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APPROXIMATION OF FUNCTIONS BY SOME MEANS OF THEIR FOURIER SERIES

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. Deviations of the integrable functions and some means for Fourier series of these functions are represented in such the forms that the appropriate reminders are estimated by moduli of smoothness for given functions from the above and below.

Let a function $f \in L_p$, $1 \le p \le \infty$, 2π-periodic and

$$(1)\qquad \begin{aligned} f(x) &\sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} A_k(x)\,, \\ \tilde{f}(x) &\sim -i \sum_{k=-\infty}^{\infty} \operatorname{sign} k\, A_k(x) \end{aligned}$$

be Fourier expansions for f and conjugate function $\tilde{f}$. Let us consider Riesz R_n^α, r_n^α and (c,α) means of (1)

$$\begin{aligned} R_n^\alpha(f;x) &= \sum_{|k|\le n} \left(1 - \frac{|k|}{n+1}\right)^\alpha A_k(x)\,, \\ r_n^\alpha(f;x) &= \sum_{|k|\le n} \left(1 - \left(\frac{|k|}{n+1}\right)^\alpha\right) A_k(x)\,, \\ \sigma_n^\alpha(f;x) &= \sum_{|k|\le n} \frac{A_{n-|k|}^\alpha}{A_n^\alpha}\, A_k(x), \end{aligned}$$

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respectively, and arithmetic ones $\sigma_n = R_n^1 = r_n^1 = \sigma_n^1$ as a special case.

Let Δ and w be symmetric difference and modulus of smoothness (of the appropriate orders and steps),

$$\Delta_\delta f(x) = f\left(x + \frac{\delta}{2}\right) - f\left(x - \frac{\delta}{2}\right), \quad \Delta_\delta^k = \Delta_\delta(\Delta_\delta^{k-1}),$$

and $w_k(f,h) = \sup\limits_{0<\delta\le h} \|\Delta_\delta^k f(0)\|$, respectively.

The deviations $f(x) - R_n^\alpha(f;x)$, $f(x) - \sigma_n^\alpha(f;x)$ were investigated by some authors in different directions. Here is one of these results due to M. M. Lekishvili [1]: *If $c > 0$ and $\alpha > 0$ we have*

$$f(x) - \sigma_n^\alpha(f;x) = -\frac{\alpha}{2\pi}\int_1^\infty \Delta_{t/(n+1)}^2 f(x) t^{-2}\, dt + \tau_n(x),$$

$$\|\tau_n(x)\|_p \le c w_2\left(f; \frac{1}{n+1}\right).$$

Such a representation was obtained by L. P. Falaleev [2] for Riesz means and earlier by H. K. Lebed' and A. A. Avdienko [3] for arithmetic means.

Now, we give some new results in this directions.

Theorem 1. *For $f \in L_p$, $1 \le p \le \infty$, $\alpha > 0$, there are $c_1(p,\alpha) > 0$ and $c_2(p,\alpha) > 0$ such that*

$$f(x) - \sigma_n^\alpha(f;x) = -\frac{\alpha}{2\pi}\int_1^\infty \Delta_{t/(n+1)}^2 f(x) t^{-2}\, dt + \tau_n(f;x),$$

where

$$c_1(p) w_2\left(f; \frac{1}{n+1}\right)_p \le \|\tau_n(f;x)\|_p \le c_2(p) w_2\left(f; \frac{1}{n+1}\right)_p.$$

The same result was earlier proved for Riesz means in [4].

The reminders in Theorems 1 and 2 are estimated from both above and below but not from above as in [1]–[3], namely the exact orders of remainders are obtained in our theorem.

Theorem 2. *For $f \in L_p$, $1 \le p \le \infty$, $m \in \mathbb{N}$, there are $c_1(p,m) > 0$ and $c_2(\dots) > 0$ such that* $\left(\sum_1^0 = 0\right)$

$$f(x) - \sigma_n(f;x) = \sum_{j=1}^{m} c_j^* \int_1^\infty \Delta_{t/(n+1)}^{2j} f(x) t^{-2j}\, dt \\ + \sum_{j=1}^{m-1} c_j^* \int_1^\infty \Delta_{t/(n+1)}^{2j+1} \tilde{f}(x) t^{-2j+1}\, dt + \tau_n(f;x),$$

where

$$c_1(p,m) w_{2m} \left(f; \frac{1}{n+1}\right)_p \le \|\tau_n(f;x)\|_p \le c_2(p,m) w_{2m} \left(f; \frac{1}{n+1}\right)_p .$$

