# DERIVATIVES OF THE MASS MOMENT VECTORS AT THE DIMENSIONAL COORDINATE SYSTEM $N$ 

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This paper is dedicated to Professor D. S. Mitrinović


#### Abstract

In the previous papers [2-5] the mass moments vectors for the pole and the axis are introduced by definitions. By using these vectors we introduced vector method for mass moment state analysis in the referential point of the body or space. In certain papers (for example, see [6-10], we pointed out that these vectors can be used for qualitative analysis of the kinetic parameters properties of the rotors dynamic as well as of the bearing kinetic pressures of the shaft. In this paper the "support" vectors of the body mass linear moment as well as of the body mass inertia moment for the pole $O$ and axis oriented by unit vector $\vec{n}$ are introduced. In this paper some knowledge about change (rate) in time and time derivatives of the body mass linear moment vectors and body mass inertia moment vectors for the pole and axis for the different properties of the body are pointed out. Body is observed for following cases: a) body is rigid and when body is rotated with angular velocity around fixed axis; b) body is with rigid structure configuration but with changeable body mass in these structure configuration; c) body is with changeable structure configuration as well as with changeable body mass in these structure configuration. This paper gives the time derivatives of the material body mass inertia moment vectors at the point and for the axis at dimensional curvillinear coordinate system N . By using mass moment vectors and their derivatives, the linear momentum and angular momentum of the rotor which rotates around one or two rotation axes are expressed simpler then the other ways, as it was shown in the paper. That fact is the main reason which for simpler qualitative analysis kinetic properties of rotor dynamic and their kinetic pressures on the shaft bearings.


[^0]
## 1. Introduction

This part introduces the vectors (see $[2-5]$ ): $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, and $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, at the dimensional curvillinear coordinate system N.

1* Vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the particle mass static (linear) moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$, in the form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{\text { def }}{=}[\vec{n}, \vec{\rho}] m \tag{*}
\end{equation*}
$$

where $\vec{\rho}$ is the vector of the mass particle position of the particle's mass $m$ with respect to the common pole $O$.
$2^{*}$ Vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of he particle mass inertia moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$ :

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{\text { def }}{=}[\vec{\rho},[\vec{n}, \vec{\rho}]] m \tag{*}
\end{equation*}
$$

$3^{*}$ Vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$ in the form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{\text { def }}{=} \iiint_{V}[\vec{n}, \vec{\rho}] d m, \quad d m=\sigma d V \tag{1}
\end{equation*}
$$

where $\rho$ is the vector of the rigid body points position of the elementary body mass $d m$ with respect to the common pole $O$. The illustration is given in the Figure 1.

4* Vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$ :

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{\text { def }}{=} \iiint_{V}[\vec{\rho},[\vec{n}, \vec{\rho}]] d m \tag{2}
\end{equation*}
$$

It can also be considered the body mass square moment vector at the point $O$ for the axis, through the pole, oriented by the unit vector $\vec{n}$.

## 2. The Dimensional Curvilinear Coordinate System $N$

According to the notation in the Fig. 1 the material point position vector $\vec{\rho}$ at the dimensional coordinate system $N$, can be written in the form:

$$
\begin{equation*}
\vec{\rho}=x^{k} \vec{g}_{k} \tag{3}
\end{equation*}
$$

while unit vector $\vec{n}$ of the axis orientation can be written in the form:

$$
\begin{equation*}
\vec{n}=\lambda^{k} \vec{g}_{k} \tag{4}
\end{equation*}
$$

In the previous expression $\vec{g}_{k}$ the basic vectors of the dimensional $N$ of the curvilinear coordinates $\vec{g}_{k}=\frac{\partial \vec{\rho}}{\partial x^{k}}$ for these vectors it stands that:

$$
\begin{equation*}
\left(\vec{g}_{k}, \vec{g}_{l}\right)=g_{k l} \tag{5}
\end{equation*}
$$

their product represents the metric tensor coordinates of the defined curvilinear coordinates system space. The position vector $\vec{\rho}$ magnitude squared is: $(\vec{\rho}, \vec{\rho})=\left(\vec{g}_{k}, \vec{g}_{l}\right) x^{k} x^{l}=g_{k l} x^{k} x^{l}$, while for the axis orientation unit vector $\vec{n}:(\vec{n}, \vec{n})=\left(\vec{g}_{k}, \vec{g}_{l}\right) \lambda^{k} \lambda^{l}=1$. By using previous expression we can write the following derivatives:

