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A NUMERICAL METHOD FOR A VOLTERRA–TYPE INTEGRAL EQUATION WITH LOGARITHMIC KERNEL

M. R. Capobianco and N. Mastronardi

This paper is dedicated to Professor D. S. Mitrinović

Abstract. We consider a class of integral equations of Volterra type with constant coefficients containing a logarithmic difference kernel. This class coincides for $a = 0$ with the Symm's equation. We can transform the general integral equation into an equivalent singular equation of Cauchy type which allows us to give the explicit formula for the solution. The numerical method proposed in this paper consists in considering the interpolation of the known function f and in substituting this in the expression of the solution g. Then, with the aid of the invariance properties of the orthogonal polynomials for the Cauchy integral equations, we obtain an approximate solution of the function g. We give weighted norm estimates for the error of this method. The paper concludes with some numerical examples.

1. Introduction

In this paper we consider the following integral equation

$$a \int_{-1}^{x} g(t)dt + \frac{b}{\pi} \int_{-1}^{1} g(t) \log|x-t|dt = f(x), \qquad -1 < x < 1, \tag{1.1}$$

under the hypotheses $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$ and $f(x) \in C^{p+\lambda}([-1,1])$, $p \geq 1$, and $0 < \lambda \leq 1$, where $C^{p+\lambda}([-1,1])$ is the usual space of all the functions f such that the p-th derivative of f is an Hölder continuous function.

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In the case $a = 0$ equation (1.1) was firstly considered by Carleman [3]. In the last decades for $a = 0$ this equation is known as the Symm's integral equation (see for example [9–10] and the references given there).

The integral equation (1.1) has also been considered in [5], but using a different approach. Here, with the aid of the relation

$$h(x) = \int_{-1}^{x} g(t)dt - h_0 \frac{1+x}{2}, \quad h_0 = \int_{-1}^{1} g(t)dt, \tag{1.2}$$

i.e., $h(-1) = h(1) = 0$, we introduce a new unknown function $h(x)$ and, after integrating by parts, equation (1.1) becomes

$$ah(x) - \frac{b}{\pi} \int_{-1}^{1} \frac{h(t)}{t-x} dt = f_0(x), \quad -1 < x < 1, \tag{1.3}$$

where $f_0(x) = f(x) - h_0 G(x)$ and

$$G(x) = a\frac{1+x}{2} - \frac{b}{\pi} + \frac{b}{2\pi}[(1-x)\log(1-x) + (1+x)\log(1+x)].$$

Now, we observe that problem (1.1) and problem (1.2)–(1.3) are equivalent. Equation (1.3) is a Cauchy singular integral equation. Taking into account the boundary conditions $h(-1) = h(1) = 0$, we look for a solution of (1.3) in the form $h(x) = w(x)\overline{h}(x)$, where $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is a Jacobi weight function with the exponents α, β, defined by the relations $a - ib = \exp\{i\pi\alpha_0\}, \quad |\alpha_0| < 1, \ -1 < \alpha := M - \alpha_0, \beta := N + \alpha_0 < 1, \quad M, N \in \mathbb{N}$ and such that the index of the integral equation (1.3) $\chi := -(\alpha+\beta) = -(M+N) = -1$, i.e., $0 < \alpha, \beta < 1$.

Now, denoting by $L^2_w = L^2_w(-1,1)$ the Hilbert space of all real-valued functions on $(-1,1)$ which are square integrable with respect to the weight w, we recall that the Cauchy singular integral equation (1.3) with index $\chi = -1$ is uniquely solvable for all $f_0 \in L^2_{w^{-1}}$ satisfying the condition

$$\int_{-1}^{1} w^{-1}(x) f_0(x) dx = 0. \tag{1.4}$$

Moreover, we have (see for more details [9])

$$\overline{h}(x) = (\widehat{A} f_0)(x) := a w^{-1}(x) f_0(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{w^{-1}(t) f_0(t)}{t-x} dt. \tag{1.5}$$

Condition (1.4) gives us the possibility to determine also the constant h_0 in relation (1.2). In fact, we obtain

$$h_0 = \frac{\int_{-1}^{1} w^{-1}(x)f(x)dx}{\int_{-1}^{1} w^{-1}(x)G(x)dx}.$$

Here, we consider the case $h_0 = 0$, i.e. $f_0(x) = f(x)$. Now, recalling relation (1.2), we obtain the exact solution $g(x)$ of the integral equation (1.1), given by

$$g(x) = h'(x) = \frac{d}{dx}\left[\, w(x)(\widehat{A}f)(x)\,\right]. \tag{1.6}$$

