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SPECTRAL ANALYSIS OF NONSELFADJOINT SCHRÖDINGER OPERATORS WITH SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. In this article we have investigated the spectrum of the one-dimensional non-selfadjoint Schrödinger operators with boundary condition involving spectral parameter. Discussing the spectrum of L, we have proved that, if the potential function q satisfies $\int_0^\infty |q(x)| \exp(\epsilon\sqrt{x})dx < \infty$, $\epsilon > 0$ then the eigenvalues and the spectral singularities are of finite number and each of them is of finite multiplicity. We have also studied the properties of principal functions and given the spectral expansion of the operator in terms of them. Lastly we have investigated the convergence of this representation.

1. Introduction

The spectral analysis of the non-selfadjoint abstract operators with purely discrete spectrum have been consider, by Keldysh [9]. In this article the spectrum, the eigenfunctions and the associated functions of the operators of the form of a polynomial in spectral parameter have been studied. Then he have proved the multiply completeness of the eigenfunctions and associated functions of these operators in Hilbert Spaces. The spectral analysis of the non-selfadjoint differential operators with continuous and discrete spectrum has been investigated by Naimark [18]. In this paper he has proved the existence of the spectral singularities on the continuous spectrum of the non-selfadjoint differential operator. Later he has shown that the spectral singularities of the operator plays an important role in the discussions of the spectral analysis. The effect of the spectral singularities on the spectral expansion have been studied by Lyance [11]. Gasymov and Maksudov [7]

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have considered the principal part of the resolvent in the neighbourhood of spectral singularities and Pavlov [20] established the dependence of the structure of the spectral singularities on the behaviour of the potential function at infinity. The spectral analysis of some types of operators with spectral singularities have also been studied in some other articles [5,6,8,12,19,21].

To investigate the spectral analysis of dissipative differential operators with finite defect of non-selfadjointness, Pavlov [22] has provided a technique which is different from the ones mentioned above, that is derived from the theories given by Lax-Phillips [10] and Nagy-Foias [4]. Using this method the spectral analysis of some dissipative operators have been investigated [1-3,13-16].

In all the above articles, the boundary condition are independent of the spectral parameter. But in most of the problems arising in Mathematical Physics and Oscillation theory, the boundary condition involve the spectral parameter. Hence, the spectral analysis of singular non-selfadjoint differential operators with spectral parameter in boundary condition are important.

In this article we consider the operator L in $L_2(\mathbb{R}_+)$, defined by the differential equation

$$-y'' + q(x)y - \lambda^2 y = 0, \quad x \in \mathbb{R}_+ = [0, \infty) \tag{1.1}$$

with the boundary condition

$$y'(0) - a\lambda y(0) = 0, \tag{1.2}$$

where q is a complex valued function and $a \in \mathbb{C}$ is a constant. First, we have investigated the spectrum of L and we have shown that it has finite number of eigenvalues and spectral singularities, each of them is of finite multiplicity, if

$$\int_0^\infty |q(x)| \exp(\epsilon\sqrt{x})dx < \infty, \quad \epsilon > 0$$

holds. Then, we have studied the properties of the principal functions of L and obtained the spectral expansion of the operator in terms of the principal functions. We also have investigated the convergence of this representation.

In the following we use the notations

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{C}_+ = \{\lambda | \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}, \quad \mathbb{C}_- = \{\lambda | \lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0\},$$
$$\mathbb{R}^\star = \mathbb{R} \setminus \{0\}, \quad \overline{\mathbb{C}}_+ = \{\lambda | \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}, \quad \overline{\mathbb{C}}_- = \{\lambda | \lambda \in \mathbb{C}, \operatorname{Im}\lambda \leq 0\}.$$

Also, $\sigma_d(L), \sigma_c(L), \sigma_{ss}(L), \rho(L)$ and $R_\lambda(L)$ will denote, the eigenvalues, the continuous spectrum, the spectral singularities, the resolvent set and the resolvent of the operator L, respectively.

2. The spectrum

Let q be a complex valued function satisfying

$$\int_0^\infty (1+x)|q(x)|dx < \infty \tag{2.1}$$

and define the functions w and w_1 by

$$w(x) = \int_x^\infty |q(t)|dt, \quad w_1(x) = \int_x^\infty w(t)dt.$$

It is well-known that [17], if (2.1) holds, for $\lambda \in \overline{\mathbb{C}}_+$, the Eq. (1.1) has the solution given by

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt. \tag{2.2}$$

This solution is analytic with respect to λ in $\mathbb{C}_+$ and continuous up to the real axis.