$$
\begin{align*}
\frac{\partial g_{j k}}{\partial x^{i}} & =\Gamma_{j i, k}+\Gamma_{k i, j}, \quad \frac{\partial \vec{g}_{i}}{\partial x^{k}}=\frac{\partial \vec{g}_{k}}{\partial x^{i}}=\Gamma_{i k}^{j} \vec{g}_{j}  \tag{6}\\
\Gamma_{i j, k} & =\Gamma_{j i, k}=[i j, k]=[j i, k]=\frac{1}{2}\left[\frac{\partial g_{j k}}{\partial x^{i}}+\frac{\partial g_{k i}}{\partial x^{j}}-\frac{\partial g_{i j}}{\partial x^{k}}\right] \\
\Gamma_{i j, k} & =\Gamma_{j i, k}=g_{l k} \Gamma_{j i}^{l}, \quad \Gamma_{j k}^{i}=g^{i l} \Gamma_{j k, l}  \tag{8}\\
\frac{d \vec{g}_{k}}{d t} & =\Gamma_{i k}^{j} \vec{g}_{j} \dot{x}^{i}+\frac{\partial \vec{g}_{k}}{\partial t}  \tag{9}\\
\frac{d g_{k p}}{d t} & =\frac{\partial g_{k p}}{\partial x^{i}} \dot{x}^{i}+\frac{\partial g_{k p}}{\partial t}=\left(\Gamma_{i k, p}+\Gamma_{i p, k}\right) \dot{x}^{i}+\frac{\partial g_{k p}}{\partial t} \tag{10}
\end{align*}
$$

## 3. The Material Particle Mass Inertia Moment Vector for the Pole and the Axis

By introducing the expression (3) and (4) into expression (2*) for the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the material particle mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}=\left[\vec{g}_{k},\left[\vec{g}_{l}, \vec{g}_{p}\right]\right] x^{k} x^{p} \lambda^{l} m \tag{11}
\end{equation*}
$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (11) can be write in the following form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}=\left(g_{k p} \vec{g}_{l}-g_{k l} \vec{g}_{p}\right) x^{k} x^{p} \lambda^{l} m . \tag{12}
\end{equation*}
$$

If we multiply scalarly the previous expression (12) with the unit vector $\vec{n}$ we obtain:

$$
\begin{equation*}
J_{\vec{n}}^{(O)}=\left(\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}\right)=\left(g_{k p} g_{1 i}-g_{k l} g_{p i}\right) x^{k} x^{p} \lambda^{l} \lambda^{i} m, \tag{13}
\end{equation*}
$$

which represent the material particle mass axial inertia moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$. This formula is the same as the formula (2.3) in [15] written by V. Vujičić.

If we now multiply the expression (13) twice vectorly with the unit vector $\vec{n}$ that is, according to the Ref. [2] or [11], we separate the material particle mass inertia moment vector deviational part for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ we obtain:

$$
\begin{align*}
\overrightarrow{\mathfrak{D}}_{\vec{n}}^{(O)} & =\left[\vec{n},\left[\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}\right]\right]  \tag{14}\\
& =\left\{g_{k p} g_{i j} \vec{g}_{i}-g_{k i} g_{l j} \vec{g}_{p}+\left(g_{k i} g_{l p}-g_{k p} g_{l i}\right) \vec{g}_{j}\right\} x^{k} x^{p} \lambda^{l} \lambda^{i} \lambda^{j} m .
\end{align*}
$$

The last expression represents the vector $\overrightarrow{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the material particles mass inertia moment at the point $O$ of the axis oriented by the unit vector $\vec{n}$ at the dimensional coordinate system $N$.

By introducing the expressions (3) and (4) into the expression (1*) for the vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the material particle mass linear moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$
\begin{equation*}
\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}=\left[\vec{g}_{i}, \vec{g}_{k}\right] x^{k} \lambda^{i} m . \tag{15}
\end{equation*}
$$

## 4. The Vector Support of the Mass Moments

Therefore we introduce the following vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ and define it by following expression:

$$
\begin{equation*}
\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)} \stackrel{\text { def }}{=} \frac{\partial \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}}{\partial m}=[\vec{\rho},[\vec{n}, \vec{\rho}]] . \tag{16}
\end{equation*}
$$

This vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ is vector "support" (carrier) of the mass inertia moment for the axis oriented by unit vector $\vec{n}$ through pole $O$, for the point $N$.