There exist many methods for the numerical resolution of equation (1.3). Among others we mention [6–7] and the book of [9] for the case $\chi = -1$. In this paper we approximate the exact solution of equation (1.1), given by (1.6), with the following procedure. Firstly, we recall that, said $\{p_m(w)\}_m$ and $\{p_m(w^{-1})\}_m$ the sequences of the orthonormal polynomials in $[-1, 1]$ with positive leading coefficient corresponding to the weight functions w and w^{-1}, respectively, we have (cf. [9])

$$\begin{aligned} (Ap_m(w))(x) &= \frac{b}{\sin(\pi\alpha)} p_{m-\chi}(w^{-1}; x), \\ (\widehat{A}p_m(w^{-1}))(x) &= \frac{b}{\sin(\pi\alpha)} p_{m+\chi}(w; x), \end{aligned} \tag{1.7}$$

where $p_{-1}(w) = p_{-1}(w^{-1}) = 0$, $\widehat{A}$ is defined by (1.5) and

$$(A\overline{h})(x) = aw(x)\overline{h}(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{w(t)\overline{h}(t)}{t-x}dt. \tag{1.8}$$

Then, our approach consists in approximating the known function f in (1.6) by its Lagrange interpolating polynomial based on a suitable matrix of knots T and in using the second of relations (1.7). In Section 2 we describe the method; in Section 3 weighted error estimate is formulated and in Section 4 we prove the convergence result of the previous Section. Finally in Section 5 we show some numerical examples.

2. Numerical Method

Let $T := \{t_k,\, k = 1, \dots, m,\, m = 1, 2, \dots\}$ be a given matrix of knots. Thus, we approximate the known function f in (1.6) by its Lagrange interpolating polynomial of degree $m-1$ based on the knots of the matrix T. Then, we write

$$f(x) \sim L_m(f;x) = \sum_{k=1}^{m} l_{m,k}(x) f(t_k), \quad l_{m,k}(x) = \prod_{\substack{r=1\\ r\neq k}}^{m} \frac{x-t_r}{t_k-t_r}. \tag{2.1}$$

Since $l_{m,k}(x)$ is a polynomial of degree $m-1$, we can write

$$l_{m,k}(x) = \sum_{j=0}^{m-1} c_j p_j(w^{-1};x), \quad c_j = \int_{-1}^{1} l_{m,k}(x) p_j(w^{-1};x) w^{-1}(x)\,dx. \tag{2.2}$$

Let $x_i = x_{m,i}(w^{-1}), i = 1, \dots, m$, be the zeros of the m-th orthogonal polynomial $p_m(w^{-1})$ with respect to the weight w^{-1} and let $\lambda_i = \lambda_{m,i}(w^{-1})$ be the related Christoffel numbers. Then, since $l_{m,k}(x)p_j(w^{-1};x)$ is a polynomial of degree at most $2m-2$, Gaussian quadrature rule with respect to the weight w^{-1} is exact; therefore

$$c_j = \int_{-1}^{1} l_{m,k}(x) p_j(w^{-1};x) w^{-1}(x)\,dx = \sum_{i=1}^{m} \lambda_i l_{m,k}(x_i) p_j(w^{-1};x_i). \tag{2.3}$$

Finally, by (2.1)–(2.3), we can write

$$f(x) \sim \sum_{k=1}^{m} f(t_k) \sum_{i=1}^{m} \lambda_i l_{m,k}(x_i) \sum_{j=0}^{m-1} p_j(w^{-1};x_i) p_j(w^{-1};x). \tag{2.4}$$