The kernel $K(x,t)$ appearing in (2.2) is continuously differentiable with respect to x and t and satisfies

$$|K(x,t)| \le \frac{1}{2}\, w\Big(\frac{x+t}{2}\Big) \exp\{w_1(x)\}, \tag{2.3}$$

$$|K_x(x,t)|, |K_t(x,t)| \le \frac{1}{4}\, \Big|q\Big(\frac{x+t}{2}\Big)\Big| + \frac{1}{2}\, w\Big(\frac{x+t}{2}\Big) w(x) \exp\{w_1(x)\}, \quad 0 \le x \le t < \infty. \tag{2.4}$$

We will employ the following notations

$$\tilde{e}(x,\lambda) = e(x,-\lambda), \beta(\lambda) = e_x(0,\lambda) - a\lambda e(0,\lambda), \tilde{\beta}(\lambda) = \tilde{e}_x(0,\lambda) - a\lambda\tilde{e}(0,\lambda).$$

Let $\varphi(x,\lambda)$ and $s(x,\lambda)$ represent the solutions of the Eq. (1.1) subject to the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi_x(0,\lambda) = a\lambda,$$
$$s(0,\lambda) = 0, \quad s_x(0,\lambda) = 1.$$

Let us introduce the sets

$$\rho_1(\lambda) = \{\lambda|\lambda \in \mathbb{C}_+, \beta(\lambda) \neq 0\}, \quad \rho_2(\lambda) = \{\lambda|\lambda \in \mathbb{C}_-, \tilde{\beta}(\lambda) \neq 0\}.$$

Using standard techniques [19] we can show that $\rho(L) = \rho_1(\lambda) \cup \rho_2(\lambda)$ and for $\lambda \in \rho(L)$, the resolvent of L is the integral operator $R_\lambda(L)$ defined as

$$R_\lambda(L)f(x) = \int_0^\infty R(x,t;\lambda)f(t)dt$$

for $f \in L_2(\mathbb{R}_+)$, where the kernel $R(x,t;\lambda)$ (i.e. the Green's function of L) is given by

$$R(x,t;\lambda) = \begin{cases} R_1(x,t;\lambda), & \lambda \in \rho_1(\lambda), \\ R_2(x,t;\lambda), & \lambda \in \rho_2(\lambda) \end{cases} \tag{2.5}$$

in which

$$R_1(x,t;\lambda) = -\frac{e(0,\lambda)}{\beta(\lambda)}\,\varphi(x,\lambda)\varphi(t,\lambda) + F(x,t;\lambda), \tag{2.6}$$

$$R_2(x,t;\lambda) = -\frac{\tilde{e}(0,\lambda)}{\tilde{\beta}(\lambda)}\varphi(x,\lambda)\varphi(t,\lambda) + F(x,t;\lambda) \tag{2.7}$$

where

$$F(x,t;\lambda) = \begin{cases} -s(x,\lambda)\varphi(t,\lambda), & 0 \le t < x, \\ \\ s(t,\lambda)\varphi(x,\lambda), & x \le t < \infty. \end{cases}$$

Using the definitions of the eigenvalues, continuous spectrum and spectral singularities [19] we obtain

$$\sigma_d(L) = \{\lambda|\lambda \in \mathbb{C}_+, \beta(\lambda) = 0\} \bigcup \{\lambda|\lambda \in \mathbb{C}_-, \tilde{\beta}(\lambda) = 0\}, \tag{2.8}$$

$$\sigma_c(L) = \mathbb{R}, \tag{2.9}$$

$$\sigma_{ss}(L) = \{\lambda|\lambda \in \mathbb{R}^\star,\ \beta(\lambda) = 0\} \cup \{\lambda|\lambda \in \mathbb{R}^\star,\ \tilde{\beta}(\lambda) = 0\}. \tag{2.10}$$

Lemma 2.1. $\{\lambda|\lambda \in \mathbb{R}^\star, \quad \beta(\lambda) = 0\} \cap \{\lambda|\lambda \in \mathbb{R}^\star, \quad \tilde{\beta}(\lambda) = 0\} = \phi.$

The proof of the lemma may easily be obtained from

$$e(0,\lambda)\tilde{\beta}(\lambda) - \tilde{e}(0,\lambda)\beta(\lambda) = -2i\lambda.$$

Definition 2.2. The multiplicity of a zero of β (or $\tilde{\beta}$) in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is called as the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Up to now, we have assumed that condition (2.1) holds. In the rest of the article we suppose that $a \neq \pm i$ and

$$\int_0^\infty |q(x)| \exp(\epsilon\sqrt{x})dx < \infty, \qquad \epsilon > 0 \tag{2.11}$$

holds.

