-3} &= 0\,,\\
&\vdots\\
k a_{k-1} + (k-1)a_{k-2} + \cdots + (2-\varepsilon)a_1 + a_0 &= 0\,,\\
a_{k-1} + a_{k-2} + \cdots + a_1 + (1-\varepsilon)a_0 &= u_k\,.
\end{aligned} \tag{6}$$

Starting from the first equation, which gives $a_{k-1} = 1$, one can in a successive way obtain solution for all $a_i = a_i(k, \varepsilon)$, $i = 0, \ldots, k-2$. The last equation in (6) serves to determine u_k. Thus we get

$$u_k = \sum_{i=0}^{k-1} a_i - \varepsilon a_0 = A_{k-1}(1) - \varepsilon A_{k-1}(0)\,. \qquad \square \tag{7}$$

Note that (4) is an identity if and only if the coefficients of the polynomial $A_{k-1}(n)$ satisfy the system of linear equations (6), where u_k is given by (7).

The first five polynomials which satisfy (4) are:

$$
\begin{aligned}
&A_0(n) = 1\,,\\
&A_1(n) = n + \varepsilon - 2\,,\\
&A_2(n) = n^2 + (\varepsilon - 3)n + 4 - 5\varepsilon\,,\\
&A_3(n) = n^3 + (\varepsilon - 4)n^2 + 7(1-\varepsilon)n + 18\varepsilon - 13\,,\\
&A_4(n) = n^4 + (\varepsilon - 5)n^3 + (11 - 9\varepsilon)n^2 + 2(16\varepsilon - 11)n + 58 - 63\varepsilon\,.
\end{aligned}
\tag{8}
$$

Theorem 1. *The summation formula*

$$
\sum_{i=0}^{n-1} \varepsilon^i i!\big[i^k + A_{k-1}(1) - \varepsilon A_{k-1}(0)\big] = -\varepsilon A_{k-1}(0) + \varepsilon^{n-1} n! A_{k-1}(n)
\tag{9}
$$

is valid if and only if the polynomials $A_{k-1}(n)$, $k \in \mathbb{N}$, *satisfy the identity* (4).

Proof. Summation of (4), previously multiplied by $\varepsilon^i i!$, gives

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \varepsilon^i i!\big[i^k + A_{k-1}(1) - \varepsilon A_{k-1}(0)\big]\\
&= \sum_{i=0}^{n-1} \varepsilon^i i!\big[(i+1)A_{k-1}(i+1) - \varepsilon A_{k-1}(i)\big]\,.
\end{aligned}
\tag{10}
$$

Since on the r. h. s. all but the first and the last term cancel, we get (9). Now one can easily show that starting from (9) one obtains (4). □

Denoting $u_k = A_{k-1}(1) - \varepsilon A_{k-1}(0)$, $v_k = -\varepsilon A_{k-1}(0)$ we can rewrite (9) in the form

$$
\sum_{i=0}^{n-1} \varepsilon^i i!(i^k + u_k) = v_k + \varepsilon^{n-1} n! A_{k-1}(n)\,, \quad k \geq 1\,.
\tag{11}
$$

Formula (9), as well as (11), is determined by polynomial $A_{k-1}(n)$ in (3), whose coefficients are solution of (6). However, for large k, (6) becomes inconvenient. Therefore, it is of interest to have another approach which is more effective to get (11).

Theorem 2. *If δ_{0k} is the Kronecker symbol and*

$$S_k^{\varepsilon}(n) = \sum_{i=0}^{n-1} \varepsilon^i i! i^k \,, \quad \varepsilon = \pm 1 \,, \quad k \in \mathbb{N}_0 \,, \tag{12}$$

then

$$S_k^{\varepsilon}(n) = \delta_{0k} + \varepsilon \sum_{l=0}^{k+1} \binom{k+1}{l} S_l^{\varepsilon}(n) - \varepsilon^n n! n^k \,, \quad k \in \mathbb{N}_0 \,, \tag{13}$$

is a recurrent relation.