According to the previous expressions (3) and (4), as well as previous notation in the Fig. 1 for the material point position vector $\rho$ and for the unit vector $\vec{n}$ axis orientation at the dimensional coordinate system $n$, the vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ "support" of the mass inertia moment for the axis oriented by unit vector $\vec{n}$ through pole $O$ for the point $N$ can be written in the form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}=\left[\vec{g}_{k},\left[\vec{g}_{l}, \vec{g}_{p}\right]\right] x^{k} x^{p} \lambda^{l} \tag{17}
\end{equation*}
$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (17) can be written in the following form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}=\left(g_{k p} \vec{g}_{l}-g_{k l} \vec{g}_{p}\right) x^{k} x^{p} \lambda^{l} \tag{18}
\end{equation*}
$$

If we multiply scalarly the previous expression (18) with the unit vector $\vec{n}$, we obtain:

$$
\begin{equation*}
\left(\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}, \vec{n}\right)=\left(g_{k p} g_{l i}-g_{k l} g_{p i}\right) x^{k} x^{p} \lambda^{l} \lambda^{i} \tag{19}
\end{equation*}
$$

which represents the axial part of the vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ "support" of the mass inertia moment for the axis oriented by unit vector $\vec{n}$ through pole $O$ for the mass point $N$ or the scalar $\mathfrak{N}_{\vec{n} \vec{n}}^{(O)}$ "support" of the mass inertia axial moment for the axis oriented by unit vector $\vec{n}$ through pole $O$, for the point $N$.

If we now multiply the expression (18) twice vectorly with the unit vector $\vec{n}$ that is, according to [2] or [11], we separate the mass inertia moment vector "support" deviational part for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, for the point $N$, we obtain:

$$
\begin{align*}
\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O) d e v} & =\left[\vec{n},\left[\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}, \vec{n}\right]\right]  \tag{20}\\
& =\left\{g_{k p} g_{i j} \vec{g}_{l}-g_{k i} g_{l j} \vec{g}_{p}+\left(g_{k i} g_{l p}-g_{k p} g_{l i}\right) \vec{g}_{j}\right\} x^{k} x^{p} \lambda^{l} \lambda^{i} \lambda^{j} .
\end{align*}
$$

## 5. The Rigid Body Mass Inertia Moment Vector for the Pole and the Axis

By introducing the expression (3) and (4) into expression (2) for the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}=\iiint_{V}\left[\vec{g}_{k},\left[\vec{g}_{l}, \vec{g}_{p}\right]\right] x^{k} x^{p} \lambda^{l} d m \tag{21}
\end{equation*}
$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (21) can be written in the following form:

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}=\iiint_{V}\left(g_{k p} \vec{g}_{l}-g_{k l} \vec{g}_{p}\right) x^{k} x^{p} \lambda^{l} d m \tag{22}
\end{equation*}
$$

If we multiply scalarly the previous expression (22) with the unit vector $\vec{n}$, we obtain:

$$
\begin{equation*}
J_{\vec{n}}^{(O)}=\left(\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}\right)=\iiint_{V}\left(g_{k p} g_{l i}-g_{k l} g_{p i}\right) x^{k} x^{p} \lambda^{l} \lambda^{i} d m \tag{23}
\end{equation*}
$$

which represent the body mass axial inertia moment at the point $O$ for the axis oriented by the unit vector $\vec{n}$.

If now we multiply the expression (22) twice vectorly with the unit vector $\vec{n}$ that is, according to the [2], we separate the body mass inertia moment vector deviational part for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, we obtain:

$$
\begin{align*}
& \overrightarrow{\mathfrak{D}}_{\vec{n}}^{(O)}=\left[\vec{n},\left[\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}\right]\right]  \tag{24}\\
& =\iiint_{V}\left\{g_{k p} g_{i j} \vec{g}_{l}-g_{k i} g_{l j} \vec{g}_{p}+\left(g_{k i} g_{l p}-g_{k p} g_{l i}\right) \vec{g}_{j}\right\} x^{k} x^{p} \lambda^{l} \lambda^{i} \lambda^{j} d m .
\end{align*}
$$

The last expression represents the vector $\overrightarrow{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the body mass inertia moment at the point $O$ of the axis oriented by the unit vector $\vec{n}$ at dimensional coordinate system $N$.