It is evident from (2.3), (2.4) and (2.11) that

$$|K(x,t)| \leq C \exp\left\{-\epsilon\sqrt{\frac{x+t}{2}}\right\}, \tag{2.12}$$

$$|K_x(x,t)|, |K_t(x,t)| \leq \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + C\exp\left\{-\epsilon\sqrt{\frac{x+t}{2}}\right\} \tag{2.13}$$

where $0 \leq x \leq t < \infty$, C is a positive constant.

Theorem 2.3. *Under the condition* (2.11) *the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Proof. In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of β in $\overline{\mathbb{C}}_+$ and $\tilde{\beta}$ in $\overline{\mathbb{C}}_-$. For the sake of simplicity, we consider only the zeros of β in $\overline{\mathbb{C}}_+$. (A similar procedure may be followed for the zeros of $\tilde{\beta}$ in $\overline{\mathbb{C}}_-$.) It is trivial from (2.12) and (2.13) that if (2.11) holds, β is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous up to the real axis. Hence we arrive to the estimates

$$\sup_{\lambda\in\overline{\mathbb{C}}_+} |\beta(\lambda) - (i-a)\lambda| < \infty, \ \sup_{\lambda\in\overline{\mathbb{C}}_+} |\beta'(\lambda)| < \infty, \tag{2.14}$$

$$|\beta^{(m)}(\lambda)| \leq C_m,\ \lambda \in \overline{\mathbb{C}}_+,\ m = 2, 3, \ldots \tag{2.15}$$

where

$$C_m = 2^m C \int_0^\infty t^{m+1} \exp(-\epsilon\sqrt{t})dt, \tag{2.16}$$

$C > 0$ is a constant. For the zeros of the function β we will define the following sets:

$$\begin{aligned}
M_1 &= \Big\{\lambda | \lambda \in \mathbb{C}_+,\ \beta(\lambda) = 0\Big\},\\
M_2 &= \Big\{\lambda | \lambda \in \mathbb{R},\ \beta(\lambda) = 0\Big\},\\
M_3 &= \Big\{\lambda | \lambda \in \overline{\mathbb{C}}_+,\ \frac{d^k}{d\lambda^k}\beta(\lambda) = 0,\ k = 0, 1, 2, \cdots\Big\},\\
M_4 &= \Big\{\lambda | \exists \{\lambda_n\}, \lambda_n \in \mathbb{C}_+, \beta(\lambda_n) = 0,\ \lambda_n \to \lambda \quad \text{for} \quad n \to \infty\Big\},\\
M_5 &= \Big\{\lambda | \exists \{\lambda_n\},\ \lambda_n \in \mathbb{R},\ \beta(\lambda_n) = 0,\ \lambda_n \to \lambda \quad \text{for} \quad n \to \infty\Big\}.
\end{aligned}$$

From the uniqueness theorem of analytic functions it is trivial that

$$M_1 \cap M_3 = \phi,\ M_3 \subset M_2,\ M_4 \subset M_2,\ M_5 \subset M_2.$$

Since all the derivatives of β are continuous up to the real axis, we simply find

$$M_4 \subset M_3,\ M_5 \subset M_3. \tag{2.17}$$

It is evident from (2.14)-(2.16) that, the function β satisfy the conditions of Pavlov theorem (see. Lemma 1.2, [20]). So $M_3 = \phi$ or $M_4 = M_5 = \phi$ by (2.17). This shows that the sets M_1 and M_2 have a finite numbers of elements. $\square$

3. Principal Functions

In this section we assume that (2.11) holds. Let $\lambda_1^+, \ldots, \lambda_j^+$ and $\mu_1^-, \ldots, \mu_l^-$ denote the zeroes of β in $\mathbb{C}_+$ and $\tilde{\beta}$ in $\mathbb{C}_-$ with multiplicities $m_1^+, \ldots, m_j^+$ and $m_1^-, \ldots, m_l^-$ respectively. Similarly let $\lambda_1, \ldots, \lambda_\alpha$ and $\mu_1, \ldots, \mu_\nu$ be the zeroes of β and $\tilde{\beta}$ on the real axis with multiplicities $m_1, \ldots, m_\alpha$ and $n_1, \ldots, n_\nu$ respectively.

