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DEGENERACY OF POSITIVE LINEAR OPERATORS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. This paper is a survey of results obtained by the authors in the topic of degeneracy of positive linear operators. A broad discussion on the topic includes some new results and applications of degeneracy.

1. Introduction

Definition 1. Let $\mathcal{Q}_m$ be the polynomial space of dimension $m+1$ and $\{L_n\}$ be the sequence of projective operators $L_n : C[0,1] \to \mathcal{Q}_m$. If exists, the set $\mathcal{D}(L_n) \subseteq C[a,b] \setminus \mathcal{Q}_m[a,b]$ such that $\deg L_n(\varphi) < n$, $\varphi \in \mathcal{D}(L_n)$, it is called the *degeneracy set* of the sequence $\{L_n\}$ and it is said that the operator sequence $\{L_n\}$ exhibits *degenerative property* or *deficiency property*.

The Bernstein operators B_n have been firstly pointed out to have degenerative property. These operators map $C[0,1]$ into $\mathcal{Q}_n$ and are defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) , \quad x \in [0,1] , \ f \in C[0,1] , \tag{1}$$

where the kernel functions

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} , \tag{2}$$

form a basis in $\mathcal{Q}_n$. It is well known [15] that $\lim B_n f = f$ uniformly on $[0,1]$.

Actually, Freedman and Passow, [9], [18], proved the following theorem:

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Theorem 1 (Freedman, Passow). *Let π_n be the set of continuous piecewise affine functions with the knots in k/n, $k = 1, ..., n-1$, $n \geq 2$, $n \in N$, and let $\{B_n\}$ be the sequence of Bernstein operators defined by* (1). *Then, for $n \geq 2$,*

$$f \in \pi_n \Rightarrow B_n f = B_{n+1} f \ . \tag{3}$$

So, $\pi_n \subseteq \mathcal{D}(B_n)$. An element of π_n is the function whose graph is the polygonal line with knots in k/n. Note that this polygonal line does not need to change slope at all of these knots. Also, it is easy to see that for every $p \in N$, $\pi_n \subseteq \pi_{pn}$, so that

$$f \in \pi_n \Rightarrow B_{pn}(f) = B_{pn+1}(f) \ , \forall p \in N \ . \tag{4}$$

In fact, this is the original form of the result of Freedman and Passow. Goodman and Sharma [10] have got the implication (4) simplified to (3).

A slightly different formulation for essentially the same fact is given in [22]. According to Schoenberg, Averbach pointed out the implication (3) as the equality case in the inequality $B_{n+1}f \leq B_n f$, for every f convex on $[0, 1]$ (cf. Temple [25]) (see section 4). The same result was given in [12, Th. 7.5].

Degenerative property of Bernstein polynomials is illustrated in Figure 1. The function $x \mapsto \varphi_c(x) = |x - c|$, has been approximated by the sequence of polynomials $B_n(x) = B_n(\varphi_c)$ for $n = 2, \ldots, 17$. For $c = 1/2$, function φ_c belongs to the degenerative set π_2, so that, in virtue of (3), i.e. (4), it must be $B_2(x) = B_3(x)$, $B_4(x) = B_5(x)$, etc. (Figure 1, left). If c differs for a small amount from $1/2$, the separation of pairs (B_{2n}, B_{2n+1}) becomes visible on the graphical presentation (Figure 1, right).

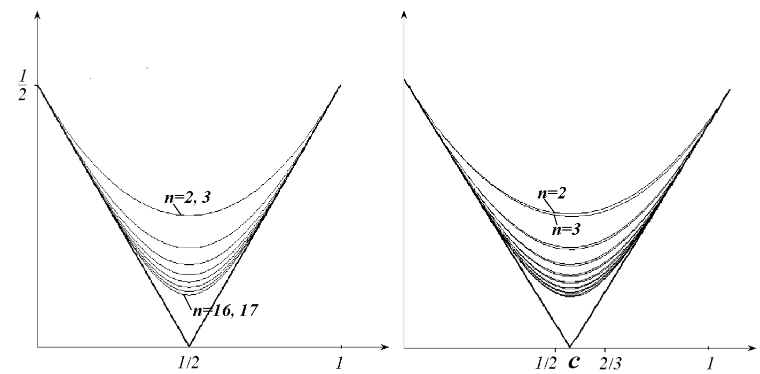

FIG. 1. Degeneracy of Bernstein polynomials

Concerning degeneracy of Bernstein polynomials, Passow [18] conjectured the inverse implication in (4):

Conjecture 1 (Passow).

$$B_{pn}(f) = B_{pn+1}(f), \forall p \in N \Rightarrow f \in \pi_n \ .$$

Goodman and Sharma [10] gave a partial answer to this conjecture by proving that if f is convex on $[0,1]$ and $B_n f = B_{n+1} f$, then $f \in \pi_n$.