By introducing the expressions (3) and (4) into the expression (1) for the vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the body mass linear moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$
\begin{equation*}
\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(O)}=\iiint_{V}\left[\vec{g}_{i}, \vec{g}_{k}\right] x^{k} \lambda^{i} d m \tag{25}
\end{equation*}
$$

## 6. Time Derivatives of the Mass Inertia Moment Vector

By using previous expressions (16), (17) and (5), (6), (7), (8), (9) and (10) for the time derivative of the vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ "support" of the mass inertia
moment for the axis oriented by unit vector $\vec{n}$ through pole $O$, for the point $N$, we can write the following:
(26) $\frac{d \overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}}{d t}=$

$$
\begin{aligned}
& {\left[\left[\left(\Gamma_{k r, p}+\Gamma_{p r, k}\right) \vec{g}_{l}-\left(\Gamma_{k r, l}+\Gamma_{l r, k}\right) \vec{g}_{p}+\left(g_{k p} \Gamma_{l r}^{s}-g_{k l} \Gamma_{p r}^{s}\right) \vec{g}_{s}\right]\right] \dot{x}^{r} x^{k} x^{p} \lambda^{l} } \\
&+\left(g_{k p} \vec{g}_{l}\right.\left.-g_{k l} \vec{g}_{p}\right)\left(\dot{x}^{k} x^{p} \lambda^{l}+x^{k} \dot{x}^{p} \lambda^{l}+x^{k} x^{p} \dot{\lambda}^{l}\right) \\
&+\left(\frac{\partial g_{k p}}{\partial t} \vec{g}_{l}+g_{k p} \frac{\partial \vec{g}_{l}}{\partial t}-\frac{\partial g_{k l}}{\partial t} \vec{g}_{p}-g_{k l} \frac{\partial \vec{g}_{p}}{\partial t}\right) x^{k} x^{p} \lambda^{l}
\end{aligned}
$$

By using the vector $\overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}$ "support" of the mass inertia moment for the axis oriented by unit vector $\vec{n}$ through pole $O$, for the point $N$, the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ at the dimensional curvillinear coordinate system $N$ we can vrite:

$$
\begin{equation*}
\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}=\iiint_{V} \overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)} d m \tag{27}
\end{equation*}
$$

Time derivative of the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ we can write in the following form:

$$
\begin{equation*}
\frac{d \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}}{d t}=\iiint_{V} \frac{d \overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}}{d t} d m+\iiint_{V} \overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)} d m \tag{28}
\end{equation*}
$$

Derivative of the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ at the dimensional curvillinear coordinate system $N$ we can write in the following form:
(29) $\frac{d \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}}{d t}=$

$$
\begin{gathered}
\iiint_{V}\left[\left[\left(\Gamma_{k r, p}+\Gamma_{p r, k}\right) \vec{g}_{l}-\left(\Gamma_{k r, l}+\Gamma_{l r, k}\right) \vec{g}_{p}+\left(g_{k p} \Gamma_{l r}^{s}-g_{k l} \Gamma_{p r}^{s}\right) \vec{g}_{s}\right]\right] \dot{x}^{r} x^{k} x^{p} \lambda^{l} d m \\
\quad+\iiint_{V}\left(g_{k p} \vec{g}_{l}-g_{k l} \vec{g}_{p}\right)\left[\left(\dot{x}^{k} x^{p} \lambda^{l}+x^{k} \dot{x}^{p} \lambda^{l}+x^{k} \dot{x}^{p} \dot{\lambda}^{l}\right) d m+x^{k} x^{p} \lambda^{l} d \dot{m}\right] \\
\quad+\iiint_{V}\left(\frac{\partial g_{k p}}{\partial t} \vec{g}_{l}+g_{k p} \frac{\partial \vec{g}_{l}}{\partial t}-\frac{\partial g_{k l}}{\partial t} \vec{g}_{p}-g_{k l} \frac{\partial \vec{g}_{p}}{\partial t}\right) x^{k} x^{p} \lambda^{l} d m
\end{gathered}
$$

