FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 13 (1998), 109–126

A GENERAL ALGORITHM FOR SOLVING STOCHASTIC HEREDITARY INTEGRODIFFERENTIAL EQUATIONS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. In this paper a general iterative method for solving stochastic hereditary integrodifferential equations is considered. Sufficient conditions under which a sequence of iterations converges almost surely to the solution of the original equation are given. The speed of convergence of these iterations is estimated. Also, some concrete iterative methods, as special cases of this general iterative procedure, are suggested.

1. Introduction

Essentially, the idea of the present paper goes back to the paper of R. Zuber ([16]) treating one general analytic iterative procedure for solving deterministic ordinary differential equations of first order. The generality of this method is in the sense that many well-known, historically important iterative methods are its special cases, for example, Picard–Lindelöf method of succesive approximations, Chaplygin methods of secants and tangents, Newton-Kantorovich method and some interpolation methods ([17]). Later, this approach was appropriatelly extended to study special classes of stochastic differential and integrodifferential equations of Ito type (see [7], [8], [9]) and immediately used, for example in [12], in which the rate of convergence of an approximate solution is estimated.

The aim of the present paper is to make an analogous iterative procedure for solving stochastic hereditary equations; precisely, for solving one very general stochastic integrodifferential equation which includes, as some

Received July 22, 1997.
1991 *Mathematics Subject Classification*. Primary 60H10.
This work was supported by grant 04M03 of MNTRS through Math. Institute SANU.

special cases, different classes of stochastic equations of Ito type. From theoretical point of view, and much more from the point of view of various applications, it is very important to construct suitable algorithms, as concrete cases of this general algorithm, for finding at least an approximate solution and to estimate an error of n-th approximation of the solution of the original equation.

Let us have in mind that the notion of hereditary phenomena are particularly convenient for studying such phenomena in continuum mechanics of materials with memories, as a version of the well-known theory or "fading memory" spaces. Mathematical models represent deterministic hereditary differential equations researched in the papers [2], [3], [4], [13], and in many others. Later, this notion was appropriatelly used in an investigation into the effect of Gaussian white noise on nonlinear hereditary phenomena, which mathematical interpretation is researched by stochastic hereditary differential equations. So, the paper [14] includes certain important results treating existence, uniqueness and stability problems for these equations.

Let us give in short some notions and results, necessary in our forthcoming investigation. For more details see previously cited papers, first of all [2] and [14].

Let $\mathbb{R}^k$ be the real k-dimensional Euclidian space and L_p^ρ, $1 \le p \le \infty$, be the usual space of classes of measurable functions, i.e.,

$$L_p^\rho = \left\{ \varphi \,|\quad \varphi : \mathbb{R}^+ \to \mathbb{R}^k; \quad \int_0^\infty |\varphi(t)|^p \rho(t)\, dt < \infty \right\},$$

where the function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, called an influence function with relaxation properties, is summable on $\mathbb{R}^+$ and for every $\sigma \ge 0$ one has

$$\overline{K}(\sigma) = \operatorname*{esssup}_{s \in R^+} \frac{\rho(s+\sigma)}{\rho(s)} \le \overline{\overline{K}} < \infty, \qquad \underline{K}(\sigma) = \operatorname*{esssup}_{s \in R^+} \frac{\rho(s)}{\rho(s+\sigma)} < \infty.$$

Also, ρ is essentially bounded, essentially strictly positive and $s\rho(s) \to 0$ as $s \to 0$ (see [2]).

Let X be *a past-history space*, i.e., a product space $X = \mathbb{R}^k \times L_p^\rho$ of elements x, $x = (\varphi(0), \varphi)$, with the norm

$$||x||_X = \left(|\varphi(0)|^p + \int_0^\infty |\,\varphi(t)|^p \rho(t)\, dt \right)^{1/p} = (|\,\varphi(0)|^p + ||\,\varphi||_r^p\,)^{1/p}\,.$$

Obviously, X is a Banach space.

An essential property of this space is the existence of strongly continuous linear semigroups of shift mappings (see [2], [3]): for $\sigma \geq 0$,

$$(T^{\sigma}\varphi)(s) = \begin{cases} \varphi(0), & s \in [0,\sigma], \\ \varphi(s-\sigma), & s \in [\sigma,\infty), \end{cases}$$
$$(\overline{T}^{\sigma}\varphi)(s) = \begin{cases} 0, & s \in [0,\sigma], \\ \varphi(s-\sigma), & s \in [\sigma,\infty), \end{cases}$$

for $s \geq 0$, $\varphi \in X$, and $\lim_{\sigma\to\infty} ||T^{\sigma}\varphi - \varphi(0)^{+}||_X = 0$, where $\varphi(0)^{+}$ is the constant function with value $\varphi(0)$. In terms of the space X, one can formulate the notion of X-admissibility for measurable functions defined on any left semiaxis of $\mathbb{R}$.

The measurable function $x : (-\infty, a] \to \mathbb{R}^k$, a =const$\in \mathbb{R}$, is X-*admissible* if for each $t \in (-\infty, a]$ the function x^t, called *its history up to* t and defined by $x^t(s) = x(t-s)$, $s \in \mathbb{R}^{+}$, is an element in X.

From the definition of the norm on the space X, we cite the following inequality, needed in our subsequent discussion: for each $x \in X$, $t_0 \in (-\infty, a]$ and $t \in [t_0, a]$,

$$(1) \qquad ||x^t||_X^2 \leq \tilde{K}\left[|x(t)|^2 + \overline{\overline{K}}^{\frac{2}{p}}||x^{t_0}||_r^2 + \left(\int_{t_0}^{t} |x(u)|^p \rho(t-u)\,du\right)^{\frac{2}{p}}\right],$$

where $\tilde{K} = 3^{\lambda l - 1} \vee 1$ (see [14]).