Now, we substitute (2.4) in (1.6) and apply the second of relations (1.7). Recalling that in this case $\alpha + \beta = 1$, i.e., $\beta = 1 - \alpha$, and

$$\begin{aligned} \varphi^2(x) \frac{d}{dx} p_n(w;x) &= (n+2)\Big(x - \frac{1-2\alpha}{2n+3}\Big) p_n(w;x) \\ &\quad - 2(n+1) A_{n+1} p_{n+1}(w;x), \quad n \geq 0, \end{aligned}$$

where

$$A_n = \frac{\sqrt{(n+\alpha)(n+1-\alpha)}}{2n+1}, \quad n \geq 1,$$

$$p_{-1}(w;x) = 0, \quad p_0(w;x) = \frac{1}{\sqrt{2\Gamma(2-\alpha)\Gamma(1+\alpha)}},$$

and, here and in the sequel, $\varphi(x) = (1-x^2)^{1/2}$, we obtain, after the derivation and other easily computations, as approximate solution of the function $g(x)$, the following

$$\varphi^2(x)g_m(x) = \frac{bw(x)}{\sin(\pi\alpha)} \left\{ \sum_{k=1}^{m} f(t_k) \sum_{i=1}^{m} \lambda_i l_{m,k}(x_i) \sum_{j=1}^{m-1} p_j(w^{-1}; x_i) j \times \left[\left(x + \frac{1-2\alpha}{2j+1} \right) p_{j-1}(w;x) - 2A_j p_j(w;x) \right] \right\}.$$

Remark. We expressly note that this procedure give us the exact solution $g(x)$ if the known function $f(x)$ is a polynomial of degree at most $m-1$.

Now we associate to the matrix of the knots T the sequence of the Lebesgue constants$\{||L_m(T)||\}_m$ defined by

$$||L_m(T)|| := \max_{|x|\le 1} \sum_{k=1}^{m} |l_{m,k}(x)|. \tag{2.5}$$

Naturally, the Lebesgue constants have a crucial role in the analysis of the convergence of the proposed method. In fact we prefer to use a matrix of knots T such that the associated Lebesgue constant is optimal, i.e.,

$$||L_m(T)|| = O(\log m). \tag{2.6}$$

Matrices that verify this condition are well known: the zeros of the Jacobi polynomials with exponents $\gamma, \delta \le -1/2$ and the set $\{-\cos(k\pi/(m+1)), k = 0, 1, \dots, m+1\}_{m\in\mathbb{N}}$. Recently in [4], [8], the authors have shown that, beginning from the zeros of the Jacobi polynomials, it is possible to obtain wide classes of knots verifying (2.6).

Now, we choose as knots of the matrix T the zeros of the m-th orthogonal polynomial $p_m(w^{-1})$ with respect to the weight w^{-1}. For $a \neq 0$, recalling that $-\beta = \alpha - 1$, we have two cases:
i) $-1/2 < -\alpha < 0$ and $-1 < -\beta < -1/2$;
ii) $-1 < -\alpha < -1/2$ and $-1/2 < -\beta < 0$.

Therefore, in these cases the Lebesgue constant associated at the matrix T is not optimal; in fact $||L_m(T)|| = O(m^{1/2+\max\{-\alpha,-\beta\}})$. Anyway, we obtain a more simple expression for the approximate solution, given by

$$\varphi^2(x)g_m(x) = \frac{bw(x)}{\sin(\pi\alpha)} \left\{ \sum_{k=1}^{m} \lambda_k f(x_k) \sum_{j=1}^{m-1} p_j(w^{-1}; x_k) j \times \left[\left(x + \frac{1-2\alpha}{2j+1} \right) p_{j-1}(w;x) - 2A_j p_j(w;x) \right] \right\}. \tag{2.7}$$

If we want to apply the method of additional knots [4], [8], we can consider the matrix $T = \{x_k = x_{m,k}(w^{-1}),\ k = 1, \dots, m,\ m = 1, 2, \dots, y_1\}$ where the point y_1 is chosen such that $y_1 \in [-1, x_1]$, $y_1 - x_1 \sim m^{-2}$ in the case i) and $y_1 \in [x_m, 1]$, $x_m - y_1 \sim m^{-2}$ in the case ii). Now, the associated Lebesgue constant is optimal, i.e., $||L_m(T)|| = O(\log m)$ and the approximate solution can be written also as

