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LEAST SQUARES APPROXIMATION OF MULTI-PORT NETWORK PARAMETERS BY RATIONAL FUNCTIONS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. The objective of this paper is to present a simple least squares approximation procedure of measured values of L–port network Z–parameters by rational functions. The polynomial orders of the numerator and denominator are chosen arbitrary or could be obtained by circuit topology. All Z–parameters have the same polynomials in the denominator. This procedure does not include computation of derivatives. The computation of polynomials in the numerator and denominator of rational functions is fully described on the passive symmetric two–port network. The procedure could be appropriate for the multi–port network element extraction.

1. Introduction

The approximation of measured L–port network parameters has attracted researchers attention for more then a decade. In paper [1], a simple procedure used for independent two–port network parameter optimization by rational function is proposed. Polynomials in the denominator of parameters, optimized with this procedure, are not equal.

L. Tiele [5] has developed the approximation procedure for measured two–port network S–parameters by rational functions in transform plane which has the same polynomial in denominator. This procedure is based on Padé's approximation. However, polynomials in the numerator and denominator have the same order, and this procedure can not be useful for equivalent circuit element extraction. In paper [4] a complicated approximation method for transistor S–parameters, involves computation of derivatives. Polynomial in the denominator and numerator is empirically obtained. In the recently

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released public paper [2], an approximation procedure for the measured values of transistor S–parameters is presented. This procedure is primarily developed for equivalent circuit element extraction, but not for the approximation of the measured values of Z–parameters by rational function.

The main aim of this paper is to present a simple least square approximation procedure for measured values of L–port network Z–parameters by rational functions. The polynomial orders of the numerator and denominator are chosen arbitrary or could be obtained by network topology. This procedure does not include computation of derivatives and all rational functions have the same coefficients in denominator.

The procedure for the computation of polynomials in the numerator and denominator of rational functions is fully described on the passive symmetric two–port network. Finally, it was shown that this procedure could be appropriate for two–port network element extraction.

2. Approximation

Let n be a number of the complex measured values for every one of L^2 network parameters on angular frequencies $\omega_k,\ k = 1, 2, \dots, n$,

$$\hat{Z}_{ij}(\omega_k) = R_{ij}(\omega_k) + jX_{i,j}(\omega_k), \quad i, j = 1, .., L \quad k = 1, \dots, n. \tag{1}$$

This measured values of Z–parameters are approximated by rational functions

$$Z_{ij}(s) = \frac{A_0^{(i,j)} + A_1^{(i,j)} s + \cdots + A_{N_{i,j}}^{(i,j)} s^{N_{i,j}}}{1 + B_1 s + \cdots + B_M s^M}, \quad i, j = 1, \dots, L, \tag{2}$$

where $N_{i,j}, M$ are known orders of the numerator and denominator, $s = \sigma + j\omega$ is complex frequency, and $n > M + N_{i,j}$.

Equation (2) could be separated on the real and imaginary part at real frequency $s = j\omega$

$$\begin{aligned} Z_{i,j}(\omega) = {} & \frac{a_0^{(i,j)} + a_2^{(i,j)}\omega^2 + a_4^{(i,j)}\omega^4 + \cdots + a_{2\lfloor \nu_{i,j}\rfloor}^{i,j}\omega^{2\lfloor \nu_{i,j}\rfloor}}{1 + b_1\omega^2 + b_2\omega^4 + \cdots + b_M\omega^{2M}} \\ & + j\frac{a_1^{(i,j)}\omega + a_3^{(i,j)}\omega^3 + \cdots a_{2\lceil \nu_{i,j}\rceil - 1}^{(i,j)}\omega^{2\lceil \nu_{i,j}\rceil - 1}}{1 + b_1\omega^2 + b_2\omega^4 + \cdots + b_M\omega^{2M}} \end{aligned} \tag{3}$$

where

$$\nu_{i,j} = \frac{M + N_{i,j}}{2},$$

$$a_k = \sum_{\mu=0}^{k} (-1)^{\lfloor \frac{k}{2} \rfloor + \mu} A_{k-\mu} B_\mu; \quad k = 1, \dots, n, \tag{4}$$

and

$$b_k = B_k^2 + \sum_{\mu=0}^{k-1} 2(-1)^{k-\mu} B_\mu B_{2k-\mu}; \quad k = 1, \dots, n. \tag{5}$$

for $B_0 = 1$.