In what follows, denote by

$$(2) \qquad ||x||_t^{*} = \sup_{t_0 \leq s \leq t} ||x^s||_X, \quad t \in [t_0, a].$$

The functional differential equation, called *the hereditary differential equation*

$$\dot{x}(t) = f(t, x^t), \quad x^0 = \varphi, \quad \varphi \in X,$$

where $f : \mathbb{R} \times X \to \mathbb{R}^k$ is the given functional, is considered in the papers [2], [4], and in many others. Its solution consists of a function $x : (-\infty, a] \to \mathbb{R}^k$, a =const> 0, such that x is X-admissible on $(-\infty, a]$, $x(t)$ is differentiable for each $t \in (0, a]$, the equation holds for $t \in [0, a]$ and $x^0 = \varphi$.

Because of the properties of the mapping $\overline{T}^{\sigma}$, in order to determine a solution of this equation, we have to find an X-admissible function $x \in$

$C^1((-\infty, a]; \mathbb{R}^k)$, such that $x(0) = \varphi(0)$ and for which this equation is valid. Here, $x^t = (x(t), x_r^t)$, where

$$x(t) = \begin{cases} x(t), & 0 \le t \le a, \\ \varphi(-t), & t \le 0, \end{cases}$$

$$x_r^t(s) = \begin{cases} x(t-s), & 0 \le s \le t, \\ \varphi(s-t), & s > t. \end{cases}$$

The continuity of the function $x(t)$ on $[0, a]$ implies that the function x^t, $t \in [0, a]$, is also continuous with respect to the norm of the space X.

All preceding notions and definitions are appropriatelly used to analyze *the stochastic hereditary integrodifferential equation*

$$\begin{aligned} (3) \qquad dx(t) = &\left[a_1(t, x^t) + \int_{t_0}^t a_2(t, s, x^s) ds + \int_{t_0}^t a_3(t, s, x^s) dW(s)\right] dt \\ &+ \left[b_1(t, x^t) + \int_{t_0}^t b_2(t, s, x^s) ds + \int_{t_0}^t b_3(t, s, x^s) dW(s)\right] dW(t), \\ &x^{t_0} = \varphi^{t_0}, \quad t \in [t_0, T], \end{aligned}$$

for which the existence, uniqueness and stability problems are studied in details in the paper [14]. Note that all considerations are on a probability space $(\Omega, \mathcal{F}, P)$. Here $W(t)$ is an $\mathbb{R}^m$-valued standard Wiener process, adapted to the family $\{\mathcal{F}_t, t \ge 0\}$ of nondecreasing sub σ-algebras of $\mathcal{F}$; $x(t)$ is an $\mathbb{R}^k$-valued stochastic process, the functionals

$$\begin{aligned} &a_1 : [t_0, T] \times X \to \mathbb{R}^k, \qquad &&b_1 : [t_0, T] \times X \to \mathbb{R}^k \times \mathbb{R}^m, \\ &a_2 : J \times X \to \mathbb{R}^k, &&b_2 : J \times X \to \mathbb{R}^k \times \mathbb{R}^m, \\ &a_3 : J \times X \to \mathbb{R}^k \times \mathbb{R}^m, &&b_3 : J \times X \to \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^m, \end{aligned}$$

where $J = \{(t, s) \in [t_0, T] \times [t_0, T] : \ s \le t\}$, are assumed to be Borel measurable on their domains. In what follows, we shall denote by $|\cdot|$ different norms on the spaces $\mathbb{R}^k, \mathbb{R}^k \times \mathbb{R}^m$ and $\mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^m$.

Note that the equation (3) presents an extension of the Ito-Volterra equations developed by Berger and Mizel in [1].

We cite the following lemma, needed to define a solution of the equation (3), First, for $t_0 \ge 0$ let $\mathcal{X}_{t_0}$ be the space of measurable random processes $x(t)$, $t \le t_0$, such that $x^{t_0} \in X$ for a.e. ω and such that for every such t, $x(t)$ is independent of $\{W_u - W_{t_0} : u \ge t_0\}$. By the structure of the space X, it follows that $x^t \in X$ for all $t \le t_0$ a.s.. Also, the initial segment φ is assumed to belong to $\mathcal{X}_{t_0}$.

Lemma. (Mizel, Trutzer, [13, p. 5]) *Let $x(t)$, $t \in R$, be a (jointly) measurable stochastic process such that $\sigma\{x(u) : u \leq t_0\} = \mathcal{G}_{t_0}$ is independent of $W_t - W_{t_0}$, $t \geq t_0$, and such that for $t \geq t_0$, $x(\cdot)$ is continuous and $\mathcal{G}_t := \mathcal{G}_{t_0} \vee \mathcal{F}_t$–progressivelly measurable. Assume that for a.e. ω the function $x^{t_0}(\cdot, \omega) \in X$. Then for $t \geq t_0$, $x^t(\omega) \in X$ for a.e. ω and the process x^t with values in X is a.s. continuous and $\mathcal{G}_t$–progressively measurable.*

A stochastic process $x(t)$, $t \in (-\infty, T]$, is *a strong solution* of the equation (3) for $t \in [t_0, T]$ if:

a) $x(t)$ is nonanticipating for $t \leq T$;
b) for a.e. ω, $x^t \in X$, $t \in [t_0, T]$;
c) $\hat{a}_1(t) = a_1(t, x^t)$, $\hat{a}_2(t,s) = a_2(t,s,x^s)$, $\hat{a}_3(t,s) = a_3(t,s,x^s)$,
$\hat{b}_1(t) = b_1(t, x^t)$, $\hat{b}_2(t,s) = b_2(t,s,x^s)$, $\hat{b}_3(t,s) = b_3(t,s,x^s)$,

are such that

$$\int_{t_0}^{T} |\hat{a}_1(t)| dt < \infty \ \text{ a.s.}, \qquad \int_{t_0}^{T} |\hat{b}_1(t)|^2 dt < \infty \ \text{ a.s.},$$
$$\int_{t_0}^{T} \int_{t_0}^{t} |\hat{a}_2(t,s)| ds dt < \infty \text{ a.s.},$$

and $\hat{a}_3$, $\hat{b}_2$, $\hat{b}_3$ satisfy $\int_{t_0}^{T} \int_{t_0}^{t} |f(t,s)|^2 ds dt < \infty$ a.s.;
d) $x^{t_0} = \varphi^{t_0}$;
e) the equation (3) holds a.s. for each $t \in [t_0, T]$.