Let us introduce the Hilbert spaces

$$H_m = \left\{ f : \int_0^\infty |(1+x)^m f(x)|^2 dx < \infty \right\}, \; m = 0, 1, 2, \ldots$$

$$H_{-m} = \left\{ g : \int_0^\infty |(1+x)^{-m} g(x)|^2 dx < \infty \right\}, \; m = 0, 1, 2, \ldots$$

with

$$||f||_m^2 = \int_0^\infty |(1+x)^m f(x)|^2 dx; \quad ||g||_{-m}^2 = \int_0^\infty |(1+x)^{-m} g(x)|^2 dx$$

respectively. It is evident that

$$H_0 = L_2(\mathbb{R}_+), \;\; H_m \subset L_2(\mathbb{R}_+) \subset H_{-m}, \;\; m = 1, 2, \ldots$$

Theorem 3.1. *We have*

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(\cdot, \lambda) \right\}_{\lambda = \lambda_k^+} \in L_2(\mathbb{R}_+), \; n = 0, 1, \cdots, m_k^+ - 1, \quad k = 1, 2, \ldots, j;$$

$$\left\{ \frac{\partial^p}{\partial \lambda^p} \varphi(\cdot, \lambda) \right\}_{\lambda = \mu_i^-} \in L_2(\mathbb{R}_+), \;\; p = 0, 1, \cdots, \; m_i^- - 1, \; i = 1, 2, \ldots, l;$$

$$\left\{ \frac{\partial^n}{\partial \alpha^n} \varphi(\cdot, \lambda) \right\}_{\lambda = \lambda_k} \in H_{-(n+1)}, \; n = 0, 1, \ldots, \; m_k - 1, \; k = 1, 2, \ldots, \alpha;$$

$$\left\{ \frac{\partial^p}{\partial \lambda^p} \varphi(\cdot, \lambda) \right\}_{\lambda = \mu_i} \in H_{-(p+1)}, \; p = 0, 1, \ldots, \; n_i - 1, \; i = 1, 2, \ldots, \nu.$$

Proof. Let $\theta(x, \lambda)$ denote the solution of (1.1) subject to the conditions

$$\lim_{x \to \infty} e^{i\lambda x} \theta(x, \lambda) = 1, \quad \lim_{x \to \infty} e^{i\lambda x} \theta_x(x, \lambda) = -i\lambda.$$

This solution is analytic with respect to λ in $\mathbb{C}_+$ and continuous in $\mathbb{C}_+ \cup \mathbb{R}^\star$ ([18]). Moreover $e(x, \lambda)$ and $\theta(x, \lambda)$ provide the fundamental solutions of (1.1) for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}^\star$. Then we find

$$\text{(3.1)} \quad \varphi(x, \lambda) = \frac{\beta(\lambda)}{2i\lambda} \theta(x, \lambda) - \frac{[\theta_x(0, \lambda) - a\lambda\theta(0, \lambda)]}{2i\lambda} e(x, \lambda), \; \lambda \in \mathbb{C}_+ \cup \mathbb{R}^\star.$$

The proof of the theorem can be completed by use of (2.2) and (3.1) easily. □

Let us choose m_0 so that

$$m_0 = \max\{m_1, \dots, m_\alpha,\ n_1, \dots, n_\nu\}.$$

In the following we use the notations

$$H_+ = H_{m_0+1}, \quad H_- = H_{-(m_0+1)}.$$

By Theorem 3.1 we have the following

Remark 3.2.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_k} \in H_-,\ \ n = 0, 1, \dots, m_k - 1,\ \ k = 1, 2, \dots, \alpha,$$

$$\left\{\frac{\partial^p}{\partial\lambda^p}\varphi(\cdot,\lambda)\right\}_{\lambda=\mu_i} \in H_-,\ \ p = 0, 1, \dots,\ \ n_i - 1,\ \ i = 1, 2, \dots, \nu.$$

4. The Spectral Expansion

Let $C_0^\infty(\mathbb{R}_+)$ denote the set of infinitely differentiable functions with compact support. Hence for each $f \in C_0^\infty(\mathbb{R}_+)$ we have

$$\text{(4.1)} \quad \int_0^\infty R(x,t;\lambda)f(t)dt = -\frac{f(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_0^\infty R(x,t;\lambda)\Big\{-f''(t) + q(t)f(t)\Big\}dt.$$

