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FRACTAL SURFACES LEVELLING

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. This paper presents some visualization techniques for data interpolation by fractal surfaces. The fractal dimension plays a role of the shape parameter. Both, regular and scattered data, are examined and the corresponding algorithms for levelling these surfaces are developed and illustrated through several examples.

1. Introduction

Euclidean geometry and elementary functions, such as polynomials, rational polynomials, trigonometric and exponential functions are the basis of the traditional methods for approximation discrete data. Graphical systems founded on traditional geometry are effective for making pictures of man-made objects, such as roads, buildings, furniture, vehicles etc. But, there are many objects in nature which can not be easily described in terms of elementary functions and Euclidean geometry only, such as, for example, profiles of mountain ranges, clouds and horizons over forests. It is desirable for graphical systems to be able to deal with this type of problem. For modelling these objects fractal functions are extremely useful [6], [8].

Here, in this paper, some levelling techniques for data interpolation by fractal surfaces are developed. These algorithms help in increasing the realism of the computer generated images of previous type. Here we restrict ourselves on fractal surfaces from $\mathbb{R}^3$ being a special case of the general theory developed by Barnsley [6] and Massopust [8].

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Let the discrete data $\{(x_n, y_n, F_n) : \ n = 0, 1, \ldots, N\}$ be known. Using affine transformations one can form an iterative function system (IFS) $\{\mathbb{R}^3; w_n; n = 1, \ldots, N\}$, where the transformation w_n is defined by

$$w_n(\mathbf{x}) = \begin{bmatrix} a_n & b_n & 0 \\ c_n & d_n & 0 \\ e_n & f_n & g_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} h_n \\ i_n \\ j_n \end{bmatrix},$$

where $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$. In other words, w_n transforms a triangle from $\mathbb{R}^3$ into a triangle from $\mathbb{R}^3$. By the analogy with 2D cases, in each transformations, there is one free parameter, namely g_n, supplying vertical scaling control. For example, the choice $g_n = 0$ will produce the piecewise linear interpolation function. This parameter also determines fractal dimension.

Choosing g_n for a free parameter allowed us 9 parameters more, which give us a chance to transform three points ($\in \mathbb{R}^3$) to another three points.

2. Regular data interpolation

In the cases of regular data, two special cases of Massopust's construction [8, Chapter 8] will be considered: 1. data can be arranged in a triangular or 2. in a rectangular mesh.

First, let us assume a triangular arrangement. Then, interpolation nodes are arranged like in the Figure 1 (left).

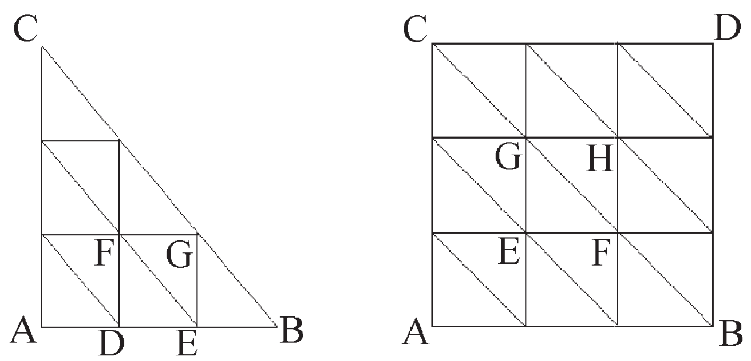

FIG. 1. Two ways of ordering the data

So, the interpolation nodes form N^2 triangles, which make up one big triangle Δ. The idea for obtaining interpolation function is to determine one transformation, which will transform big triangle onto the smaller one. After that, in each step, algorithm arbitrarily chooses between different transformations.

The nodes in the mesh are distanced for h_x on x-axis, and for h_y on y-axis. The lengths of the triangle's sides along the x and y-axis are H_x and H_y. The small triangles can be divided in two groups: 1. Homothetic images of the triangle ABC, or 2. Mirror images of the triangles of the group 1 w.r.t. hypotenuse.