As in the case of the deterministic hereditary differential equation, if $x(t)$, $t \in (-\infty, T]$, is a strong solution of the equation (3), then the Lemma and the properties of the mapping $\overline{T}^{\sigma}$ imply that x^t is almost surely continuous and $\mathcal{G}_t$-measurable stochastic process.

Also, we cite the following existence and uniqueness theorem for the equation (3), appearing in the paper [14], needed in our future investigation.

Theorem A. (V.J. Mizel and V. Trutzer, [14; Theorem 2.1', p. 18]) *Assume that there exists a constant L such that for all $(t,s) \in J$ and $x, y \in X$*

$$\begin{aligned} (4) \qquad & |a_1(t,x) - a_1(t,y)| + |a_2(t,s,x) - a_2(t,s,y)| \\ & + |a_3(t,s,x) - a_3(t,s,y)| + |b_1(t,x) - b_1(t,y)| \\ & + |b_2(t,s,x) - b_2(t,s,y)| + |b_3(t,s,x) - b_3(t,s,y)| \\ & \leq L ||x - y||_X; \end{aligned}$$

$$
\begin{aligned}
(5) \qquad & |a_1(t,x)| + |a_2(t,s,x)| + |a_3(t,s,x)| \\
& + |b_1(t,x)| + |b_2(t,s,x)| + |b_3(t,s,x)| \\
& \le L(1 + ||x||_X).
\end{aligned}
$$

Then there exists a unique almost surely continuous strong solution $x(t)$, $t \in (-\infty, T]$, *of the equation* (3) *and*

$$E\{sup_{t\in[t_0,T]}|x(t)|^2\} < \infty.$$

Note that the proof of the existence of a solution is based, as in the classical stochastic case, on Picard–Lindelöf method of sucessive approximations, for $n = 0, 1, 2, \ldots$:

$$
\begin{aligned}
x_n(t) &= \varphi(t), \quad t \le t_0; \\
x_0(t) &= \varphi(t_0), \quad t \in [t_0, T]; \\
x_n(t) &= \varphi(t_0) + \Big[\int_{t_0}^t a_1(s, x_{n-1}^s) + \int_{t_0}^s a_2(s,u,x_{n-1}^u)du \\
& \quad + \int_{t_0}^s a_3(s,u,x_{n-1}^u)dW(u)\Big] ds + \Big[\int_{t_0}^t b_1(s, x_{n-1}^s) \\
& \quad + \int_{t_0}^s b_2(s,u,x_{n-1}^u)du + \int_{t_0}^s b_3(s,u,x_{n-1}^u)dW(u)\Big] dW(s), \quad t \ge t_0.
\end{aligned}
$$

Note, also, that the existence and uniqueness problem for this equation is considered in the paper [10], using the concept of a random bounded integral contractor, which includes the Lipschitz condition as a special case.

2. Main results

Together with the equation (3) we consider the sequence of stochastic hereditary integrodifferential equations

$$
\begin{aligned}
(6) \qquad dx_{n+1}(t) &= \Big[a_{1,n}(t, x_{n+1}^t) + \int_{t_0}^t a_{2,n}(t,s,x_{n+1}^s)ds \\
& \quad + \int_{t_0}^t a_{3,n}(t,s,x_{n+1}^s)dW(s)\Big] dt + \Big[b_{1,n}(t, x_{n+1}^t) \\
& \quad + \int_{t_0}^t b_{2,n}(t,s,x_{n+1}^s)ds + \int_{t_0}^t b_{3,n}(t,s,x_{n+1}^s)dW(s)\Big] dW(t), \\
& \quad x_{n+1}^{t_0} = \varphi^{t_0}, \quad t \in [t_0, T], \quad n \in \mathbb{N},
\end{aligned}
$$

where all functionals $a_{i,n}$, $b_{i,n}$, $i = 1, 2, 3$, $n \in \mathbb{N}$, are defined as the functionals a_i, b_i, $i = 1, 2, 3$, of the equation (3) and satisfy the conditions (4) and (5) of the Theorem A. Therefore, for every $n \in \mathbb{N}$ the equation (6) has a unique strong solution $x_{n+1}(t)$, $t \in (-\infty, T]$, as almost surely continuous $\mathcal{G}_t$-measurable stochastic process.

Our main purpose here is to give sufficient conditions of closeness of the functionals $a_{i,n}$, $b_{i,n}$, $i = 1, 2, 3$, $n \in \mathbb{N}$, with the corresponding functionals a_i, b_i, $i = 1, 2, 3$, such that the sequence of processes $\{x_n(t), t \in (-\infty, T], n \in \mathbb{N}\}$ converges in some sense to the solution $x(t)$, $t \in (-\infty, T]$, of the equation (3) as $n \to \infty$.