The operator $\lceil x \rceil$–rounds x to the nearest integer towards infinity, and $\lfloor x \rfloor$–rounds x to the nearest integer towards minus infinity.

First, we divide the imaginary and real part of (3) and equalize the result with the quotient of the imaginary and real part of measured values

$$\frac{a_1^{(i,j)} \omega_k + a_3^{(i,j)} \omega_k^3 + \cdots + a_{2\lceil \nu_{i,j} \rceil - 1}^{(i,j)} \omega_k^{2\lceil \nu_{i,j} \rceil - 1}}{a_0^{(i,j)} + a_2^{(i,j)} \omega_k^2 + a_4^{(i,j)} \omega_k^4 + \cdots + a_{2\lfloor \nu_{i,j} \rfloor}^{i,j} \omega_k^{2\lfloor \nu_{i,j} \rfloor}} = p_k^{(i,j)}, \tag{6}$$

for $k = 1, \dots, n$, where

$$p_k^{(i,j)} = \frac{X_{i,j}(\omega_k)}{R_{i,j}(\omega_k)}, \quad k = 1, \dots, n,$$

for $R_{i,j}(\omega_k) \neq 0$. If $R_{i,j}(\omega_k) = 0$ to avoid dividing by zero, we divide real and imaginary part of Eqn. (3)

$$p_k^{(i,j)} = \frac{R_{i,j}(\omega_k)}{X_{i,j}(\omega_k)}, \quad k = 1, \dots, n.$$

Equation (6) contains only numerator coefficients of the real and imaginary part of network parameter $Z_{i,j}(\omega)$.

After polynomial coefficient normalization with $a_0^{(i,j)}$, both in the numerator and denominator, (6) can be rewritten as

$$\frac{\alpha_1^{(i,j)} \omega_k + \alpha_3^{(i,j)} \omega_k^3 + \cdots + \alpha_{2\lceil \nu_{i,j} \rceil - 1}^{(i,j)} \omega_k^{2\lceil \nu_{i,j} \rceil - 1}}{1 + \alpha_2^{(i,j)} \omega_k^2 + \alpha_4^{(i,j)} \omega_k^4 + \cdots + \alpha_{2\lfloor \nu_{i,j} \rfloor}^{i,j} \omega_k^{2\lfloor \nu_{i,j} \rfloor}} = p_k^{(i,j)}, \tag{7}$$

for $k = 1, \dots, n$. Unknown coefficients $\alpha_1^{(i,j)}, \alpha_2^{(i,j)}, \dots, \alpha_{M+N_{i,j}}^{(i,j)}$ of network parameters $Z_{i,j}$ are determined by solving following linear system of $n > N_{i,j} + M$ equations

$$\alpha_1^{(i,j)} \omega_k - \alpha_2^{(i,j)} p_k^{(i,j)} \omega_k^2 + \cdots + \alpha_\eta^{(i,j)} \omega_k^\eta (-p_k^{(i,j)})^{\frac{1+(-1)^\eta}{2}} = p_k^{(i,j)}, \tag{8}$$

where $\eta = \max(2\lfloor \nu_{i,j} \rfloor, 2\lceil \nu_{i,j} \rceil - 1)$ and $k = 1, \dots, n$.