The corresponding transformation will be different for each group, i.e.

$$w_{nm}(\mathbf{x}) = \begin{bmatrix} a_{nm} & 0 & 0 \\ 0 & b_{nm} & 0 \\ c_{nm} & d_{nm} & e_{nm} \end{bmatrix} \mathbf{x} + \begin{bmatrix} f_{nm} \\ g_{nm} \\ h_{nm} \end{bmatrix} \tag{1}$$

and

$$\bar{w}_{pq}(\mathbf{x}) = \begin{bmatrix} 0 & \bar{a}_{pq} & 0 \\ \bar{b}_{pq} & 0 & 0 \\ \bar{c}_{pq} & \bar{d}_{pq} & \bar{e}_{pq} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \bar{f}_{pq} \\ \bar{g}_{pq} \\ \bar{h}_{pq} \end{bmatrix}. \tag{2}$$

Free parameters are e_{nm} and $\bar{e}_{pq}$. Other parameters are determined so that the transformation w_{nm} maps $A \to D,\ B \to E,\ C \to F$ and the transformation $\bar{w}_{pq}$ maps $A \to E,\ B \to F,\ C \to H$. The coordinates of the points A, B and C are $A(x_0, y_0, z_0)$, $B(x_0 + H_x, y_0, z_1)$, and $C(x_0, y_0 + H_y, z_2)$.

By solving the corresponding linear systems, one gets

$$\begin{aligned} a_{nm} &= \frac{h_x}{H_x}, \qquad f_{nm} = x_{n-1} - \frac{h_x}{H_x} x_0, \\ b_{nm} &= \frac{h_y}{H_y}, \qquad g_{nm} = y_{m-1} - \frac{h_y}{H_y} y_0, \\ c_{nm} &= \frac{1}{H_x}((z_{n,m-1} - z_{n-1,m-1}) - e_{nm}(z_1 - z_0)), \\ d_{nm} &= \frac{1}{H_y}((z_{n-1,m} - z_{n-1,m-1}) - e_{nm}(z_2 - z_0)), \\ h_{nm} &= z_{n-1,m-1} - e_{nm} z_0 - y_0 d_{nm} - x_0 c_{nm}, \end{aligned} \tag{3}$$

for the coefficients in (1), and

$$\begin{aligned} \bar{a}_{pq} &= -\frac{h_x}{H_y}, \qquad \bar{f}_{pq} = x_{p-1} + h_x + \frac{h_x}{H_y} y_0, \\ \bar{b}_{pq} &= -\frac{h_y}{H_x}, \qquad \bar{g}_{pq} = y_{q-1} + h_y + \frac{h_y}{H_x} x_0, \\ \bar{c}_{pq} &= \frac{1}{H_x}((z_{p,q-1} - z_{pq}) - \bar{e}_{pq}(z_1 - z_0)), \\ \bar{d}_{pq} &= \frac{1}{H_y}((z_{p-1,q} - z_{pq}) - \bar{e}_{pq}(z_2 - z_0)), \\ \bar{h}_{pq} &= z_{pq} - \bar{e}_{pq} z_0 - y_0 \bar{d}_{pq} - x_0 \bar{c}_{pq}, \end{aligned} \tag{4}$$

for the coefficients in (2).

The next theorems are special cases of theorems in [8]. They give the conditions for existence of interpolating fractal functions of two variables.

Theorem 1. *Let* $N \in \mathbb{N} \backslash \{1\}$ *and let* $S = \{\mathbb{R}^3; w_{nm}, \bar{w}_{pq} : n, m \geq 1, n+m = 2, \ldots, N+1, p, q \geq 1, p+q = 2, \ldots, N\}$ *denote the IFS S, associated with the data set* $\{(x_n, y_m, F_{nm}) : n, m \geq 0, n+m = 0, \ldots, N\}$. *Let the vertical scaling factors obey* $|e_{nm}| < 1$ *and* $|\bar{e}_{pq}| < 1$. *Then, there is a metric d on* $\mathbb{R}^3$, *equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. In particular, there is a unique non-empty compact set* $G \subset \mathbb{R}^3$, *such that*

$$G = \bigcup_{\substack{n,m \geq 1 \\ n+m=2,\ldots,N+1}} w_{nm}(G) \bigcup_{\substack{p,q \geq 1 \\ p+q=2,\ldots,N}} \bar{w}_{pq}(G).$$