In the case for pure rotation of the rigid body we can write the following:

$$
\begin{align*}
& \frac{d \vec{\rho}}{d t}=\dot{x}^{k}\left(\vec{g}_{k}+x^{p} \Gamma_{p k}^{j} \vec{g}_{k}\right)+x^{k} \frac{\partial \vec{g}_{k}}{\partial t}  \tag{30}\\
& \frac{d \vec{n}}{d t}=\dot{x}^{i}\left(\frac{\partial \lambda^{k}}{\partial x^{i}} \vec{g}_{k}+\lambda^{k} \Gamma_{i k}^{j} \vec{g}_{l}\right)+\lambda^{k} \frac{\partial \vec{g}_{k}}{\partial t}, \quad \vec{\omega}=\omega \vec{n}=\omega \lambda^{k} \vec{g}_{k}  \tag{31}\\
& \frac{d \vec{\rho}}{d t}=[\vec{\omega}, \vec{\rho}]=\omega x^{p} \lambda^{l}\left[\vec{g}_{l}, \vec{g}_{p}\right], \quad \frac{d \vec{\omega}}{d t}=\dot{\omega} \vec{n}=\dot{\omega} \lambda^{k} \vec{g}_{k}, \quad \frac{d \vec{n}}{d t}=0 \tag{32}
\end{align*}
$$

and the time derivative of the vector $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole $O$ and the axis oriented by the unit vector $\vec{n}$ we can write in following form:

$$
\begin{align*}
\frac{d \overrightarrow{\mathfrak{N}}_{\overrightarrow{\vec{n}}}^{(O)}}{d t} & =\left[\vec{\omega}, \overrightarrow{\mathfrak{N}}_{\vec{n}}^{(O)}\right]=\omega x^{k} x^{p} \lambda^{l} \lambda^{i}\left\{g_{k p}\left[\vec{g}_{i}, \vec{g}_{l}\right]-g_{k l}\left[\vec{g}_{i}, \vec{g}_{p}\right]\right\}  \tag{33}\\
& =\omega x^{k} x^{p} \lambda^{l} \lambda^{i}\left\{e_{i l s} g_{k p} \vec{g}^{s}-e_{i p j} g_{k l} \vec{g}^{j}\right\} \sqrt{g} .
\end{align*}
$$

## 7. Concluding Remarks

We are following the classic definition, we write for the linear momentum following expression:

$$
\begin{equation*}
\overrightarrow{\mathfrak{K}}=\iiint_{V} \vec{\nu}_{N} d m=\iiint_{V}\left(\vec{\nu}_{A}+[\vec{\omega}, \vec{\rho}]\right) d m=M \vec{\nu}_{A}+\omega \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)} \tag{34}
\end{equation*}
$$

The expression (34) of the linear momentum $\overrightarrow{\mathfrak{K}}$ of the rigid body whose points have the translation velocity $\vec{\nu}_{A}$ of the referential point $A$ and the relative velocity $[\vec{\omega}, \vec{\rho}]$ due to the rotation around the axis oriented by the vector $\vec{\omega}=\omega \vec{n}$ through the point $A$ has two parts: $1^{*}$ the translatory one equal to the product of the referential point velocity and the body mass - the linear momentum due to the translation motion with the velocity of the referential point $A$; and the rotatory one equal to the product of the magnitude $\omega$ of the angular velocity $\vec{\omega}=\omega \vec{n}$ and the vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point $A$ for the axis oriented by the unit vector $\vec{n}$.

If the pole $A$ is the body mass center $C$ then the linear momentum is equal only in the translatory part since the vector $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment for the pole in the body mass center is equal to zero regardless of its orientation so that the linear momentum is equal to the product of this velocity $\vec{\nu}_{C}$ of the body mass center and the rigid body mass: $\overrightarrow{\mathfrak{K}}=M \vec{\nu}_{C}$.

The same stands for if the pole $A$ is not the body mass center but if the axis oriented with $\vec{\omega}=\omega \vec{n}$ trough pole $A$ passes trough the mass center.

The second kinetic vector connected to the referential point which plays an important part (role) in the rigid body dynamics is the rigid body angular momentum (motion quantity moment) for the given pole, $\overrightarrow{\mathfrak{L}}_{0}$.

Following the classic definition according to the [1] according to the notation given in the Fig. 2 the rigid body angular momentum is calculated by means of the following expression:

$$
\begin{equation*}
\overrightarrow{\mathfrak{L}}_{0}=\iiint_{V}\left[\vec{r}, \vec{\nu}_{N}\right] d m=\iiint_{V}\left[\vec{r}+\vec{\rho}, \vec{\nu}_{A}+[\vec{\omega}, \vec{\rho}]\right] d m \tag{35}
\end{equation*}
$$