Proof. We have

$$\begin{aligned} S_k^{\varepsilon}(n) &= \delta_{0k} + \sum_{i=0}^{n-2} \varepsilon^{i+1} (i+1)! (i+1)^k \\ &= \delta_{0k} + \varepsilon \sum_{i=0}^{n-1} \varepsilon^i i! (i+1)^{k+1} - \varepsilon^n n! n^k \\ &= \delta_{0k} + \varepsilon \sum_{l=0}^{k+1} \binom{k+1}{l} S_l^{\varepsilon}(n) - \varepsilon^n n! n^k \,. \qquad \square \end{aligned}$$

Relation (13) gives a simpler way to find (11) in the explicit form for a particular index $k \geq 0$.

From (13) one can obtain recurrent relations for u_k, v_k and $A_{k-1}(n)$. In particular, when $\varepsilon = 1$, we have

$$u_{k+1} = -k u_k - \sum_{l=1}^{k-1} \binom{k+1}{l} u_l + 1 \,, \quad u_1 = 0 \,, \quad k \geq 1 \,, \tag{13.a}$$

$$v_{k+1} = -k v_k - \sum_{l=1}^{k-1} \binom{k+1}{l} v_l - \delta_{0k} \,, \quad k \geq 0 \,. \tag{13.b}$$

Some first values of u_k and v_k ($\varepsilon = 1$) are:

k	1	2	3	4	5	6	7	8	9	10	11
u_k	0	1	-1	-2	9	-9	-50	267	-413	$-2\,180$	$17\,731$
v_k	-1	1	1	-5	5	21	-105	141	777	$-5\,513$	$13\,209$

As an illustration of the above summation formulae, the first four examples ($\varepsilon = 1$) are:

$$\sum_{i=0}^{n-1} i!i = -1 + n!\,, \tag{14.a}$$

$$\sum_{i=0}^{n-1} i!(i^2+1) = 1 + n!(n-1)\,, \tag{14.b}$$

$$\sum_{i=0}^{n-1} i!(i^3-1) = 1 + n!(n^2-2n-1)\,, \tag{14.c}$$

$$\sum_{i=0}^{n-1} i!(i^4-2) = -5 + n!(n^3-3n^2+5)\,. \tag{14.d}$$

Note that $i^k + u_k$ in (11) is a polynomial $P_k(i)$ in (2) in a reduced form and suitable for generalization. Namely, (11) can be generalized to

$$\sum_{i=0}^{n-1} \varepsilon^i i! P_k(i) = V_k + \varepsilon^{n-1} n! B_{k-1}(n)\,, \quad k \geq 1\,, \tag{15}$$

where $P_k(i) = \sum\limits_{r=0}^{k} C_r i^r$ with

$$C_0 = \sum_{r=1}^{k} C_r u_r\,, \quad V_k = \sum_{r=1}^{k} C_r v_r\,, \quad B_{k-1}(n) = \sum_{r=1}^{k} C_r A_{r-1}(n)$$

and $C_1, \dots, C_k \in \mathbb{Z}$. Polynomials $P_k(i)$ which do not have the above form do not yield (15).

3. Divisibility

The above results enable us to investigate some divisibility properties of $\sum\limits_{i=0}^{n-1} \varepsilon^i i! P_k(i)$ with respect to all factors contained in $n!$. According to (15) we have that $\sum\limits_{i=0}^{n-1} \varepsilon^i i! P_k(i)$ and V_k are equally divisible with respect to factors of $n!$, as well as those of $B_{k-1}(n)$.