In order to state the main result of this paper, denote by

$$\begin{aligned} F_n(t,s,x) = {} & |a_1(t,x) - a_{1,n}(t,x)| + |a_2(t,s,x) - a_{2,n}(t,s,x)| \\ & + |a_3(t,s,x) - a_{3,n}(t,s,x)| + |b_1(t,x) - b_{1,n}(t,x)| \\ & + |b_2(t,s,x) - b_{2,n}(t,s,x)| + |b_3(t,s,x) - b_{3,n}(t,s,x)|, \end{aligned}$$

where $n \in \mathbb{N}$ and $x_1(t)$, $t \in (-\infty, T]$, is any process having the properties of the solution of the equation (3) and $E\{\sup_{t_0 \le t \le T} |x_1(t)|^2\} < \infty$.

Theorem B. *Let the functionals a_i, b_i, $a_{i,n}$, $b_{i,n}$, $i = 1, 2, 3$, $n \in \mathbb{N}$, and $\varphi \in X$ satisfy the conditions of the Theorem A and let*

$$\sum_{n=1}^{\infty} \sup_{J \times X} F_n(t,s,x) < \infty. \tag{7}$$

Then the sequence of solutions $\{x_{n+1}(t), t \in (-\infty, T], n = 2, 3, \cdots\}$ of the equations (6) *converges almost surely, uniformly on $[t_0, T]$, to the solution $x(t)$, $t \in (-\infty, T]$, of the equation* (3)*.*

Proof. For $(t, s) \in J,\ x \in X$, denote by

$$\begin{aligned} G_n(t,s,x) = {} & |a_1(t,x) - a_{1,n}(t,x)|^2 + |a_2(t,s,x) - a_{2,n}(t,s,x)|^2 \\ & + |a_3(t,s,x) - a_{3,n}(t,s,x)|^2 + |b_1(t,x) - b_{1,n}(t,x)|^2 \\ & + |b_2(t,s,x) - b_{2,n}(t,s,x)|^2 + |b_3(t,s,x) - b_{3,n}(t,s,x)|^2, \quad n \in \mathbb{N}. \end{aligned}$$

and by

$$\epsilon_n = E\left\{\sup_J G_n(t,s,x_n^t)\right\}, \quad n \in \mathbb{N}, \tag{8}$$

where $x_1(t)$, $t \in (-\infty, T]$, is any process having the properties of the solution of the equation (3) and $E\{\sup_{t_0\le t\le T}|x_1(t)|^2\} < \infty$.

So, the condition (7) implies $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

By using the integral form of the equations (3) and (6) and by adding some terms, we have for $t \in [t_0, T]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&x(t) - x_{n+1}(t) \\
=&\int_{t_0}^{t} \{[a_1(s, x^s) - a_1(s, x_n^s)] + [a_1(s, x_n^s) - a_{1,n}(s, x_n^s)] \\
&+[a_{1,n}(s, x_n^s) - a_{1,n}(s, x^s)] + [a_{1,n}(s, x^s) - a_{1,n}(s, x_{n+1}^s)]\} ds \\
&+\cdots + \int_{t_0}^{t}\int_{t_0}^{s} \{[b_3(s, u, x^u) - b_3(s, u, x_n^u)] \\
&+[b_3(s, u, x_n^u) - b_{3,n}(s, u, x_n^u)] + [b_{3,n}(s, u, x_n^u) - b_{3,n}(s, u, x^u)] \\
&+[b_{3,n}(s, u, x^u) - b_{3,n}(s, u, x_{n+1}^u)]\}\, dW(u)dW(s).
\end{aligned}$$

From the inequality $(p_1 + p_2 + \cdots + p_k)^2 \le k\,(p_1^2 + p_2^2 + \cdots + p_k^2)$, we obtain

$$\begin{aligned}
&E\{\sup_{t_0\le s\le t} |x(s) - x_{n+1}(s)|^2\} \\
\le& 24\left[E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[a_1(u, x^u) - a_1(u, x_n^u)]du\right|^2\right\} + \cdots\right. \\
&+ \left. E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[b_{3,n}(u, v, x^v) - b_{3,n}(u, v, x_{n+1}^v)]dW(v)dW(u)\right|^2\right\}\right],
\end{aligned}$$

and we estimate each of these integrals. So, by applying Cauchy–Schwarz inequality, the usual stochastic integral isometry, the Lipschitz condition (4), (2) and (8), we find:

$$\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[a_1(u, x^u) - a_1(u, x_n^u)]du\right|^2\right\} \\
\le&(t - t_0)\int_{t_0}^{t} E|a_1(u, x^u) - a_1(u, x_n^u)|^2\, ds \\
\le&(T - t_0)L^2\int_{t_0}^{t} E||x - x_n||_s^{*2} ds;
\end{aligned}$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[a_1(u,x_n^u)-a_{1,n}(u,x_n^u)]du\right|^2\right\}\\
\le&(t-t_0)\int_{t_0}^{t}E|a_1(u,x_n^u)-a_{1,n}(u,x_n^u)|^2du\\
\le&(T-t_0)\int_{t_0}^{t}E\{\sup_J G_n(u,v,x_n^u)\}du=(t-t_0)(T-t_0)\epsilon_n;
\end{aligned}
$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[a_2(u,v,x^v)-a_2(u,v,x_n^v)]dvdu\right|^2\right\}\\
\le&(t-t_0)\int_{t_0}^{t}E\left|\int_{t_0}^{u}[a_2(u,v,x^v)-a_2(u,v,x_n^v)]dv\right|^2du\\
\le&(t-t_0)\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}E|a_2(u,v,x^v)-a_2(u,v,x_n^v)|^2dvdu\\
\le&(t-t_0)L^2\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}E||x-x_n||_v^{*2}dvdu.
\end{aligned}
$$