Let Γ_r denote the disc with center at the origin and radius $r; \partial\Gamma_r$ be the boundary of Γ_r. r will be chosen so that all eigenvalues and spectral singularities of L are in Γ_r. $\sqcap_{r,\eta}$ denotes the part of Γ_r lying in the strip $|Im\lambda| \le \eta$, (the shade part in fig.1) and $\Gamma_{r,\eta} := \Gamma_{r,\eta}^+ \cup \Gamma_{r,\eta}^-$ where $\Gamma_{r,\eta}^+$ and $\Gamma_{r,\eta}^-$ be the parts of $\Gamma_r \setminus \sqcap_{r,\eta}$ in the upper and lower half-planes respectively. Let us choose η so small that $\sqcap_{r,\eta}$ does not contain any eigenvalues of L

As it is seen from Fig. 1

$$\text{(4.2)} \quad \partial\Gamma_{r,\eta} = \partial\Gamma_r \setminus \partial\sqcap_{r,\eta}.$$

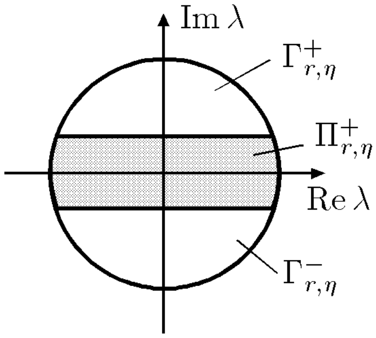

FIG. 1

From (4.1) and (4.2) we have

$$
\begin{aligned}
(4.3)\qquad f(x) &= \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\partial\Gamma_r}\left\{\frac{1}{\lambda}\int_0^\infty R(x,t;\lambda)[-f''(t)+g(t)f(t)]dt\right\}d\lambda\\
&-\lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int_{\partial\Gamma_{r,\eta}}\lambda\int_0^\infty R(x,t;\lambda)f(t)dtd\lambda\\
&-\lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int_{\partial\Pi_{r,\eta}}\lambda\int_0^\infty R(x,t;\lambda)f(t)dtd\lambda.
\end{aligned}
$$

Using Jordan's lemma we see that the first term of the right hand side of (4.3) vanishes as $r\to\infty$.

Let γ^+ be the contour which isolates the real zeros of β by semicircles with centers at $\lambda_k, k=1,2,\cdots,\alpha$ having the same radius in the upper half-plane in which β is analytic. Similarly γ^- will denote the corresponding contour for real zeros of $\tilde{\beta}$ in the lower half-plane. The radius will be chosen so small that two neighboring semicircles have no common points (see Fig. 2).

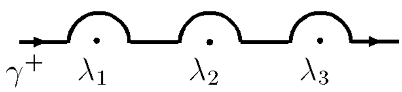

FIG. 2a

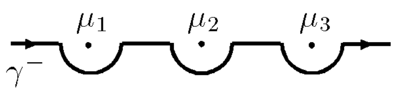

FIG. 2b

From (2.5) – (2.7) and (4.3) we obtain

$$f(x) = \sum_{k=1}^{j} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_k^+ - 1} M_k^+(\lambda)\varphi(x,\lambda)\varphi(f,\lambda) \right\}_{\lambda=\lambda_k^+} - \frac{1}{2\pi i}\int_{\gamma^+} \frac{\lambda e(0,\lambda)}{\beta(\lambda)}\varphi(x,\lambda)\varphi(f,\lambda)d\lambda$$
$$= \sum_{k=1}^{l} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_k^- - 1} M_k^-(\lambda)\varphi(x,\lambda)\varphi(f,\lambda) \right\}_{\lambda=\mu_k^-} + \frac{1}{2\pi i}\int_{\gamma^-} \frac{\lambda \tilde{e}(0,\lambda)}{\tilde{\beta}(\lambda)}\varphi(x,\lambda)\varphi(f,\lambda)d\lambda \tag{4.4}$$

where

$$M_k^+(\lambda) = \frac{\lambda(\lambda-\lambda_k)^{m_k^+} e(0,\lambda)}{(m_k^+ - 1)!\beta(\lambda)}, \quad k = 1, 2, \dots, j,$$