Proposition 1. *If the polynomial $A_{k-1}(n)$ satisfies the identity* (4) *then we have the following congruence*

$$\sum_{i=0}^{n-1} \varepsilon^i i! \big[i^k + A_{k-1}(1) - \varepsilon A_{k-1}(0)\big] \equiv -\varepsilon A_{k-1}(0) \pmod{n!}\,. \tag{16}$$

Proof. Congruence (16) is a direct consequence of (9). □

From (16) it follows

$$\sum_{i=0}^{n-1} \varepsilon^i i! i^k \equiv -[A_{k-1}(1) - \varepsilon A_{k-1}(0)] \sum_{i=0}^{n-1} \varepsilon^i i! - \varepsilon A_{k-1}(0) \pmod{n}$$

and this property can be used to simplify numerical investigation of divisibility of $\sum\limits_{i=0}^{n-1} \varepsilon^i i! i^k$ by n.

There is a simple example of (16), *e.g.*

$$\sum_{i=0}^{n-1} i! i \equiv -1 \pmod{n!}\,, \tag{17}$$

what follows from (14.a).

Proposition 2. *The following statements are valid:*

$$\begin{gathered} \sum_{i=0}^{n-1} i! i \not\equiv 0 \pmod{n}\,, \quad n > 1\,, \\ \sum_{i=0}^{p-1} i! i \not\equiv 0 \pmod{p}\,, \quad p \in P\,, \\ \Big(\sum_{i=0}^{n-1} i! i, n!\Big) = 1\,, \quad n > 1\,, \end{gathered} \tag{18}$$

where (a, b) denotes the greatest common divisor of $a, b \in \mathbb{Z}$, and P is the set of prime numbers.

Proof. Every of equation in (18) follows from (14.a). One can also show that these statements are equivalent. □

Due to (16) divisibility of $\sum\limits_{i=0}^{n-1} \varepsilon^i i! i^k$, $k \geq 1$, by factors of $n!$ is in some relation to divisibility of $\sum\limits_{i=0}^{n-1} \varepsilon^i i!$ except for the case $A_{k-1}(1) = \varepsilon A_{k-1}(0)$.

4. On Kurepa's Hypothesis

Kurepa in [1] introduced a hypothesis

$$(!n, n!) = 2\,, \quad 2 \le n \in \mathbb{N}\,, \tag{19}$$

where

$$!n = \sum_{i=0}^{n-1} i! \tag{20}$$

has been called the left factorial. In spite of many papers (for a review see [2] and references therein) on KH it is still an open problem in number theory ([3]). Many equivalent statements to KH have been obtained (for some of them see [4]). Among very simple assertions equivalent to (19) are ([1]):

$$\begin{aligned} &!n \not\equiv 0 \pmod{n}\,, \quad n > 2\,, \\ &!p \not\equiv 0 \pmod{p}\,, \quad p > 2\,. \end{aligned} \tag{21}$$

KH is verified by computer calculations (see [2]) for $n < 2^{23}$ ([5]).

The above obtained summation formulae give us possibility to introduce infinitely many new statements equivalent to KH. The first three of them, which follow from (14), are:

$$\begin{aligned} &\sum_{i=0}^{p-1} i!i^2 \not\equiv 1 \pmod{p}\,, \quad p > 2\,, \\ &\sum_{i=0}^{p-1} i!i^3 \not\equiv 1 \pmod{p}\,, \quad p > 2\,, \\ &\sum_{i=0}^{p-1} i!i^4 \not\equiv -5 \pmod{p}\,, \quad p > 2\,. \end{aligned} \tag{22}$$

Theorem 3. *If u_k and v_k satisfy* (13.a) *and* (13.b) *then*

$$\sum_{i=0}^{p-1} i!i^k \not\equiv v_k \pmod{p}\,, \quad p > 2\,, \tag{23}$$

is equivalent to KH for such $k \in \mathbb{N}$ for which u_k is not divisible by p.