$$\begin{aligned}
\varphi^2(x) g_m(x) = \frac{b w(x)}{\sin(\pi\alpha)} \Bigg\{ & \sum_{k=1}^{m} \lambda_k \frac{f(x_k)}{x_k - y_1} \sum_{j=1}^{m-1} p_j(w^{-1}; x_k) \\
& \times \Bigg[(1 - x^2) p_{j-1}(w; x) + (x - y_1) j \left(\left(x + \frac{1 - 2\alpha}{2j + 1} \right) \right. \\
& \left. \times p_{j-1}(w; x) - 2A_j p_j(w; x) \right) \Bigg] \\
& + (\beta - \alpha - x) p_0^2(w^{-1}) \sum_{k=1}^{m} \frac{\lambda_k f(x_k)}{x_k - y_1} + \frac{f(y_1) m}{p_m(w^{-1}; y_1)} \\
& \times \left[\left(x + \frac{1 - 2\alpha}{2m + 1} \right) p_{m-1}(w; x) - 2A_m p_m(w; x) \right] \Bigg\} .
\end{aligned}$$

The case $\boldsymbol{a = 0}$. In this case the integral equation (1.1), as we note in Section 1, is the well known Symm's integral equation. Moreover, this means that $w(x) = (1 - x^2)^{1/2}$ and $w^{-1}(x) = 1/(1 - x^2)^{1/2}$. Thus, if we choose the knots of the matrix T as the zeros of the m-th Chebyshev polynomial of first kind, firstly we have that the associated Lebesgue constant is optimal ($||L_m(T)|| = O(\log m)$); moreover we have a more simple expression for the approximate solution, given by

$$\begin{aligned}
\varphi(x) g_m(x) &= \frac{b\pi}{m} \sum_{k=1}^{m} f(x_k) \sum_{j=1}^{m-1} j T_j(x_k) \left(x U_{j-1}(x) - U_j(x) \right) \\
&= \frac{-b\pi}{m} \sum_{k=1}^{m} f(x_k) \sum_{j=1}^{m-1} j T_j(x_k) T_j(x),
\end{aligned} \tag{2.8}$$

where, now $x_k = \cos(2k - 1)\pi/2m$, $k = 1, \dots, m$, and T_m and U_m denote the m-th orthonormal Chebyshev polynomials of first and second kind, respectively.

In this case the method is not new; in fact it coincides with the collocation method with Chebyshev polynomials for the Symm's integral equation on an interval (see [10] and the references given by the authors). Anyway, with Theorem 3.1, we establish for this method the rate of convergence in a weighted uniform norm.

3. Main Results

Since, as we can deduce from Section 1, the solution $g(x)$ of (1.1) is not bounded in ± 1, it does not make sense to estimate the error uniformly on $[-1,1]$. Nevertheless, the following theorem provides a weighted estimate on the whole interval.

Theorem 3.1. *Let $T = \{t_k,\ k = 1, \dots, m,\ m = 1, 2, \dots\}$ be a given matrix of knots. If $f(x) \in C^{p+\lambda}([-1,1])$, $p \geq 1$, $0 < \lambda \leq 1$, then*

$$||\varphi^2(g - g_m)|| \leq \frac{C \log m ||L_m(T)||}{m^{p+\lambda-1}}, \tag{3.1}$$

where $\varphi(x) = (1-x^2)^{1/2}$ and $||L_m(T)||$ is defined by (2.5).

Remark. Sometimes it is sufficient to know the values of the solution in closed subintervals contained in $(-1,1)$. Then, by the previous theorem, the following estimate holds

$$|g(x) - g_m(x)| \leq \frac{C \log m ||L_m(T)||}{m^{p+\lambda-1}}, \quad \forall x \in [c,d] \subset (-1,1).$$

4. The Proof of the Main Result

In order to prove the main result of this paper described in the previous Section, some notations and preliminary results are needed. In the following the symbol C stands for a generic constant taking different values at different places. It will always clear what variables and indices the constants are independent of. If A and B are two expressions depending on some variables, then we write: $A \sim B$ if and only if $|AB^{-1}| \leq C$ and $|A^{-1}B| \leq C$, uniformly for the variables under consideration.

Introduce the remainder term by

$$R_m(f;x) = f(x) - L_m(f;x). \tag{4.1}$$

Lemma 4.1. *Let $T = \{t_k,\, k = 1, \dots, m,\, m = 1, 2, \dots\}$ be a given matrix of knots. Then for every function $f \in C^{p+\lambda}([-1,1])$, $p \geq 0$, $0 < \lambda \leq 1$, we have*

$$||w\widehat{A}(R_m(f))|| \leq \frac{C \log m ||L_m(T)||}{m^{p+\lambda}}, \tag{4.2}$$

where $\widehat{A}$ is the operator defined by (1.5), *$||L_m(T)||$ is defined by* (2.5) *and $R_m(f)$ denotes the Lagrange interpolation error* (4.1).