By using partial integration, we obtain

$$
\begin{aligned}
&\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}E||x-x_n||_v^{*2}dvdu=\frac{(t-t_0)^2}{2}\int_{t_0}^{t}E||x-x_n||_u^{*2}du\\
-&\int_{t_0}^{t}\frac{(u-t_0)^2}{2}\int_{t_0}^{u}E||x-x_n||_v^{*2}dvdu\le\frac{(T-t_0)^2}{2}\int_{t_0}^{t}E||x-x_n||_u^{*2}du.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[a_2(u,v,x^v)-a_2(u,v,x_n^v)]dvdu\right|^2\right\}\\
\le&\frac{(T-t_0)^3}{2}L^2\int_{t_0}^{t}E||x-x_n||_s^{*2}ds;
\end{aligned}
$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[a_2(u,v,x_n^v)-a_{2,n}(u,v,x_n^v)]dvdu\right|^2\right\}\\
\le&(t-t_0)\int_{t_0}^{t}E\left|\int_{t_0}^{u}[a_2(u,v,x_n^v)-a_{2,n}(u,v,x_n^v)]dv\right|^2du\\
\le&(t-t_0)\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}E|a_2(u,v,x_n^v)-a_{2,n}(u,v,x_n^v)|^2dvdu\\
\le&(t-t_0)\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}\epsilon_n dvdu\le(t-t_0)\frac{(T-t_0)^3}{3}\epsilon_n;
\end{aligned}
$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[a_3(u,v,x^v)-a_3(u,v,x_n^v)]dW(v)du\right|^2\right\}\\
\le&(t-t_0)\int_{t_0}^{t}E\left|\int_{t_0}^{u}[a_3(u,v,x^v)-a_3(u,v,x_n^v)]dW(v)\right|^2du\\
=&(t-t_0)\int_{t_0}^{t}\int_{t_0}^{u}E|a_3(u,v,x^v)-a_3(u,v,x_n^v)|^2dvdu\\
\le&(t-t_0)L^2\int_{t_0}^{t}\int_{t_0}^{u}E||x-x_n||_u^{*2}dvdu\\
\le&(T-t_0)^2L^2\int_{t_0}^{t}E||x-x_n||_s^{*2}ds;
\end{aligned}
$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[a_3(u,v,x_n^v)-a_{3,n}(u,v,x_n^v)]dW(v)du\right|^2\right\}\\
\le&(t-t_0)\int_{t_0}^{t}E\left|\int_{t_0}^{u}[a_3(u,v,x_n^v)-a_{3,n}(u,v,x_n^v)]dW(v)\right|^2du\\
=&(t-t_0)\int_{t_0}^{t}\int_{t_0}^{u}E|a_3(u,v,x_n^v)-a_{3,n}(u,v,x_n^v)|^2dvdu\\
\le&(t-t_0)\frac{(T-t_0)^2}{2}\epsilon_n.
\end{aligned}
$$

By applying Doob inequality for Ito integrals and by using the preceding estimations, we get:

$$E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[b_1(u,x^u)-b_1(u,x_n^u)]dW(u)\right|^2\right\}$$
$$\le 4\int_{t_0}^{t}E|b_1(u,x^u)-b_1(u,x_n^u)|^2du\le 4L^2\int_{t_0}^{t}E||x-x_n||_s^{*2}ds;$$

$$E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[b_1(u,x_n^u)-b_{1,n}(u,x_n^u)]dW(u)\right|^2\right\}\le 4(t-t_0)\epsilon_n;$$

$$E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[b_2(u,v,x^v)-b_2(u,v,x_n^v)]dvdW(u)\right|^2\right\}$$
$$\le 4\int_{t_0}^{t}E\left|\int_{t_0}^{u}[b_2(u,v,x^v)-b_2(u,v,x_n^v)]dv\right|^2du$$
$$\le 4\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}E||x-x_n||_v^{*2}dvdu\le 4\frac{(T-t_0)^2}{2}L^2\int_{t_0}^{t}E||x-x_n||_s^{*2}ds;$$

$$E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[b_2(u,v,x_n^v)-b_{2,n}(u,v,x_n^v)]dvdW(u)\right|^2\right\}$$
$$\le 4\int_{t_0}^{t}E\left|\int_{t_0}^{u}[b_2(u,v,x_n^v)-b_{2,n}(u,v,x_n^v)]dv\right|^2du$$
$$\le 4\int_{t_0}^{t}(u-t_0)\int_{t_0}^{u}\epsilon_n dvdu\le 4(t-t_0)\frac{(T-t_0)^2}{3}\epsilon_n;$$

$$E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[b_3(u,v,x^v)-b_3(u,v,x_n^v)]dW(v)dW(u)\right|^2\right\}$$
$$\le 4\int_{t_0}^{t}E\left|\int_{t_0}^{u}[b_3(u,v,x^v)-b_3(u,v,x_n^v)]dW(v)\right|^2du$$
$$=4\int_{t_0}^{t}\int_{t_0}^{u}E|b_3(u,v,x^v)-b_3(u,v,x_n^v)|^2dvdu$$
$$\le 4(T-t_0)L^2\int_{t_0}^{t}E||x-x_n||_s^{*2}ds;$$

$$
\begin{aligned}
&E\left\{\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}\int_{t_0}^{u}[b_3(u,v,x_n^v)-b_{3,n}(u,v,x_n^v)]dW(v)dW(u)\right|^2\right\}\\
\le& 4\int_{t_0}^{t}E\left|\int_{t_0}^{u}[b_3(u,v,x_n^v)-b_{3,n}(u,v,x_n^v)]dW(v)\right|^2du\\
=&4\int_{t_0}^{t}\int_{t_0}^{u}E|b_3(u,v,x_n^v)-b_{3,n}(u,v,x_n^v)|^2dvdu\le 4(t-t_0)\frac{(T-t_0)}{2}\epsilon_n.
\end{aligned}
$$