Proof. The metric

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + \theta|z_1 - z_2|, \quad \theta \in \mathbb{R}^+,$$

is equivalent to the Euclidean metric in $\mathbb{R}^3$. Let the transformations w_{nm} and $\bar{w}_{pq}$ be defined by (1) and (2). Then

$$\begin{aligned} &d(w_{nm}(x_1, y_1, z_1), w_{nm}(x_2, y_2, z_2)) \\ \leq &(|a_{nm}| + \theta_1|c_{nm}|)|x_1 - x_2| + (|b_{nm}| + \theta_1|d_{nm}|)|y_1 - y_2| + \theta_1|e_{nm}||z_1 - z_2|, \end{aligned}$$

and

$$\begin{aligned} &d(\bar{w}_{pq}(x_1, y_1, z_1), \bar{w}_{pq}(x_2, y_2, z_2)) \\ \leq &(|\bar{b}_{pq}| + \theta_2|\bar{c}_{pq}|)|x_1 - x_2| + (|\bar{a}_{pq}| + \theta_2|\bar{d}_{pq}|)|y_1 - y_2| + \theta_2|\bar{e}_{pq}||z_1 - z_2|. \end{aligned}$$

Here,

$$\begin{aligned} |a_{nm}| &= \frac{|x_n - x_{n-1}|}{|x_N - x_0|} < 1, \quad |b_{nm}| = \frac{|y_m - y_{m-1}|}{|y_N - y_0|} < 1, \\ |\bar{a}_{pq}| &= \frac{|y_q - y_{q-1}|}{|y_N - y_0|} < 1, \quad |\bar{b}_{pq}| = \frac{|x_p - x_{p-1}|}{|x_N - x_0|} < 1, \end{aligned}$$

because $N \geq 2$.

If $c_{nm} = d_{nm} = 0$, for all $n, m \geq 1, n+m = 2, \ldots, N+1$. Then one can choose $\theta_1 = 1$, otherwise

$$\theta_1 = \frac{\min\{1 - |a_{nm}|, 1 - |b_{nm}|\}}{2\max\{|c_{nm}|, |d_{nm}|\}}.$$

Similarly, if $\bar{c}_{pq} = \bar{d}_{pq} = 0$, for all $p, q \geq 0, p+q = 0, \ldots, N$, then $\theta_2 = 1$, otherwise

$$\theta_2 = \frac{\min\{1 - |\bar{b}_{pq}|, 1 - |\bar{a}_{pq}|\}}{2\max\{|\bar{c}_{pq}|, |\bar{d}_{pq}|\}}.$$

Finally, $\theta = \min\{\theta_1, \theta_2\}$.

Further

$$d(w_{nm}(x_1,y_1,z_1), w_{nm}(x_2,y_2,z_2)) \le \max\{a,b,\delta\} d((x_1,y_1,z_1),(x_2,y_2,z_2))$$

and

$$d(\bar{w}_{pq}(x_1,y_1,z_1), \bar{w}_{pq}(x_2,y_2,z_2)) \le \max\{\bar{a},\bar{b},\bar{\delta}\} d((x_1,y_1,z_1),(x_2,y_2,z_2)),$$

where

$$\begin{aligned} a &= \left(\frac{1}{2} + \frac{\max\{|a_{nm}|\}}{2}\right) < 1, & \bar{a} &= \left(\frac{1}{2} + \frac{\max\{|\bar{b}_{pq}|\}}{2}\right) < 1, \\ b &= \left(\frac{1}{2} + \frac{\max\{|b_{nm}|\}}{2}\right) < 1, & \bar{b} &= \left(\frac{1}{2} + \frac{\max\{|\bar{a}_{pq}|\}}{2}\right) < 1, \\ \delta &= \max\{|e_{nm}|\} < 1, & \bar{\delta} &= \max\{|\bar{e}_{pq}|\} < 1. \quad \square \end{aligned}$$

Let Δ be a triangle ABC.