$$M_k^-(\lambda) = \frac{\lambda(\lambda-\mu_k^-)^{m_k^-} \tilde{e}(0,\lambda)}{(m_k^- - 1)!\tilde{\beta}(\lambda)}, \quad k = 1, 2, \dots, l,$$

and

$$\varphi(f,\lambda) = \int_0^\infty f(x)\varphi(x,\lambda)dx.$$

Lemma 4.1. *For any $f \in C_0^\infty(\mathbb{R}_+)$, there exists a constant $C > 0$ so that*

$$\int_{\mathbb{R}^\star} |\varphi(f,\lambda)|^2 d\lambda \le C \int_0^\infty |f(x)|^2 dx. \tag{4.5}$$

Proof. It is known that for $\lambda \in \mathbb{R}^\star$

$$\varphi(x,\lambda) = \frac{\beta(\lambda)}{2i\lambda}\tilde{e}(x,\lambda) - \frac{\tilde{\beta}(\lambda)}{2i\lambda}e(x,\lambda). \tag{4.6}$$

Using (2.2), (4.6) and Parseval's equation for the Fourier transformation we get (4.5). □

By the preceding lemma, for every function $f \in L_2(\mathbb{R}_+)$

$$\varphi(f,\lambda) = \lim_{N\to\infty} \int_0^N \varphi(x,\lambda)f(x)dx$$

exists with respect to the norm in $L_2(\mathbb{R}^\star)$; i.e.

$$\lim_{N\to\infty}\int_{\mathbb{R}^\star}\Big|\varphi(f,\lambda)-\int_0^N\varphi(x,\lambda)f(x)dx\Big|^2 d\lambda=0. \tag{4.7}$$

Since $C_0^\infty(\mathbb{R}_+)$ is dense in $L_2(\mathbb{R}_+)$, the estimate (4.5) may be extended onto $L_2(\mathbb{R}_+)$; i.e. for any $f\in L_2(\mathbb{R}_+)$

$$\int_{-\infty}^{\infty}|\varphi(f,\lambda)|^2 d\lambda\le c\int_0^\infty|f(x)|^2dx, \tag{4.8}$$

where $\varphi(f,\lambda)$ must be understood in the sense of (4.7). We shall need a generalization of this estimate.

Lemma 4.2. *If $f\in H_m$, $\varphi(f,\lambda)$ has the derivatives $\dfrac{d^s}{d\lambda^s}\varphi(f,\lambda)$ with $s=1,2,\dots,m-1$ on $\mathbb{R}^\star$. The derivative $\dfrac{d^m}{d\lambda^m}\varphi(f,\lambda)$ exists a.e. and satisfies*

$$\int_{\mathbb{R}^\star}\Big|\frac{d^\nu}{d\lambda^\nu}\varphi(f,\lambda)\Big|^2d\lambda\le C_\nu\int_0^\infty\big|(1+x)^\nu f(x)\big|^2dx,\ \nu=1,2,\dots,m, \tag{4.9}$$

with $C_\nu>0$.

The proof is similar to that of Lemma 4.1.

Let us define

$$\wedge_k^{(1)}=(\lambda_k-\delta,\lambda_k+\delta),\ k=1,2,\dots,\alpha,\ \wedge_k^{(2)}=(\mu_k-\delta,\mu_k+\delta),\ k=1,2,\dots,\nu,$$

with $\delta>0$. We can choose δ so small that

$$\wedge_i^{(1)}\cap\wedge_j^{(1)}=\phi,\quad \wedge_i^{(2)}\cap\wedge_j^{(2)}=\phi$$

holds for every $i\ne j$. Let us denote the functions $F_{kj}^{(1)}$ and $F_{kj}^{(2)}$ by

$$F_{kj}^{(1)}(\lambda)=\begin{cases}\frac{(\lambda-\lambda_k)^j}{j!}, & \lambda\in\overline{\mathbb{C}}_+\cap\{\lambda:|\lambda-\lambda_k|<\delta\},\ k=1,\dots,\alpha,\\ 0, & \lambda\in\wedge_0^{(1)},\end{cases} \tag{4.10}$$

$$F_{kj}^{(2)}(\lambda)=\begin{cases}\frac{(\lambda-\mu_k)^j}{j!}, & \lambda\in\overline{\mathbb{C}}_-\cap\{\lambda:|\lambda-\mu_k|<\delta\},\ k=1,\dots,\nu,\\ 0, & \lambda\in\wedge_0^{(2)},\end{cases} \tag{4.11}$$

where

$$\wedge_0^{(1)} = \mathbb{R} \setminus \bigcup_{k=1}^{\alpha} \wedge_k^{(1)}, \quad \wedge_0^{(2)} = \mathbb{R} \setminus \bigcup_{k=1}^{\nu} \wedge_k^{(2)}.$$