Constants c_i^* in this theorem are constructive. The special case of Theorem 2 is $(m = 2)$

$$f(x) - \sigma_n(f;x) = c_1^* \int_1^\infty \Delta_{t/(n+1)}^{2} f(x) t^{-2}\, dt + c_2^* \int_1^\infty \Delta_{t/(n+1)} \tilde{f}(x) t^{-3}\, dt \\ + c_3^* \int_1^\infty \Delta_{t/(n+1)}^{4} f(x) t^{-4}\, dt + \tau_n(f;x)\,,$$

where

$$c_1 w_4(f; 1/(n+1))_p \le \|\tau_n(f;x)\|_p \le c_2 w_4(f; 1/(n+1))_p. \tag{2}$$

We use another technique to prove Theorems 1 and 2 than that one in the cases of [1]–[3]. This technique (comparison and theorems on multiplicators) was developed by R. M. Trigub [5].

Let $\Lambda_1 = \|\lambda_k^{(n)}\|$, $\Lambda_2 = \|\tilde{\lambda}_k^{(n)}\|$ be given matrices with elements depending on $n \in \mathbb{N}$ and $\tau_n(f;\Lambda_1,x) \sim \sum_k \lambda_k^{(n)} A_k(x)$, $\tau_n(f;\Lambda_2,x) \sim \sum_k \tilde{\lambda}_k^{(n)} A_k(x)$ be different means of (1) and functions $\tau_n(f,\Lambda_i,x)$ belong to L_p, $1 \le p \le \infty$. Then the inequality holds (comparison principle)

$$\|f(\cdot) - \tau_n(f,\Lambda_1,\cdot)\| \le \tau(\Lambda) \|f(\cdot) - \tau_n(f;\Lambda_2,\cdot)\|$$

where $\Lambda = \|\lambda_k^{*(n)}\|$, $\lambda_k^{*(n)} = (1 - \lambda_k^{(n)})/(1 - \tilde{\lambda}_k^{(n)})$ is transitional matrix $\tau(\Lambda)$ – a norm of an appropriate operator (Lebesque constant). So to prove the inequalities in Theorems 1 and 2 one needs to construct at first the transitional matrix and then check the boundness of norms of operators for transitional matrices.

To prove (2) we use the well-known result (see [5] for references) with $c_1 > 0$, $c_2 > 0$,

$$w_4\Big(f; \frac{1}{n+1}\Big) \le |f(\cdot) - r_n^4(f,\cdot)| \le c_2 w_4(\cdots)$$

Then we compare the means $f(x) - r_n^2(f;x)$ and

$$f(x) - \sigma(f;x) - c_1^* \int_1^\infty \Delta^2_{t/(n+1)} f(x) t^{-2}\,dt - c_2^* \int_1^\infty \Delta_{t/(n+1)} \tilde{f}(x) t^{-3}\,dt \\ - c_3^* \int_1^\infty \Delta^4_{t/(n+1)} f(x) t^{-4}\,dt.$$

Let

$$J_n = \int_1^\infty u^{-n} \sin^n u\,du\,, \quad i_n(x) = \int_0^1 t^{-u} \sin^n \frac{xt}{2}\,dt.$$

The transitional function $\Lambda(x)$ (matrix for $x = k/(n+1)$ for the right-side inequality in (2), $c_1 = 1/2J_2$, $c_2 = 1/4J_2J_4$, $c_3 = -1/8J_1J_3J_4$) is

$$\Lambda(x) = x^{-4}\Big(x - 2c_1 x J_2 + 4c_1 i_2(x) - 2c_2 x^2 J_3 + 8c_2 i_3(x) + 2c_3 x^3 J_4 - 16 c_3 i_4(x)\Big)$$

for $0 < x < 1$ and

$$\Lambda(x) = 1 + c_1(4i_2(x) - 2xJ_2) + c_2\left(8i_2(x) - 2x^3 J_3\right) + c_3\left(2x^3 J_4 - 16 i_4(x)\right)$$

for $x \ge 1$. Then it remains to use one of the conditions for boundness of Lebesque constants (for example Sidon-Telyakovskii, see [5]).

The proofs of the left-hand side inequality of (2) in Theorems 1 and 2 are analogous to given one.

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