If we apply (1) and if we use (2), for $s\in[t_0,T]$ we have

$$
\begin{aligned}
||x-x_n||_s^{*2}=\sup_{t_0\le u\le s}||x^u-x_n^u||_X^2\le \tilde K\Bigg[&\sup_{t_0\le u\le s}|x(u)-x_n(u)|^2\\
&+\overline{\overline{K}}^{\frac{2}{p}}||x^{t_0}-x_n^{t_0}||_r^2\left(\int_{t_0}^{s}|x(v)-x_n(v)|^p\rho(t-v)dv\right)^{\frac{2}{p}}\Bigg].
\end{aligned}
$$

Since $||x^{t_0}-x_n^{t_0}||_r^p=0$, we get easy

$$
||x-x_n||_s^{*2}\le \tilde K(1+||\rho||_{L_1}^{\frac{2}{p}})\sup_{t_0\le u\le s}|x(u)-x_n(u)|^2. \tag{9}
$$

Now, from preceding estimations we obtain

$$
\begin{aligned}
&E\{\sup_{t_0\le s\le t}|x(s)-x_{n+1}(s)|^2\}\\
\le& 24L^2\cdot\left[\frac{(T-t_0)^3}{2}+3(T-t_0)^2+5(T-t_0)+4\right]\\
&\cdot\left[2\int_{t_0}^{t}E||x-x_n||_s^2ds+\int_{t_0}^{t}E||x-x_{n+1}||_s^2ds\right]\\
&+24(t-t_0)\cdot\left[\frac{(T-t_0)^3}{3}+\frac{11}{6}(T-t_0)^2+3(T-t_0)+4\right]\cdot\epsilon_n.
\end{aligned}
$$

Denote by

$$
\begin{aligned}
\alpha&=24L^2\cdot\left[\frac{(T-t_0)^3}{2}+3(T-t_0)^2+5(T-t_0)+4\right]\tilde K_2(1+||\rho||_{L_1}^{\frac{2}{p}})\\
\beta&=24\cdot\left[\frac{(T-t_0)^3}{3}+\frac{11}{6}(T-t_0)^2+3(T-t_0)+4\right]
\end{aligned}
$$

and

$$u_n(t) = E\{\sup_{t_0 \le s \le t} |x(s) - x_n(s)|^2\}, \quad n \in \mathbb{N}.$$

Then from (9) we have

$$u_{n+1}(t) \le 2\alpha \int_{t_0}^{t} u_n(s)ds + \alpha \int_{t_0}^{t} u_{n+1}(s)ds + \beta\epsilon_n(t - t_0),\ t \in [t_0, T],\ n \in \mathbb{N}.$$

Now we apply one version of the well–known Gronwall–Bellman inequality (see, for example, [11]):

Let $U : [a, b] \to \mathbb{R},\ V : [a, b] \to \mathbb{R}$ be nonnegative integrable functions and H be a positive constant, such that the inequality

$$U(t) \le V(t) + H \int_{a}^{t} U(s)ds, \quad t \in [a, b],$$

holds. Then

$$U(t) \le V(t) + H \int_{a}^{t} V(s)e^{H(t-s)}ds, \quad t \in [a, b].$$

So, we come to the following recurence formula

$$\begin{aligned} u_{n+1}(t) \le 2\alpha \int_{t_0}^{t} u_n(s)ds + \beta\epsilon_n(t - t_0) \\ + \alpha \int_{t_0}^{t} \left[2\alpha \int_{t_0}^{s} u_n(v)dv + \beta\epsilon_n(s - t_0) \right] e^{\alpha(s-t_0)}ds. \end{aligned}$$

This formula is considered in the paper [7], where the following upper bound for $u_{n+1}(t)$ is obtained by induction, repeating integrations,

$$u_{n+1}(t) < \left[2\alpha M \cdot \frac{[2\alpha(t - t_0)]^{n-1}}{(n-1)!} + \beta \sum_{k=1}^{n} \epsilon_k \frac{[2\alpha(t - t_0)]^{n-k}}{(n-k)!} \right] \cdot \frac{e^{\alpha(t-t_0)} - 1}{\alpha}$$

where

$$E\{\sup_{t_0 \le t \le T} |x(t) - x_1(t)|^2\} \le 2 \left[E\{\sup_{t_0 \le t \le T} |x(t)|^2\} + E\{\sup_{t_0 \le t \le T} |x_1(t)|^2\} \right] = M.$$

Denote the polunomial

$$P_{n-1}(q) \equiv 2\alpha M \cdot \frac{q^{n-1}}{(n-1)!} + \beta \sum_{k=1}^{n} \epsilon_k \frac{q^{n-k}}{(n-k)!}.$$

Then we have

$$U_{n+1} < P_{n-1}(2\alpha(t-t_0)) \cdot \frac{e^{\alpha(t-t_0)}-1}{\alpha}, \quad n \in \mathbb{N}. \tag{10}$$

According to the Chebyshev inequality, for arbitrary $\epsilon > 0$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} P\left\{\sup_{t_0\le t\le T} |x(t)-x_n(t)| \ge \epsilon\right\} &\le \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} E\{\sup_{t_0\le t\le T}|x(t)-x_n(t)|^2\} \\ = \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} u_n(T) &< \frac{1}{\epsilon^2}\left[M + \sum_{n=2}^{\infty} P_{n-2}(2\alpha(T-t_0)) \cdot \frac{e^{\alpha(T-t_0)}-1}{\alpha}\right]. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=2}^{\infty} P_{n-2}(2\alpha(T-t_0)) \\ &< 2\alpha M \sum_{n=2}^{\infty} \frac{[2\alpha(T-t_0)]^{n-2}}{(n-2)!} + \beta \sum_{n=1}^{\infty} \epsilon_n \cdot \sum_{n=2}^{\infty} \frac{[2\alpha(T-t_0)]^{n-2}}{(n-2)!} < \infty, \end{aligned}$$

it follows by Borel–Cantelli lemma

$$P\{\sup_{t_0\le t\le T} |x(t)-x_n(t)| \ge \epsilon \text{ infinitely often }\} = 0,$$

i.e. for all large n

$$\sup_{t_0\le t\le T} |x(t)-x_n(t)| < \epsilon \text{ almost surely }. \tag{11}$$