Theorem 2. *Let $N \in \mathbb{N}\backslash\{1\}$ and let $S = \{\mathbb{R}^3; w_{nm}, \bar{w}_{pq} \; : \; n,m \ge 1, n+m = 2,\dots,N+1, p,q \ge 1, p+q = 2,\dots,N\}$ denote the IFS S, associated with the data set $\{(x_n, y_m, z_{nm}) : n,m \ge 0, n+m = 0,\dots,N\}$. Let the vertical scaling factors obey $|e_{nm}| < 1$ and $|\bar{e}_{pq}| < 1$, so that the IFS is hyperbolic. Let G denote the attractor of the IFS. Then G will be the graph of a continuous function $f : \Delta \to \mathbb{R}$ which interpolates the data $\{(x_n, y_m, F_{nm}) : n,m \ge 0, n+m = 0,1,\dots,N\}$ if one of the following conditions is fulfilled:*

a) *All interpolating nodes on the sides AB, BC and CA are collinear;*

b) *All interpolating nodes on the side AB and AC are collinear and all vertical scaling factors are mutually equal;*

c) *All vertical scaling factors vanish.*

Proof. Let $\mathcal{T}$ denote the set of continuous functions $f : \Delta \to \mathbb{R}$, so that

$$\begin{aligned} f(x_0, y_m) &= F_{0m}, \; m = 0,\dots,N, \\ f(x_n, y_0) &= F_{n0}, \; n = 0,\dots,N, \\ f(x_n, y_{N-n}) &= F_{n,N-n}, \; n = 0,\dots,N \end{aligned}$$

and let the metric d on $\mathcal{T}$ be defined by

$$d(f,g) = \max_{(x,y)\in T}\{|f(x,y) - g(x,y)|\} \quad \forall f,g \in \mathcal{T}.$$

Then $(\mathcal{T}, d)$ is a complete metric space.

Let $u_{nm} : [x_0, x_N] \to [x_{n-1}, x_n]$, $v_{nm} : [y_0, y_N] \to [y_{m-1}, y_m]$, $n, m \geq 1$, $n + m = 2, \ldots, N + 1$ and $\bar{u}_{pq} : [x_0, x_N] \to [x_{p-1}, x_p]$, $\bar{v}_{pq} : [y_0, y_N] \to [y_{q-1}, y_q]$, $p, q \geq 1$, $p + q = 2, \ldots, N$ are invertible transformations:

$$\begin{aligned} u_{nm}(x) =& a_{nm}x + f_{nm}, \quad v_{nm}(y) = b_{nm}y + g_{nm}, \\ \bar{u}_{pq}(x) =& \bar{b}_{pq}x + \bar{f}_{pq}, \quad \bar{v}_{pq}(y) = \bar{a}_{pq}y + \bar{g}_{pq}, \end{aligned}$$

where $a_{nm}, b_{nm}, f_{nm}, g_{nm}$ are ordered by (3) and $\bar{a}_{pq}, \bar{b}_{pq}, \bar{f}_{pq}, \bar{g}_{pq}$ are ordered by (4). Further, let the mappings $T, \bar{T} : \mathcal{T} \to \mathcal{T}$, be defined by

$$(Tf)(x, y) = c_{nm}u_{nm}^{-1}(x) + d_{nm}v_{nm}^{-1}(y) + e_{nm}f(u_{nm}^{-1}(x), v_{nm}^{-1}(y)) + h_{nm}$$

and

$$(\bar{T}f)(x, y) = \bar{c}_{pq}\bar{u}_{pq}^{-1}(x) + \bar{d}_{pq}\bar{v}_{pq}^{-1}(y) + \bar{e}_{pq}f(\bar{u}_{pq}^{-1}(x), \bar{v}_{pq}^{-1}(y)) + \bar{h}_{pq}.$$

The function Tf satisfies the following boundary conditions:

$$\begin{aligned} (TF)(x_0, y_{m-1}) =& F_{0,m-1}, \quad m = 1, \ldots, N + 1, \\ (TF)(x_{n-1}, y_0) =& F_{n-1,0}, \quad n = 1, \ldots, N + 1, \\ (TF)(x_n, y_{N-n}) =& F_{n,N-n}, \quad n = 0, \ldots, N + 1. \end{aligned}$$

Similar is valid for the function $\bar{T}f$.