Another partial answer is given by Passow [19], under the assumptions that $f \in C[0,1]$ and $f \in C^2(\frac{i-1}{p}, \frac{i}{p})$ for $i = 1, 2, \ldots, p$. Then, $B_{pn}(f) = B_{pn+1}(f)$, $n = 1, 2, \ldots$, implies $f \in \pi_n$.

According to the authors' knowledge, the conjecture of Passow is still open.

The result of Freedman and Passow (Theorem 1) was generalized in two ways. In [6] it is shown that degeneracy is not a privilege of Bernstein operators only. Other way of generalization [8] deals with multidimensional Bernstein polynomials over simplices.

2. Other operators

The first remark of the authors in [6] was that Theorem 1 was a simple consequence of the difference formula

$$\Delta B_n(f;x) = -\frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right] \ , \tag{5}$$

which seems to be firstly published by Aramă [1]. Here,

$$\delta_k^n f = \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right] \ , \ k = 0, \ldots, n-1 \ ,$$

stands for the second divided difference of the function f with respect to the points k/n, $(k+1)/(n+1)$ and $(k+1)/n$. Now, one has

Proof of Theorem 1. If $f \in \pi_n \subseteq C[0,1]$, then $\delta_k^n f = 0$, $k = 0, \ldots, n-1$, so from (5) it follows that $\Delta B_n(f;x) = 0$, i.e. (3) is valid. $\square$

Similar proof, given by Passow in [19] is based on Averbach formula ([12])

$$\frac{-\Delta B_n(f;x)}{(1-x)^{n+1}} = \sum_{k=1}^{n} \left\{ \binom{n}{k} f\left(\frac{k}{n}\right) + \binom{n}{k-1} f\left(\frac{k-1}{n}\right) - \binom{n+1}{k} f\left(\frac{k}{n+1}\right) \right\} z^k \ ,$$

where $z = x/(1-x)$.

It is shown in [6] that difference formula similar to (5) is valid for more general classes of positive linear operators (PLO), and thus, the degeneracy property as well.

Let the sequence of PLO's $\{L_n^\alpha\}_{n=1}^{+\infty}$ be given by

$$L_n^\alpha(f;x) = \sum_{k=0}^{\infty} \ell_{n,k}^\alpha(x) f(\frac{k}{n}) \ , \ x \in [0,+\infty) \ , \tag{6}$$

where $\ell_{n,k}^\alpha$ be a kernel and α be a parameter. The operators L_n^α cover several important, well known cases. Thus, for $\ell_{n,k}^\alpha(x) = \binom{n}{k} x^k (1-x)^{n-k}$, L_n^α are Bernstein operators while for $\ell_{n,k}^\alpha(x) = e^{-nx}\frac{(nx)^k}{k!}$, operators (4) reduce to Favard-Szász-Mirakyan operators [25]. If one specifies

$$\ell_{n,k}^\alpha(x) = \frac{\binom{n}{k} x^{(k,-\alpha)} (1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}} \ ,$$

where

$$x^{(k,\alpha)} = \begin{cases} 1, & k = 0 \\ x(x-\alpha)...(x-(k-1)\alpha), & k \geq 1, \end{cases}$$

the Stancu operators ([23]) are obtained. By the choice

$$\ell_{n,k}^\alpha(x) = (1+n\alpha)^{-\frac{x}{\alpha}} (\alpha + \frac{1}{n})^{-k} \frac{x^{(k,-\alpha)}}{k!} \ ,$$

(7) gives generalized Favard-Szász-Mirakyan operators [24], [16], [20].

Theorem 2 ([6]**).** *For the sequence of operators* (6) *the implication*

$$f \in \pi_n \Rightarrow L_n^\alpha f = L_{n+1}^\alpha f \ , \tag{7}$$

is valid.