For the proof of this Lemma we refer the reader to [1, Lemma 4.2, p. 52]. Setting $I_m = [x - (1+x)/2m, x + (1-x)/2m]$, $I'_m = [-1,1] \setminus I_m$ we have the following results that can be found in [2]

$$\int_{I'_m} \frac{w^{-1}(t)}{|t-x|} dt \leq C w^{-1}(x) \log m, \tag{4.3}$$

$$\int_{I'_m} \frac{w^{-1}(t)}{(t-x)^2} dt \leq C \frac{w^{-1}(x) m}{1-x^2}. \tag{4.4}$$

Lemma 4.2. *Let $T = \{t_k,\, k = 1, \dots, m,\, m = 1, 2, \dots\}$ be a given matrix of knots. If $f(x) \in C^{p+\lambda}([-1,1])$, $p \geq 1$, $0 < \lambda \leq 1$, then*

$$||w\varphi^2(\widehat{A}R_m(f))'|| \leq \frac{C \log m ||L_m(T)||}{m^{p+\lambda-1}}, \tag{4.5}$$

where $\varphi(x) = (1-x^2)^{1/2}$, $||L_m(T)||$ and $R_m(f)$ are defined by (2.5) *and* (4.1), *respectively.*

Proof. Since by the second of relations (1.7) it results $(\widehat{A}\ 1)(x) = 0$, we can write

$$|w\varphi^2(\widehat{A}R_m(f))'(x)| = \frac{b}{\pi} w(x)\varphi^2(x) \left| \frac{d}{dx} \int_{-1}^{1} \frac{w^{-1}(t)[R_m(f;t) - R_m(f;x)]}{t-x} \, dt \right|,$$

i.e.,

$$\begin{aligned}
&|w\varphi^2(\widehat{A}R_m(f))'(x)| \\
=&A(x) \left| \int_{-1}^{1} \frac{w^{-1}(t)[-R'_m(f;x)(t-x) + R_m(f;t) - R_m(f;x)]}{(t-x)^2} \, dt \right| \\
=&A(x) \left| \left\{ \int_{I_m} + \int_{I'_m} \right\} \frac{-R'_m(f;x)(t-x) + R_m(f;t) - R_m(f;x)}{(t-x)^2} \, w^{-1}(t) \, dt \right| \\
=&I_1 + I_2,
\end{aligned}$$

where we put $A(x) = (b/\pi)w(x)\varphi^2(x)$. We remember that $1 \pm t \sim 1 \pm x$ for $x, t \in I_m$,

$$||\varphi^i(R_m(f))^{(i)}|| \leq C||L_m(T)||E_{m-i-1}(f^{(i)}), \quad \text{for} \quad p \geq i \geq 0,$$

$$\varphi^2(t)\frac{|R'_m(f;t) - R'_m(f;x)|}{|t-x|} \leq C\Big(|t-x|^{\lambda-1} + m^2\omega\big(f;m^{-1}\big) + m^2||L_m(T)||E_{m-1}(f)\Big), \quad \text{for } p = 1,$$

where $E_m(f) = \inf_{P \in P_m} ||f - P||$ and P_m is the set of all polynomials of degree at most m. Thus, for $p = 1$, there exists a point $\xi_t \in I_m$, such that

$$I_1 \leq \int_{I_m} \frac{\varphi^2(t)|R'_m(f;\xi_t) - R'_m(f;x)|}{|t-x|}dt \leq \frac{C||L_m(T)||}{m^\lambda}. \tag{4.6}$$

For $p \geq 2$ there exists a point $\xi'_t \in I_m$ such that

$$I_1 \leq \int_{I_m} \varphi^2(t)|R''_m(\xi'_t)|dt \leq \frac{C||L_m(T)||}{m^{p+\lambda-1}}. \tag{4.7}$$