The mappings Tf and $\bar{T}f$ are continuous on each sub-triangle of Δ. Then it remains to demonstrate that Tf and $\bar{T}f$ are continuous at the edge points of each sub-triangle. At the point (x, y) from the edge, the values of $(Tf)(x, y)$ and $(\bar{T}f)(x, y)$ are apparently defined in two different ways: the values on the side EF (see Figure 1) are

$$\begin{aligned} F_z =& c_{nm}x^{-1} + d_{nm}y^{-1} + e_{nm}z^{-1} + h_{nm} \\ =& \frac{1}{H_x}[(z_{n,m-1} - z_{n-1,m-1}) - e_{nm}(z_1 - z_0)](x_0 + H_x - H_x t) \\ &+ \frac{1}{H_y}[(z_{n-1,m} - z_{n-1,m-1}) - e_{nm}(z_2 - z_0)](y_0 + H_y t) \\ &+ e_{nm}f + z_{nm} - e_{nm}z_0 - y_0 d_{nm} - x_0 c_{nm} \\ =& [-(z_{n,m-1} - z_{n-1,m-1}) + e_{nm}(z_1 - z_0) + (z_{n-1,m} - z_{n-1,m-1}) \\ &- e_{nm}(z_2 - z_0)]t + (z_{n,m-1} - z_{n-1,m-1}) - e_{nm}(z_1 - z_0) \\ &+ z_{n-1,m-1} - e_{nm}z_0 + e_{nm}f \\ =& (z_{n-1,m} - z_{n,m-1})t + z_{n,m-1} + e_{nm}[f - (z_2 - z_1)t - z_1] \end{aligned}$$

and

$$\begin{aligned}
\bar{F}_z =&\bar{c}_{pq}x^{-1} + \bar{d}_{pq}y^{-1} + \bar{e}_{pq}z^{-1} + \bar{h}_{pq} \\
=&\frac{1}{H_x}[(z_{p,q-1} - z_{pq}) - \bar{e}_{pq}(z_1 - z_0)](x_0 + H_x - H_x t) \\
&+ \frac{1}{H_y}[(z_{p-1,q} - z_{pq}) - \bar{e}_{pq}(z_2 - z_0)](y_0 + H_y t) \\
&+ \bar{e}_{pq}f + z_{pq} - \bar{e}_{pq}z_0 - y_0\bar{d}_{pq} - x_0\bar{c}_{pq} \\
=&[-(z_{p,q-1} - z_{pq}) + \bar{e}_{pq}(z_1 - z_0) + (z_{p-1,q} - z_{pq}) - \bar{e}_{pq}(z_2 - z_0)]t \\
&+ (z_{p,q-1} - z_{pq}) - \bar{e}_{pq}(z_1 - z_0) + z_{pq} - \bar{e}_{pq}z_0 + \bar{e}_{pq}f \\
=&(z_{p-1,q} - z_{p,q-1})t + z_{p,q-1} + \bar{e}_{pq}[f - (z_2 - z_1)t - z_1].
\end{aligned}$$

Those values will be equal if one of the following conditions fulfilled:
a) Vertical scaling factors are mutually equal ($e_{nm} = \bar{e}_{pq}$);
b) All interpolating nodes on the side BC are collinear ($f = (z_2 - z_1)t + z_1$).

Similarly, the values on the sides EG or FG will be mutually equal if one of the following is fulfilled:
a) Vertical scaling factors vanish;
b) All interpolating nodes on the sides AB and AC are collinear.

Finally, the values on the all sub-triangles edges will be equal if one of the following conditions is fulfilled:
a) All interpolating nodes on the sides AB, BC and CA are collinear;
b) All interpolating nodes on the side AB and AC are collinear and all vertical scaling factors are mutually equal;
c) All vertical scaling factors vanish.

We conclude that, under these conditions, T and $\bar{T}$ take $\mathcal{T}$ into $\mathcal{T}$.

Now, let us show that T and $\bar{T}$ are a contraction mapping on the metric space $(\mathcal{T}, d)$. Let $f, g \in \mathcal{T}$ and let $x \in [x_{n-1}, x_n]$; $y \in [y_{m-1}, y_m]$; $n, m \geq 1$; $n + m = 2, \ldots, N + 1$). Then,

$$|(Tf)(x, y) - (Tg)(x, y)| \leq |e_{nm}|d(f, g),$$

so that

$$d(Tf, Tg) \leq \delta d(f, g),$$

where $\delta = \max\{|e_{nm}| \,:\, n, m \geq 1,\ n + m = 2, \ldots, N + 1\} < 1$. So, T and $\bar{T}$ are contractive mapping of the metric space $(\mathcal{T}, d)$. Therefore each of them have a unique fixed points in $\mathcal{T}$. That is, there exists a function $f \in \mathcal{T}$ such that