Now we will define the functionals

$$F_1\{g_1(\lambda)\} = g_1(\lambda) - \sum_{k=1}^{\alpha} \sum_{j=0}^{m_k-1} \left\{ \frac{d^j}{d\lambda^j} g_1(\lambda) \right\}_{\lambda=\lambda_k} F_{kj}^{(1)}(\lambda), \tag{4.12}$$

$$F_2\{g_2(\lambda)\} = g_2(\lambda) - \sum_{k=1}^{\nu} \sum_{j=0}^{n_k-1} \left\{ \frac{d^j}{d\lambda^j} g_2(\lambda) \right\}_{\lambda=\mu_k} F_{kj}^{(2)}(\lambda), \tag{4.13}$$

where g_1 and g_2 are chosen so that the right hand side of the above formulas are meaningful. It is trivial from (4.10) and (4.11) that $\lambda_k,\ k = 1, 2, \ldots, \alpha$ are roots of $F_1\{g_1(\lambda)\} = 0$ of order at least m_k and $\mu_k,\ k = 1, 2, \ldots, \nu$ are roots of $F_2\{g_2(\lambda)\} = 0$ of order at least n_k.

Now let us define the operators

$$I^{(1)} f(x) = \frac{1}{2\pi i} \int_{\gamma^+} \frac{\lambda e(0,\lambda)}{\beta(\lambda)} \varphi(x,\lambda)\varphi(f,\lambda) d\lambda, \tag{4.14}$$

$$I^{(2)} f(x) = \frac{1}{2\pi i} \int_{\gamma^-} \frac{\lambda \tilde{e}(0,\lambda)}{\tilde{\beta}(\lambda)} \varphi(x,\lambda)\varphi(f,\lambda) d\lambda. \tag{4.15}$$

Lemma 4.3. *For each $f \in H_+$, there exists constants $C_1 > 0$ and $C_2 > 0$ such that*

$$||I^{(1)} f||_- \le C_1 ||f||_+, \tag{4.16}$$

$$||I^{(2)} f||_- \le C_2 ||f||_+ \tag{4.17}$$

hold.

Proof. Let us start with the inequality (4.16). Since $f \in H_+$ we can apply F_1 to $\varphi(x,\lambda)\varphi(f,\lambda)$. From (4.12) and (4.14) we have

$$\begin{aligned} I^{(1)} f(x) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e(0,\lambda)}{\beta(\lambda)} F_1\{\varphi(x,\lambda)\varphi(f,\lambda)\} d\lambda \\ &+ \frac{1}{2\pi i} \sum_{k=1}^{\alpha} \sum_{j=0}^{m_k-1} \left\{ \frac{\partial^j}{\partial\lambda^j} [\varphi(x,\lambda)\varphi(f,\lambda)] \right\}_{\lambda=\lambda_k} \int_{\gamma^+} \frac{\lambda e(0,\lambda) F_{kj}^{(1)}(\lambda)}{\beta(\lambda)} d\lambda. \end{aligned} \tag{4.18}$$

By the definition of F_1 and using the integral form of the remainder in the Taylor formula, we get

$$(4.19)\qquad F_1\{\varphi(x,\lambda)\varphi(f,\lambda)\} = \begin{cases} \varphi(x,\lambda)\varphi(f,\lambda), & \lambda \in \wedge_0^{(1)}, \\ \frac{1}{(m_k-1)!}\int_{\lambda k}^{\lambda}(\lambda-\mu)^{m_k-1}\left\{\frac{\partial^{m_k}}{\partial\mu^{m_k}}[\varphi(x,\mu)\varphi(f,\mu)]\right\}d\mu, & \lambda \in \wedge_k^{(1)}, \end{cases}$$

where $k = 1, \ldots, \alpha$. If we us the notations

$$I_k^{(1)} f(x) = \frac{1}{2\pi i}\int_{\wedge_k^{(1)}} \frac{\lambda e(0,\lambda)}{\beta(\lambda)} F_1\{\varphi(x,\lambda)\varphi(f,\lambda)\} d\lambda, \quad k = 0, 1, 2, \ldots, \alpha,$$