Therefore, $x_n(t) \to x(t)$ as $n \to \infty$ almost surely, uniformly on $[t_0, T]$. □

Therefore, the solution of the equation (3) is approximated by the solutions of the equations (6). Obviously, from (10) we have obtained a mean square error of n–th approximation

$$E\{\sup_{t_0\le t\le T}|x(t)-x_n(t)|^2\} < P_{n-2}(2\alpha(T-t_0)) \cdot \frac{e^{\alpha(T-t_0)}-1}{\alpha}, \quad n = 2, 3, \ldots .$$

Note that the condition (7) is very strict, but it is easy to investigate it in concrete situations. Note also that it could be modified and weakened, ie. the Theorem B could be proved with the assumption $\sum_{k=1}^{\infty}$ instead of the condition (7). At least theoreticaly, this fact gives us a possibility to construct an ϵ–approximation for the solution by a suitable choice of a sequence of functionals $a_{i,n},\ b_{i,n},\ i = 1, 2, 3,\ n \in \mathbb{N}$, in the following way:

First, let $\sum_{n=1}^{\infty} c_n,\ c_n = \text{const} \geq 0$, be any convergent series. Let $x_1(t),\ t \in (-\infty, T]$,be an arbitrary X-admisible, almost surely continuons $\mathcal{G}_t$-measurable prosess with $x_1^{t_0} = \varphi^{t_0},\ E\{\sup_{t_0 \leq t \leq T} |x_1(t)|^2\} < \infty$.

Next, we choose functionals $a_{i,1},\ b_{i,1},\ i = 1, 2, 3$, satisfying the conditions (4) and (5), such that $\sup_J G_1(t, s, x_1^t) \leq c_1$ almost surely and we determine a solution $x_2(t),\ t \in (-\infty, T]$, of the equation (6) for $n = 1$. Inductively, if we have a solution $x_n(t),\ t \in (-\infty, T]$, we choose functionals $a_{i,n},\ b_{i,n}$, $i = 1, 2, 3$, satisfying (4) and (5), such that $\sup_J G_n(t, s, x_n^t) \leq c_n$ almost surely. Now, a process $x_{n+1}(t),\ t \in (-\infty, T]$, is defined as a solution of the equation (6), etc.. Clearly, $\sum_{n=1}^{\infty} \epsilon_n \leq \sum_{n=1}^{\infty} c_n < \infty$.

Thus, for arbitrary $\epsilon > 0$ there exists an ϵ–approximation of the solution, i.e., there exists a number $m \in N$, such that from (11),

$$\sup_{t_0 \leq t \leq T} |x(t) - x_m(t)| < \epsilon \quad \text{almost surely.}$$

Analogously to the papers [7] and [17], we use the notion *Z–algorithm* for the described iterative method. Since the functionals $a_{i,n},\ b_{i,n},\ i = 1, 2, 3$, define $(n+1)$–th approximation, the sequence of the set of functionals

$$\{\,(a_{1,n}, a_{2,n}, a_{3,n}, b_{1,n}, b_{2,n}, b_{3,n}), n \in \mathbb{N}\} \tag{12}$$

is called *the determining sequence for the Z–algorithm.*

Naturally, the speed of convergence depends on first approximation and on the choice of determining sequence. We consider the Z–algorithm to be good enough if the equations (6) can be effectively solvable. Certainly, in the case of stochastic hereditary integrodifferential equations this requirement is too strong and, from a practical point of view, it is almost impossible to form such an algorithm. This fact suggests us to construct simple forms of linearization of the functionals $a_i,\ b_i,\ i = 1, 2, 3$, as it will be shown in the next examples. Also, our forthcoming study is mostly based on theoretical considerations that some well–known iterative methods are concrete Z–algorithms.

Example 1. Let the functionals a_i, b_i, $i = 1,2,3$, satisfy the conditions of the Theorem A and let the functions $\alpha_{i,n} : [t_0, T] \to \mathbb{R}^k$, $\beta_{i,n} : [t_0, T] \to \mathbb{R}^k$, $i = 1,2,3$, $n \in \mathbb{N}$, be uniformly bounded with some constant l. Then the sequence of functionals (12), defined by

$$\begin{aligned}
a_{1,n}(t,x) &= \alpha_{1,n}(t)||x - x_n^t||_X + a_1(t, x_n^t),\\
a_{i,n}(t,s,x) &= \alpha_{i,n}(t)||x - x_n^t||_X + a_i(t,s,x_n^s), \quad i = 2,3,\\
b_{1,n}(t,x) &= \beta_{1,n}(t)||x - x_n^t||_X + b_1(t, x_n^t),\\
b_{i,n}(t,s,x) &= \beta_{i,n}(t)||x - x_n^t||_X + b_i(t,s,x_n^s), \quad i = 2,3
\end{aligned}$$