$$(Tf)(x,y) = f(x,y), \quad \forall (x,y) \in \Delta.$$

Let $\tilde{G}$ be a graph of function f. Then $(Tf)(x,y)$ can be expressed as

$$(Tf)(a_{nm}x + f_{nm}, b_{nm}y + g_{nm}) = c_{nm}x + d_{nm}y + e_{nm}f(x,y) + h_{nm},$$

where $(x,y) \in \Delta$ and $n, m \geq 1$; $n + m = 2, \ldots, N+1$. This implies that

$$\tilde{G} = \bigcup_{\substack{n,m\geq 1 \\ n+m=2,\ldots,N+1}} w_{nm}(\tilde{G}) \bigcup_{\substack{p,q\geq 1 \\ p+q=2,\ldots,N}} \bar{w}_{pq}(\tilde{G}). \tag{5}$$

Since, by Theorem 1 there exists only one non-empty compact set G, the attractor of the $IFS\ S$, which satisfies equation (5), it follows that $\tilde{G} = G$. □

Definition 1. The function $f(x)$ whose graph is the attractor of an IFS as described in Theorems 1 and 2 above, is called the fractal interpolation function corresponding to the data $\{(x_n, y_m, F_{nm}) : nm \geq 0, n+m = 0, 1, \ldots, N\}$.

In the cases of rectangular arrangement, interpolation nodes are arranged, like in Figure 1 (right).

So, the interpolating nodes form NM rectangles. The distances between nodes are h_x and h_y. The distances between vertices $A - D$ are H_x and H_y. So, the big rectangle is transformed onto the smaller one. But, by affine transformations one can transform only triangle to triangle (there are only 9 parameters to be determined). Therefore, each rectangle must be subdivides onto 2 triangles. Now, an affine transformation can be performed for each triangle separately.

Suppose that the rectangle subdivision along the main diagonal is performed. Now, if the point lies in the upper big triangle, than that point will be transformed onto the corresponding upper smaller rectangle. On the contrary, if the point lies in the lower big triangle, than that point will be transformed onto the corresponding lower smaller triangle. In each step algorithm choose arbitrarily between the small rectangle, where the transformation will be done. The corresponding transformations are now,

$$w_{nm}(\mathbf{x}) = \begin{bmatrix} a_{nm} & 0 & 0 \\ 0 & b_{nm} & 0 \\ c_{nm} & d_{nm} & e_{nm} \end{bmatrix} \mathbf{x} + \begin{bmatrix} f_{nm} \\ g_{nm} \\ h_{nm} \end{bmatrix} \tag{6}$$

and

$$\bar{w}_{pq}(\mathbf{x}) = \begin{bmatrix} \bar{a}_{pq} & 0 & 0 \\ 0 & \bar{b}_{pq} & 0 \\ \bar{c}_{pq} & \bar{d}_{pq} & \bar{e}_{pq} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \bar{f}_{pq} \\ \bar{g}_{pq} \\ \bar{h}_{pq} \end{bmatrix}. \tag{7}$$

The coordinates of the vertex points $A - D$ are $A(x_0, y_0, z_0), B(x_0 + H_x, y_0, z_1), C(x_0 + H_x, y_0 + H_y, z_2)$, and $D(x_0 + H_x, y_0, z_4)$. The unknown parameters are

$$\begin{aligned}
a_{nm} =& \frac{h_x}{H_x}, \quad f_{nm} = x_{n-1} - \frac{h_x}{H_x} x_0, \quad b_{nm} = \frac{h_y}{H_y}, \quad g_{nm} = y_{m-1} - \frac{h_y}{H_y} y_0, \\
c_{nm} =& \frac{1}{H_x}((z_{n,m-1} - z_{n-1,m-1}) - e_{nm}(z_1 - z_0)), \\
d_{nm} =& \frac{1}{H_y}((z_{nm} - z_{n,m-1}) - e_{nm}(z_2 - z_1)), \\
h_{nm} =& z_{n-1,m-1} - e_{nm} z_0 - y_0 d_{nm} - x_0 c_{nm},
\end{aligned}$$