Following the idea of this paper that at the basis of the rigid body motion interpretation there are rigid body dynamic parameters which express the mass inertia moment properties and the kinematic parameters, translation velocity $\vec{\nu}_{A}$ of the rigid body referential point and the angular velocity $\vec{\omega}$ of the relative momentary rotation around the axis oriented with $\vec{\omega}$ and through the referential point $A$ then the angular momentum for the point $A, \overrightarrow{\mathfrak{L}}_{A}$ is connected not only to the pole but to the axis oriented by the momentary angular velocity vector to which we connect the vectors $\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}$ and $\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the rigid body mass linear and inertia moments by connecting the body mass to the translation velocity of the referential point $A$. Therefore we write that it is:

$$
\begin{equation*}
\overrightarrow{\mathfrak{L}}_{A}=\left[\overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}, \vec{\nu}_{A}\right]+\omega \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)} . \tag{36}
\end{equation*}
$$



Fig. 1


Fig. 2

For the case of the rigid body rotation around the fixed axis the linear momentum and angular momentum are:

$$
\begin{align*}
\overrightarrow{\mathfrak{K}} & =\left[\vec{\omega}, \vec{\rho}_{C}\right] M=\omega \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}  \tag{37}\\
\overrightarrow{\mathfrak{L}}_{A} & =\vec{\omega}\left(\vec{n}, \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}\right)+\omega\left[\vec{n}\left[\overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{n}\right]\right]=\vec{\omega}\left(\vec{n}, \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}\right)+\omega \overrightarrow{\mathfrak{D}}_{\vec{n}}^{(A)} \tag{38}
\end{align*}
$$

Since the velocity $\vec{\nu}$ and the acceleration $\vec{a}$ of the body elementary mass at the point $N$ are:

$$
\begin{equation*}
\vec{\nu}=[\vec{\omega}, \vec{\rho}], \quad \vec{a}=[\dot{\vec{\omega}}, \vec{\rho}]+[\vec{\omega},[\vec{\omega}, \vec{\rho}]] \tag{39}
\end{equation*}
$$

then for the main vector $\vec{F}_{r j}$ the inertia force of the overall rigid body rotating around the axis with the angular velocity $\vec{\omega}$ we obtain:

$$
\begin{equation*}
\vec{F}_{r j}=-\iiint_{V} \vec{a} d m=-\dot{\omega} \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}-\omega \frac{d \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}}{d t}=-\dot{\omega} \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}-\omega\left[\vec{\omega}, \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}\right] \tag{40}
\end{equation*}
$$

For the main moment of the inertia forces of the overall rigid body rotating around the axis and for the point $A$ we calculate the following:

$$
\begin{equation*}
\overrightarrow{\mathfrak{M}}_{A j}=\iiint_{V}\left[\vec{\rho}, d \vec{F}_{r j}\right]=-\dot{\omega} \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}-\dot{\omega} \frac{d \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}}{d t}=-\dot{\omega} \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}-\omega\left[\vec{\omega}, \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}\right] . \tag{41}
\end{equation*}
$$

The dynamic equations of the body rotation around fixed axis can be obtained by differentiating in time the expression (38) for the linear momentum and expression (38) for angular momentum on the basis of which we obtain:

$$
\begin{equation*}
1 * \quad \frac{d \overrightarrow{\mathfrak{K}}}{d t}=\dot{\omega} \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}+\omega\left[\vec{\omega}, \overrightarrow{\mathfrak{S}}_{\vec{n}}^{(A)}\right]=-\vec{F}_{r j}=\vec{F}_{r} . \tag{42}
\end{equation*}
$$

The equation (42) for the linear momentum change which is equal to the main vector (resultant) of the active and reactive forces shows that the motion linear momentum changes the vector normal to the rotation axis and has two components: one due to the angular velocity change which is normal to the rotation axis and the plane which contains the body mass center and the rotation axis, and the other component which depends on the angular velocity square which is normal to the rotation axis and lie in the plane formed by rotation axis and the rigid body mass center doing rotation.

$$
\begin{equation*}
2 * \quad \frac{d \overrightarrow{\mathfrak{L}}_{A}}{d t}=\dot{\omega} \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}+\omega\left[\vec{\omega}, \overrightarrow{\mathfrak{J}}_{\vec{n}}^{(A)}\right]=-\overrightarrow{\mathfrak{M}}_{A j}=\overrightarrow{\mathfrak{M}}_{A} \tag{43}
\end{equation*}
$$

We see that by using mass moment vectors and their derivatives, the linear momentum and angular momentum of the rotor which rotates around one or two rotation axes are expressed simpler then the other ways, as it was shown in the paper. That fact is the main reason which for simpler qualitative analysis kinetic properties of rotor dynamic and their kinetic pressures on the shaft bearings.

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