The result of the over-determined linear system (8) for every given pair (i, j), in the least squares sense is

$$[\alpha] = ([S]^{\mathrm{T}}[S])^{-1}[S]^{\mathrm{T}}[p], \tag{9}$$

where

$$[S] = \begin{bmatrix} \omega_1 & -\omega_1^2 p_1^{(i,j)} & \dots & \omega_1^\eta (-p_1^{(i,j)})^{\frac{1+(-1)^\eta}{2}} \\ \omega_2 & -\omega_2^2 p_2^{(i,j)} & \dots & \omega_2^\eta (-p_2^{(i,j)})^{\frac{1+(-1)^\eta}{2}} \\ \vdots & & & \\ \omega_n & -\omega_n^2 p_n^{(i,j)} & \dots & \omega_n^\eta (-p_n^{(i,j)})^{\frac{1+(-1)^\eta}{2}} \end{bmatrix}$$

and

$$[\alpha] = \begin{bmatrix} \alpha_1^{(i,j)} & \alpha_2^{(i,j)} & \dots & \alpha_{N_{i,j}+M}^{(i,j)} \end{bmatrix}^T, \; [p] = \begin{bmatrix} p_1^{(i,j)} & p_2^{(i,j)} & \dots & p_n^{(i,j)} \end{bmatrix}^T.$$

Repeating this procedure L^2 times, we obtain coefficients $\alpha_1^{(i,j)}$, $\alpha_2^{(i,j)}$, $\dots$, $\alpha_{N_{i,j}+M}^{(i,j)}$ for all L^2 network parameters.

In the second step of approximation one can obtain the denominator coefficients $b_1, b_2, \dots, b_M$ and coefficients $a_0^{(i,j)}$ for every one of L^2 network parameters. Equalizing absolute values of network parameter (3) with absolute measured values (1), we obtain linear system of equations

$$\frac{\sqrt{u_{i,j}^2(\omega_k) + v_{i,j}^2(\omega_k)}}{1 + b_1\omega_k^2 + b_2\omega_k^4 + \dots + b_M\omega_k^{2M}} = \frac{q_k^{(i,j)}}{a_0^{(i,j)}}, \quad k = 1, \dots, n, \tag{10}$$

where

$$q_k^{(i,j)} = \sqrt{R_{i,j}^2(\omega_k) + X_{i,j}^2(\omega_k)}$$

and

$$u_{i,j}(\omega_k) = 1 + \dots + \alpha_{2\lfloor \nu_{i,j} \rfloor}^{i,j} \omega_k^{2\lfloor \nu_{i,j} \rfloor},$$
$$v_{i,j}(\omega_k) = \alpha_1^{(i,j)}\omega_k + \dots + \alpha_{2\lceil \nu_{i,j} \rceil - 1}^{(i,j)} \omega_k^{2\lceil \nu_{i,j} \rceil - 1}.$$

The coefficients $\alpha_1^{(i,j)}, \alpha_2^{(i,j)}, \dots, \alpha_{N_{i,j}+M}^{(i,j)}$ are computed in the previous step. The denominator coefficients $b_1, b_2, \dots, b_M$ and coefficients $a_0^{(i,j)}, i, j = 1, \dots, L$, are determined by solving linear system of $nL^2 > M + L^2$ equations

$$b_1\omega_k^2 + b_2\omega_k^4 + \dots + b_M\omega_k^{2M} + a_0^{(i,j)} h_k^{(i,j)} = -1, \tag{11}$$

for $i, j = 1, \dots, L$ and $k = 1, \dots, n$, where

$$h_k^{(i,j)} = -\frac{\sqrt{u_{i,j}^2(\omega_k) + v_{i,j}^2(\omega_k)}}{q_k^{(i,j)}} \qquad (k = 1, \dots, n). \tag{12}$$

The result of (11) is

$$[b] = ([H]^T[H])^{-1}[H][J], \tag{13}$$

where

$$[H] = \begin{bmatrix} \omega_1^2 & \dots & \omega_1^{2M} & h_1^{(1,1)} & 0 & \dots & 0 \\ \omega_1^2 & \dots & \omega_1^{2M} & 0 & h_1^{(1,2)} & \dots & 0 \\ \vdots & & & & & & \\ \omega_1^2 & \dots & \omega_1^{2M} & 0 & 0 & \dots & h_1^{(L,L)} \\ \omega_2^2 & \dots & \omega_2^{2M} & h_2^{(1,1)} & 0 & \dots & 0 \\ \omega_2^2 & \dots & \omega_2^{2M} & 0 & h_2^{(1,2)} & \dots & 0 \\ \vdots & & & & & & \\ \omega_2^2 & \dots & \omega_2^{2M} & 0 & 0 & \dots & h_2^{(L,L)} \\ \vdots & & & & & & \\ \omega_n^2 & \dots & \omega_n^{2M} & 0 & 0 & \dots & h_n^{(L,L)} \end{bmatrix},$$