in relation (6) and

$$\begin{aligned}
\bar{a}_{pq} =& \frac{h_x}{H_x}, \quad \bar{f}_{pq} = x_{p-1} - \frac{h_x}{H_x} x_0, \quad \bar{b}_{pq} = \frac{h_y}{H_y}, \quad \bar{g}_{pq} = y_{q-1} - \frac{h_y}{H_y} y_0, \\
\bar{c}_{pq} =& \frac{1}{H_x}((z_{pq} - z_{p-1,q}) - \bar{e}_{pq}(z_2 - z_3)), \\
\bar{d}_{pq} =& \frac{1}{H_y}((z_{p-1,q} - z_{p-1,q-1}) - \bar{e}_{pq}(z_3 - z_0)), \\
\bar{h}_{pq} =& z_{p-1,q-1} - \bar{e}_{pq} z_0 - y_0 \bar{d}_{pq} - x_0 \bar{c}_{pq},
\end{aligned}$$

for relation (7). The free parameters are e_{nm} and $\bar{e}_{pq}$.

If one choose the rectangle subdivision by another diagonal, then the corresponding transformations will be

$$t_{nm}(\mathbf{x}) = \begin{bmatrix} a_{nm} & 0 & 0 \\ 0 & b_{nm} & 0 \\ c_{nm} & d_{nm} & e_{nm} \end{bmatrix} \mathbf{x} + \begin{bmatrix} f_{nm} \\ g_{nm} \\ h_{nm} \end{bmatrix} \tag{8}$$

and

$$\bar{t}_{pq}(\mathbf{x}) = \begin{bmatrix} \bar{a}_{pq} & 0 & 0 \\ 0 & \bar{b}_{pq} & 0 \\ \bar{c}_{pq} & \bar{d}_{pq} & \bar{e}_{pq} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \bar{f}_{pq} \\ \bar{g}_{pq} \\ \bar{h}_{pq} \end{bmatrix}, \tag{9}$$

where

$$
\begin{aligned}
a_{nm} =& \frac{h_x}{H_x}, \quad f_{nm} = x_{n-1} - \frac{h_x}{H_x} x_0, \quad b_{nm} = \frac{h_y}{H_y}, \quad g_{nm} = y_{m-1} - \frac{h_y}{H_y} y_0,\\
c_{nm} =& \frac{1}{H_x}((z_{n,m-1} - z_{n-1,m-1}) - e_{nm}(z_1 - z_0)),\\
d_{nm} =& \frac{1}{H_y}((z_{n-1,m} - z_{n-1,m-1}) - e_{nm}(z_3 - z_0)),\\
h_{nm} =& z_{n-1,m-1} - e_{nm} z_0 - y_0 d_{nm} - x_0 c_{nm},
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{a}_{pq} =& \frac{h_x}{H_x}, \quad \bar{f}_{pq} = x_{p-1} - \frac{h_x}{H_x} x_0, \quad \bar{b}_{pq} = \frac{h_y}{H_y}, \quad \bar{g}_{pq} = y_{q-1} - \frac{h_y}{H_y} y_0,\\
\bar{c}_{pq} =& \frac{1}{H_x}((z_{pq} - z_{p-1,q}) - \bar{e}_{pq}(z_2 - z_3)),\\
\bar{d}_{pq} =& \frac{1}{H_y}((z_{pq} - z_{p,q-1}) - \bar{e}_{pq}(z_2 - z_1)),\\
\bar{h}_{pq} =& (z_{p-1,q} - z_{pq} + z_{p,q-1}) - \bar{e}_{pq}(z_1 - z_2 + z_3) - y_0 \bar{d}_{pq} - x_0 \bar{c}_{pq},
\end{aligned}
$$

The numbers e_{nm} and $\bar{e}_{pq}$ are the free parameters.

Of course, combinations are possible. In different subrectangles one can choose arbitrarily between $w_{nm}, \bar{w}_{pq}$ and $t_{nm}, \bar{t}_{pq}$. This gives 2^{NM} different interpolants.

Like in the case of triangular meshes, the theorems similar to Theorems 1 and 2 can be proved and the definition which corresponds to Definition 1 can be made.