$$\tilde{I}^{(1)} f(x) = \frac{1}{2\pi i}\sum_{k=1}^{\alpha}\sum_{j=0}^{m_k-1}\left\{\frac{\partial^j}{\partial\lambda^j}[\varphi(x,\lambda)\varphi(f,\lambda)]\right\}_{\lambda=\lambda_k}\int_{\gamma^+}\frac{\lambda e(0,\lambda)F_{kj}^{(1)}(\lambda)}{\beta(\lambda)}d\lambda,$$

we obtain

$$(4.20)\qquad I^{(1)} = I_0^{(1)} + I_1^{(1)} + I_2^{(1)} + \cdots + I_\alpha^{(1)} + \tilde{I}^{(1)}$$

from (4.18) and (4.19). We will easily prove that each of the operators $I_0^{(1)}, I_1^{(1)}, \cdots, I_\alpha^{(1)}$ and $\tilde{I}^{(1)}$ are continuous from H_+ into H_-. So we get (4.16).

In a similar way (4.17) can be proved. □

Since we know that $C_0^\infty(\mathbb{R}_+) \subset H_m$, by Lemma 4.3 we have following remark.

Remark 4.4. For each $f \in C_0^\infty(\mathbb{R}_+)$ the integrals in (4.4) are convergent the norm of H_-.

Theorem 4.5. *Under the condition* (2.11)*, for any* $f \in H_+$*, the spectral* *expansion of L, in terms of the principal functions is*

$$
\begin{aligned}
(4.21)\qquad & f(x)\\
&=\sum_{k=1}^{j}\left\{(\frac{\partial}{\partial\lambda})^{m_k^+-1}M_k^+(\lambda)\varphi(x,\lambda)\varphi(f,\lambda)\right\}_{\lambda=\lambda_k^+}\\
&+\sum_{k=1}^{l}\left\{(\frac{\partial}{\partial\lambda})^{m_k^--1}M_k^-(\lambda)\varphi(x,\lambda)\varphi(f,\lambda)\right\}_{\lambda=\mu_k^-}\\
&+\frac{1}{2\pi i}\sum_{p=1}^{\alpha}\sum_{j=0}^{m_p-1}\left\{(\frac{\partial}{\partial\lambda})^{j}\varphi(x,\lambda)\varphi(f,\lambda)\right\}_{\lambda=\lambda_p}\int_{\gamma^+}\frac{\lambda e(0,\lambda)F_{pj}^{(1)}(\lambda)}{\beta(\lambda)}d\lambda\\
&+\frac{1}{2\pi i}\sum_{p=1}^{\nu}\sum_{j=0}^{n_p-1}\left\{(\frac{\partial}{\partial\lambda})^{j}\varphi(x,\lambda)\varphi(f,\lambda)\right\}_{\lambda=\mu_p}\int_{\gamma^-}\frac{\lambda \tilde{e}(0,\lambda)F_{pj}^{(2)}(\lambda)}{\tilde{\beta}(\lambda)}d\lambda\\
&-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda e(0,\lambda)}{\beta(\lambda)}F_1\{\varphi(x,\lambda)\varphi(f,\lambda)\}d\lambda\\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda \tilde{e}(0,\lambda)}{\tilde{\beta}(\lambda)}F_2\{\varphi(x,\lambda)\varphi(f,\lambda)\}d\lambda
\end{aligned}
$$

in since of the norm of H_-.

Proof. We can easily obtain (4.21) for $f\in C_0^\infty(\mathbb{R}_+)$, by use of (4.14), (4.15) and (4.18) in (4.4). The integrals over the contours γ^+ and γ^- are absolutely convergent. We have shown in Lemma 4.3 that the integrals

$$
\int_{-\infty}^{\infty}\frac{\lambda e(0,\lambda)}{\beta(\lambda)}F_1\{\varphi(x,\lambda)\varphi(f,\lambda)\}d\lambda,\quad \int_{-\infty}^{\infty}\frac{\lambda \tilde{e}(0,\lambda)}{\tilde{\beta}(\lambda)}F_2\{\varphi(x,\lambda)\varphi(f,\lambda)\}d\lambda
$$

are convergent in the sense of the norm of H_- for $f\in C_0^\infty(\mathbb{R}_+)\subset H_m$. Since $C_0^\infty(\mathbb{R}_+)$ is dense in H_+ the proof is completed. □

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