$$[b] = \begin{bmatrix} b_1 & b_2 & \dots & b_M & a_0^{(1,1)} & a_0^{(1,2)} & \dots & a_0^{(L,L)} \end{bmatrix}^T,$$

and

$$[J] = \begin{bmatrix} -1 & -1 & \dots & -1 \end{bmatrix}^T.$$

The real and imaginary parts (3) of rational functions (2) are obtained by computing coefficients $b_1, b_2, \dots, b_M$ and coefficients $a_0^{(i,j)}$, $i, j = 1, \dots, L$.

$$a_k^{(i,j)} = a_0^{(i,j)} \alpha_k^{(i,j)}, \quad k = 1, 2, \dots, N_{i,j} + M.$$

In the third step of approximation we compute coefficients B_i and $A_k^{i,j}$.

The coefficients B_i in the denominator of the rational functions (1), are computed by solving nonlinear system (5). This system consists of M equations with M unknowns.

After computing coefficients B_i, it is possible to compute numerator coefficients $A_k^{i,j}$ by solving linear system (4). This system of equations is over-determined, so we can use least square method

$$[A] = ([B]^T[B])^{-1}[B]^T[a], \tag{14}$$

where

$$[B] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -B_1 & 1 & \dots & 0 \\ -B_2 & B_1 & \dots & 0 \\ \vdots & & & \\ (-1)^{\lfloor \frac{M+1}{2} \rfloor} B_M & (-1)^{\lfloor \frac{M-1}{2} \rfloor} B_{M-1} & \dots & (-1)^{\lfloor \frac{-M+1}{2} \rfloor} \end{bmatrix},$$

and

$$\begin{aligned} [A] &= [A_0^{(i,j)} A_1^{(i,j)} A_2^{(i,j)} \dots A_N^{(i,j)}]^T, \\ [a] &= [a_0^{(i,j)} a_1^{(i,j)} a_2^{(i,j)} \dots a_N^{(i,j)}]^T. \end{aligned} \tag{15}$$

Repeating (14) for every one of L–port network Z–parameters, approximation procedure is completed.

3. Example

We will perform the above procedure on the symmetric two–port network (L=2), shown on Figure 1. RC circuit element values are not known.

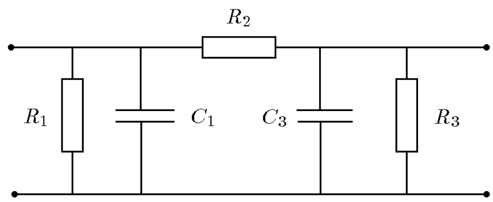

FIG. 1. The symmetric two–port network

The symmetric two–port network satisfies the conditions $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$. Thus, we will approximate only $Z_{11} = R_{11} + jX_{11}$ and $Z_{12} = R_{12} + jX_{12}$ parameters. The complex normalized measured values of parameters Z_{11} and Z_{12}, on the normalized frequencies f are shown in Table 1.