3. Scattered data interpolation

Suppose that $\{(x_i, y_i) : i = 1, \ldots, N\}$ are scattered interpolating nodes in Δ and $\{f(x_i, y_i) : i = 1, \ldots, N\}$ are corresponding data set. In the cases of scattered data two different schemes are examined.

Recursive scheme. The (x_1, y_1) node forms 3 subtriangle with the vertices $A-C$. Next, (x_2, y_2) lies in one of them forming another 3 sub-sub-triangles. This procedure continues as soon as all interpolation nodes are examined. Now, interpolant can be obtained in few steps. First, by corresponding affine transformation, the initial point ($\in \Delta$) maps into some of three subtriangles being derived in the first subdivision. Secondly, if this subtriangle contains sub-sub-triangles the procedure continues until the final point is obtained.

Triangulation scheme. By this approach all points are examined at the same time. Using the method of Akima [1], [2], [3], [4] for bivariate interpolation, triangulation of the domain is performed. This method is based on *max-min angle triangulation* of the points $\{(x_i, y_i)\}$ suggested by Lawson [7]. As it is shown by G. Nielson [9], [10] max-min triangulation of Lawson, is equivalent to min-max criteria of Little and Barnhill [5]. Characterization of these two triangulations is similar: each triangulation is associated with a vector, having n_i entries representing either the largest or the smallest angle of each triangle. These entries are ordered by the using a lexicographic rule. In the case of min-max criteria, the smallest of these vectors based on their lexicographic coordinate gives the optimal triangulation, while in the case of max-min criteria, the largest vector is associated with the optimal triangulation. After the optimal triangulation is obtained one can transform a big triangle onto the arbitrarily small one by a suitable transformation.

Which between these two methods are better? The preference is surely at the second one due to optimal triangulation. By this method every new point is obtained using only one step, while the first method gains the same result in a few steps. However, if the data structure is not constant, i.e. if the interpolation problem must be solved repeatedly with the addition of new interpolation nodes than, by the using of the second algorithm, one must redo the triangulation process from the beginning each time the new interpolation node adds, while the first method starts from the structure already existed.

4. Examples

We consider two examples.

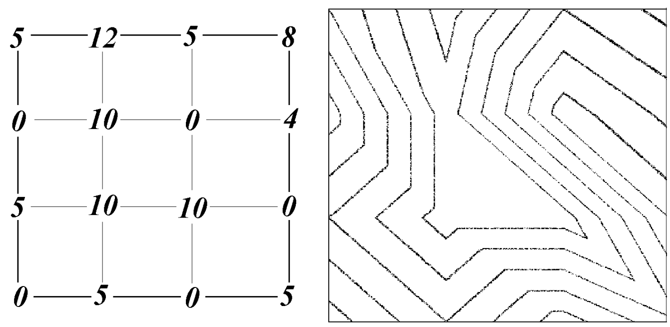

FIG. 2

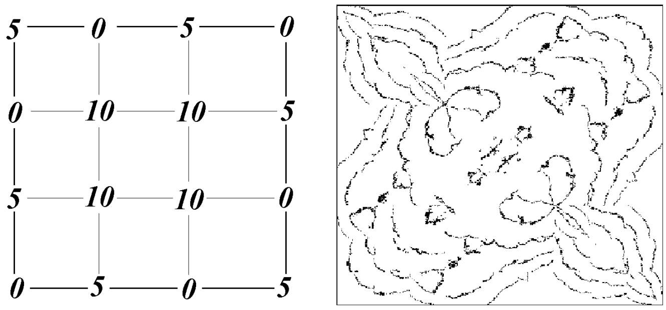

Fig. 3

Example 1. In the case of the rectangular arrangement, the interpolation nodes are arranged like in Figure 2 (left). The vertical scaling factors are zero in all subrectangles. The corresponding level lines map is shown in Figure 2 (right). The continuous interpolant is obtained.

Example 2. In the case of the rectangular arrangement, the interpolation nodes are arranged like in Figure 3 (left). The vertical scaling factors are now $e_{nm} = 0.3$ in all subrectangles. The corresponding level lines map is shown in Figure 3 (right). In this example, the conditions for obtaining the continuous interpolant are not satisfied (neither all vertical scaling factors are vanished, nor all interpolating nodes on the sides AB, BC, CD and DA are collinear) and the obtained interpolant is not continuous.

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