Proof. Consider (11) for $\varepsilon = 1$ and $n = p$. According to KH one has $u_k \sum_{i=0}^{p-1} i! \not\equiv 0 \pmod p$ for $p > 2$ and p which does not divide $u_k \neq 0$. For such primes p it holds (23). □

Starting from the Fermat little theorem, i.e. $i^{p-1} = 1$ in the Galois field GF(p) if $i = 1, 2, \dots, p-1$, one can easily show that assertion

$$\sum_{i=0}^{p-1} i! i^{r(p-1)} \not\equiv -1 \pmod p\,, \quad p > 2\,, \quad r \in \mathbb{N} \tag{24}$$

is equivalent to KH. This can be regarded as a special case of the Theorem 3. Since r may be any positive integer it means that there are infinitely many equivalents to KH.

Note that on the basis of Fermat's theorem one can also obtain

$$\sum_{i=0}^{p-1} i! i^{k+r(p-1)} = \sum_{i=0}^{p-1} i! i^k - \delta_{0k}\,, \quad k \in \mathbb{N}_0\,, \quad r \in \mathbb{N}\,. \tag{25}$$

Combining (11) and (25) we find in GF(p):

$$u_{k+r(p-1)} = u_k\,, \quad v_{k+r(p-1)} = v_k\,, \quad k, r = 0, 1, 2, \dots\,. \tag{26}$$

Proposition 3. *If u_k and v_k satisfy* (13.a) *and* (13.b)*, respectively, the following relations in GF(p) are valid:*

$$(u_{p-1} + 1) \sum_{i=0}^{p-1} i! = v_{p-1} + 1\,, \tag{27.a}$$

$$u_p \sum_{i=0}^{p-1} i! = v_p + 1\,, \tag{27.b}$$

$$(u_{p+1} - 1) \sum_{i=0}^{p-1} i! = v_{p+1} - 1\,, \tag{27.c}$$

$$(u_{p+2} + 1) \sum_{i=0}^{p-1} i! = v_{p+2} - 1\,. \tag{27.d}$$

Proof. One can start from (11), then use (25) and (14). □

From equations (13.a), (13.b) and (26) one obtains in GF(p):

$$\begin{aligned} (u_{p+2}, v_{p+2}) &= (-u_p - 1, -v_p)\,, \\ (u_{p+1}, v_{p+1}) &= (1, 1)\,, \\ (u_p, v_p) &= (u_{p-1} + 1, v_{p-1}) = (0, -1)\,. \end{aligned}$$

Thus (27.a)–(27.d) are equivalent identities which are always satisfied owing to the values of u_k and v_k and they do not depend on validity of KH.

5. Concluding Remarks

It is worth noting that for every $k \in \mathbb{N}$ there is a unique pair (u_k, v_k) of integers u_k and v_k which connect $\sum_{i=0}^{n-1} \varepsilon^i i! i^k$ and $\sum_{i=0}^{n-1} \varepsilon^i i!$ into simple summation formula (11). All other results of the present paper are mainly various consequences of this fact.

Formula (11) is also suitable to consider its limit when $n \to \infty$ in p–adic analysis. Namely, since $|n!|_p \to 0$ as $n \to \infty$, one obtains

$$\sum_{i=0}^{\infty} \varepsilon^i i!(i^k + u_k) = v_k\,,$$

valid in $\mathbb{Q}_p$ for every p. Some p–adic aspects of the series $\sum_{i=0}^{\infty} \varepsilon^i i! P_k(i)$ and their possible role in theoretical physics are considered in [6].

Having infinitely many new equivalents, Kurepa's hypothesis becomes more challenging. Moreover, KH itself seems to be the simplest among all its equivalents. In p–adic case KH can be also formulated as follows:

$$\sum_{i=0}^{\infty} i! = a_0 + a_1 p + a_2 p^2 + \cdots\,, \quad p \in P\,,$$

where a_i are definite digits with $a_0 \neq 0$ for all $p \neq 2$.

Acknowledgment: The author thanks Ž. Mijajlović for useful discussions.

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