describes a determining sequence of the Z–algorithm for the equation (3). Really, the functionals $a_{i,n}$, $b_{i,n}$, $i = 1,2,3$, $n \in \mathbb{N}$, satisfy the Lipschitz condition (4) with the constant l. It is easy to obtain $G_n(t,s,x_n^t) \equiv 0$ almost surely for every $n \in \mathbb{N}$ and, therefore, $\sum_{n=1}^{\infty} \epsilon_n < \infty$. But the condition (5) does not hold directly. By following the procedure as in the paper [7], by using the results of the paper [14; Theorem 2.3', p. 31 and Theorem 2.4', p. 33] and the usual properties of stopping times, we come to the conclusion that the Theorem B can be applied and, therefore, this iterative method describes the Z–algorithm. So, from (10) we find

$$E\{\sup_{t_0 \le t \le T} |x(t) - x_n(t)|^2\} < 2\alpha M \cdot \frac{[2\alpha(T - t_0)]^{n-2}}{(n-2)!} \cdot \frac{e^{\alpha(T-t_0)} - 1}{\alpha}, \; n \in \mathbb{N} \backslash \{1\}.$$

In particular, if $\alpha_{i,n} = \beta_{i,n} \equiv 0$, $i = 1,2,3$, $n \in \mathbb{N}$, then this algorithm is reduced to the Picard–Lindelöf method of succesive approximations.

Example 2. Let the functionals a_i, b_i, $i = 1,2,3$, satisfy the conditions of the Theorem A and let $\alpha_{i,n} : [t_0, T] \to \mathbb{R}$, $\beta_{i,n} : [t_0, T] \to \mathbb{R}$, $i = 1,2,3$, $n \in \mathbb{N}$, be uniformly bounded functions. Define the sequence of functionals

$$\begin{aligned}
a_{1,n}(t,x) &= \alpha_{1,n}(t)[x(t) - x_n(t)] + a_1(t, x_n^t),\\
a_{i,n}(t,s,x) &= \alpha_{i,n}(t)[x(t) - x_n(t)] + a_i(t,s,x_n^s), \quad i = 2,3,\\
b_{1,n}(t,x) &= \beta_{1,n}(t)[x(t) - x_n(t)] + b_1(t, x_n^t),\\
b_{i,n}(t,s,x) &= \beta_{i,n}(t)[x(t) - x_n(t)] + b_i(t,s,x_n^s), \quad i = 2,3.
\end{aligned}$$

Similarly to the Example 1, it is easy to prove that (12) describes a determining sequence of the Z–algorithm. Therefore, the sequence of solutions of suitable linear stochastic integrodifferential equations (6), not hereditary with respect to unknown processes $x_n(t)$, $n \in \mathbb{N}$, presents iterations of the solution $x(t)$ of the equation (3). Obviously, if $\alpha_{i,n} = \beta_{i,n} \equiv 0$, $i = 1, 2, 3$, $n \in \mathbb{N}$, we obtain also Picard–Lindelöf method of iterations.

As we saw earlier, our further intention is to construct some other determining sequences and to choose the best one, in the sence that the equations (6) could be effectively solveble with the fastest convergence of their solutions to the solution of the equation (3). Also, having in mind the results of the paper [10], a subject of forthcoming investigation is to give sufficient conditions under which the iterative procedure used to prove the existence of a solution of the equation (3), describes a special Z–algorithm.

Acknowledgement. The authors are grateful to the referee for helpful suggestions on the improvement of the paper.

REFERENCES

1. M. A. Berger and V. J. Mizel: *Volterra equations with Ito integrals, I.* J. of Integral equations **2** (1980), 187–245.
2. B. D. Coleman and V. J. Mizel: *Norms and semigroups in the theory of fading memory.* Arch. Rational Mech. Anal. **23** (1966), 87–123.
3. B. D. Coleman and V. J. Mizel: *On the general theory of fading memory.* Arch. Rational Mech. Anal. **29** (1968), 18–31.
4. B. D. Coleman and D. R. Owen: *On the initial value problem for a class of functional-differential equations.* Arch. Rational Mech. Anal. **55** (1974), 275–299.
5. J. Hale: *Theory of Functional Differential Equations.* Springer, 1977.
6. N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes.* Nort Holland, 1981.
7. S. Janković: *Iterative procedure for solving stochastic differential equations.* Mathematica Balkanica, New Series, **1** (1) (1987) 64–71.
8. S. Janković: *One general iterative method treating stochastic integrodifferential equations.* Zbornik radova Filozofskog fakulteta u Nišu, Serija Mat., **1** (11) (1987), 31–39.
9. S. Janković: *Some special iterative procedure for solving stochastic differential equations of Ito type.* Mathematica Balkanica, New Series, **3** (1) (1989), 44–50.
10. M. Jovanović and S. Janković: *On a class of nonlinear stochastic hereditary integrodifferential equations.* Indian J. pure appl. Math. **28** (8) (1997), 1061–1082.

11. G. Kallianpur: *Stochastic Filtering Theory.* Springer-Verlag, New York Inc., 1980.
12. S. Kanagawa: *The rate of convergence for approximate solutions of stochastic differential equations.* Tokyo J. Math. **12** (1) (1989), 31–48.
13. M. J. Leitman and V. Mizel: *Hereditary laws and nonlinear integral equations on the line.* Adv. in Math. **22** (1976), 220–266.
14. V. J. Mizel and V. Trutzer: *On stochastic hereditary equations: existence and asymptotic stability.* J. of Integral Equations **7** (1984), 1–72.
15. A. N. V. Rao and W. J. Padgett: *Stability for a class of stochastic nonlinear feedback systems.* J. of Integral Equations **6** (1984), 159–173.
16. R. Zuber: *About one algorithm for solving of first order differential equations I.* Zastosow. Mat. VIII **4** (1966), 351-363 (in Polish).
17. R. Zuber: *About one algorithm for solving of first order differential equations II.* Zastosow. Mat.IX **1** (1966), 85–97 (in Polish).

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