On the other hand, applying inequalities (4.3) and (4.4), we obtain

$$I_2 \leq w(x)||\varphi(R_m(f))'||\int_{I'_m} \frac{w^{-1}(t)}{|t-x|}\,dt + w(x)\varphi^2(x)||R_m(f)|| \times \int_{I'_m} \frac{w^{-1}(t)}{(t-x)^2}dt \leq \frac{C\log m||L_m(T)||}{m^{p+\lambda-1}}. \tag{4.8}$$

Therefore, by inequalities (4.6)–(4.8), Lemma 4.2 follows. $\square$

Proof of Theorem 3.1. We find that

$$\varphi^2(x)|g(x) - g_m(x)| \leq C\Big\{w(x)|(\widehat{A}R_m(f))(x)| + w(x)\varphi^2(x)\,\Big|\,\frac{d}{dx}(\widehat{A}R_m(f))(x)\,\Big|\Big\}.$$

Therefore, by relations (4.2) and (4.5), inequality (3.1) easily follows and this theorem is completely proved. $\square$

5. Numerical Examples

In this section we give some numerical results obtained by using algorithms described by relations (2.7) and (2.8), for $a \neq 0$ and $a = 0$, respectively. For all examples we report in tables the maximum of the weighted errors $\varphi^2(x)\,(g(x) - g_m(x))$, evaluated at $x = \pm 2j/m$, $j = 0, 1, \dots, \lfloor m/2 \rfloor - 1$, for various values of m. All computations were carried out in double precision with a machine precision approximately equal to 2.22×10^{-16}. We observe that the numerical results are in accordance to the theoretical ones.

Example 5.1. Let us consider the integral equation

$$a \int_{-1}^{x} g(t)dt + \frac{b}{\pi} \int_{-1}^{1} g(t) \log|x - t| dt = w(x)x(1 - x^2), \quad -1 < x < 1.$$

Since in this case $h_0 = 0$, the analytical solution is

$$\begin{aligned} g(x) &= aw(x)[-4x^2 + (\beta - \alpha)x + 1] + \frac{b}{\pi} w(x)(\beta - \alpha - x) \\ &\times \left[x \log \frac{1 - x}{1 + x} + \frac{-6x^2 + 4}{3(1 - x^2)} \right] + \frac{b}{\pi} w(x) \left[-6x + (1 - 3x^2) \log \frac{1 - x}{1 + x} \right]. \end{aligned}$$

Table 5.1 shows the results obtained choosing $a = b = \sqrt{2}/2$ and $\alpha = 1/4$, $\beta = 3/4$. Thus, $f(x) \in C^{1+1/4}([-1, 1])$.

Table 5.1: Weighted errors $\|(1 - x^2)(g - g_m)\|_\infty$

m	Example 5.1	Example 5.2
8	$.972330 \times 10^{-2}$	$.670934 \times 10^{-2}$
16	$.857902 \times 10^{-3}$	$.318672 \times 10^{-3}$
32	$.114778 \times 10^{-3}$	$.415256 \times 10^{-4}$
64	$.118823 \times 10^{-4}$	$.415256 \times 10^{-4}$
128	$.888457 \times 10^{-6}$	$.450591 \times 10^{-6}$

Example 5.2. Here, we consider the same integral equation as in Example 5.1, but choosing $a = 0$, $b = 1$ and $\alpha = \beta = 1/2$. Thus, $f(x) \in C^{1+1/2}([-1, 1])$. The corresponding results are given also in Table 5.1.

Example 5.3. We consider the integral equation

$$\frac{1}{\pi} \int_{-1}^{1} g(t) \log|x - t| dt = x^2 \sin x, \qquad -1 < x < 1.$$

The known function $x^2 \sin x \in C^\infty([-1,1])$. By Theorem 3.1, the numerical solution exponentially coverges to the solution of the integral equation. Since the analytical solution is unknown, we report in Table 5.2 the weighted error between the numerical solution computed for $m = 128$, and the numerical solution computed for $m = 4, 8, 16$.

Table 5.2: Weighted errors for Example 5.3

m	$\|(1-x^2)(g_{128} - g_m)\|_\infty$
4	3.89255×10^{-2}
8	6.02510×10^{-6}
16	6.43540×10^{-13}

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Istituto per Applicazioni della Matematica
C.N.R., Via P. Castellino 111
80131 Napoli
Italy

Universitá degli Studi della Basilicata
Dipartimento di Matematica
Via Nazario Sauro 85
85100 Potenza
Italy