Table 1:

f	R_{11}	X_{11}	R_{12}	X_{12}
0.01	0.748	-0.039	0.249	-0.023
0.05	0.700	-0.180	0.211	-0.104
0.09	0.610	-0.270	0.147	-0.147
0.13	0.515	-0.330	0.085	-0.157
0.17	0.426	-0.354	0.039	-0.145
0.21	0.355	-0.355	0.008	-0.126
0.25	0.297	-0.347	-0.010	-0.104
0.29	0.253	-0.336	-0.020	-0.086
0.33	0.215	-0.320	-0.026	-0.070
0.37	0.184	-0.304	-0.028	-0.057
0.41	0.159	-0.289	-0.028	-0.047
0.45	0.139	-0.279	-0.027	-0.039

The Z–parameters of two–port network, displayed on Figure 1, coritten in the form

$$Z_{11}(s) = \frac{A_0^{(11)} + A_1^{(11)} s}{1 + B_1 s + B_2 s^2}, \qquad Z_{12}(s) = \frac{A_0^{(12)}}{1 + B_1 s + B_2 s^2},$$

where $N_{11} = 1$, $N_{12} = 0$ and $M = 2$. Computing (9), one can obtain values for coefficients $\alpha_k^{(i,j)},\ k = 1, 2, \ldots, N_{i,j} + M$.

$$\begin{aligned} \alpha_1^{(11)} &= -0.8215; & \alpha_1^{(12)} &= -1.5174; \\ \alpha_2^{(11)} &= 0.3656; & \alpha_2^{(12)} &= -0.4998; \\ \alpha_3^{(11)} &= -0.2404; & & \end{aligned}$$

In the second step of approximation, using (12), we compute values for $h^{(i,j)}$. Coefficients $a_0^{(i,j)}$ and coefficients $b_i,\ i = 1, 2$ are computed by solving (10).

After de-normalization of coefficients $\alpha_1^{(1,1)}, \alpha_2^{(1,1)}$, $\alpha_3^{(1,1)}$ with $a_0^{(1,1)}$ and coefficients $\alpha_1^{(1,2)}$, $\alpha_2^{(1,2)}$ with $a_0^{(1,2)}$, one obtain values for the real and imaginary part of the numerator coefficients of parameters Z_{11} and Z_{12}.

$$Z_{11}(\omega) = \frac{0.7817 + 0.2858\omega^2}{1 + 1.0519\omega^2 + 0.2016\omega^4} - j\frac{0.6422\omega + 0.1879^{(1,1)}\omega^3}{1 + 1.0519\omega^2 + 0.2016\omega^4}$$

and

$$Z_{12}(\omega) = \frac{0.2036 - 0.1018\omega^2}{1 + 1.0519\omega^2 + 0.2016\omega^4} - j\frac{0.309\omega}{1 + 1.0519\omega^2 + 0.2016\omega^4}.$$

Solving (4) i (5), we finally obtain coefficients for Z–parameter rational functions

$$Z_{11}(s) = \frac{0.7817 + 0.4516s}{1 + 1.3964s + 0.449s^2}, \qquad Z_{12}(s) = \frac{0.2165}{1 + 1.3964s + 0.449s^2}.$$

Figure 2 shows the real and imaginary part of the measured and approximated values of the two–port network parameters.

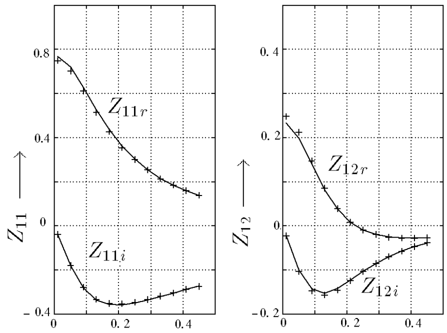

FIG. 2. The measured and approximated values of two–port network "Z"–parameters

Analyzing the Figure 2, it can be concluded that it is possible to approximate measured values of rational function parameters with procedure described above.

Coefficients obtained with this procedure could be further used in two–port network element extraction. There are 5 coefficients and 3 unknown elements of the two–port network. We have nonlinear and over-determined system of equations. The values of the network elements could be obtained from the computed coefficients using least squares method and by solving nonlinear system of equations.

4. Conclusions

In this paper, a simple approximation procedure of the measured values of Z–parameters by rational functions is presented. This procedure is based on the least squares approximation method. Orders of the numerator and denominator are obtained by circuit topology. All Z–parameters have the same polynomial in the denominator. As an example, the measured values of two–port network Z–parameters are approximated. Finally, this procedure could be appropriate for the multi–port network element extraction.

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