Proof. The proof is based on the difference formula which holds for all $x \geq 0$ and $\alpha \geq 0$

$$\Delta L_n^\alpha(f;x) = \frac{x^2 - L_n^\alpha(e_2;x)}{(n+1)(1+n\alpha)} \sum_{k=0}^{+\infty} s_k^\alpha(x) \ell_{n-1,k}^\alpha(x) \delta_k^n f \ ,$$

where $e_2(t) = t^2$, and the sequence $s_k^\alpha(x)$ satisfies

$$\sum_{k=0}^{+\infty} s_k^\alpha(x)\ell_{n-1,k}^\alpha(x) = 1 \ .$$

For the details of the proof, see [6]. $\square$

In [8], the authors determined all PLO's of interpolation type

$$L_n(f;x) = \sum_{k=0}^{n} f(x_k^n)b_k^n(x), x \in [0,1], \ n \in N \ , \tag{8}$$

where the kernel functions $\{b_k^n(x)\}$ form a basis in $\mathcal{Q}_n$ and $0 = x_0^n < \ldots < x_n^n = 1$. Note that $L_n : C[0,1] \to \mathcal{Q}_n$. It is clear that $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$, which means that $b_k^n(x) \in$span$\{b_j^{k+1}(x)\}$, i.e. the degree elevation formula

$$b_k^n(x) = \sum_{j=0}^{n+1} m_{j,k} b_j^{n+1}(x) \ ,$$

takes place. The matrix of coefficients $[m_{j,k}]$, in many known cases has a three-diagonal form, so that the degree elevation formula has the form

$$b_k^n(x) = m_{k,k} b_k^{n+1}(x) + m_{k+1,k} b_{k+1}^{n+1}(x) \ , \tag{9}$$

where the coefficients $m_{i,j}$ satisfy the following conditions

$$m_{i,j} \geq 0, \tag{10}$$

$$\sum_{j=0}^{n} m_{i,j} = 1, \quad i = 0, ..., n+1, \tag{11}$$

$$\sum_{j=0}^{n} m_{i,j} x_i^n = x_i^{n+1}, \ i = 0, ..., n+1. \tag{12}$$

Besides, it is natural that the structure of the matrix of mesh knots $[x_k^n]$ affects the convergence of the sequence $\{L_n f\}$. For the majority of interpolation type PLO's, this matrix has *dyadic structure*, which means that it always exists the continuous monotone function g such that $g(x_k^n) = k/n$ for $k = 0, \ldots, n$ and $n \in N$. It is easy to show, that for each dyadic matrix of nodes, $x_{pk}^{pn} = x_k^n$ for $p \in N$.

Let Π_n be the set of continuous piecewise affine functions with the knots in $\{x_k^n\}_{k=1}^{n-1}$, $n \in N$, then the following theorem generalizes Theorem 1.

Theorem 3 ([8]). *Let the sequence of PLO* (8) *be defined by the dyadic node matrix* $[x_k^n]$. *Let for every* $n \in N$

a) $L_n\varphi = \varphi,\ \varphi(x) = \alpha x + \beta,\ \alpha, \beta \in \mathbf{R}$;

b) $L_n(f; 0) = f(0), \quad L_n(f; 1) = f(1)$;

c) the kernel functions $b_k^n(x)$ *satisfy three-term recurrence relation* (9). *Then the sequence* $\{L_n\}$ *exhibits degeneracy property, i.e.*

$$f \in \Pi_n \Rightarrow L_{n+1}f = L_n f \ .$$

Proof. It is based on the difference formula

$$L_n(f; x) - L_{n+1}(f; x) = \sum_{k=0}^{n-1} a_{n,k} b_{k+1}^{n+1}(x)[x_k^n, x_{k+1}^{n+1}, x_{k+1}^n; f], \tag{13}$$

where $a_{n,k} = (x_{k+1}^n - x_{k+1}^{n+1})(x_{k+1}^{n+1} - x_k^n)$. Note that the divided difference in (13) is well defined on the dyadic node matrix. Further, from $f \in \Pi_n$ it is obvious that $[x_k^n, x_{k+1}^{n+1}, x_{k+1}^n; f] = 0$, i.e. $L_{n+1}(f) = L_n(f)$. For details of this proof, see [8]. $\square$

The Theorem 3 contains following important special cases:

1) By choosing $b_k^n(x) = p_{n,k}(x)$ (given by (2)) and $x_k^n = k/n$, the degree raising formula is

$$b_k^n(x) = \frac{n-k+1}{n+1} b_k^{n+1}(x) + \frac{k+1}{n+1} b_{k+1}^{n+1}(x), k = 0, ..., n \ . \tag{14}$$

Thus, by Theorem 3, $D(B_n) \supseteq \pi_n$.

2) For

$$b_k^n(x) = \frac{\binom{n}{k} x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}} \ ,$$

($\alpha \geq 0$ is fixed), and $x_k^n = k/n$, the Stancu operators are obtained. The degree raising formula is just the same as (14), and the degeneracy set includes π_n, as above.

3) Let $\mu = (\mu_0, ..., \mu_{n-1}), \nu = (\nu_0, ..., \nu_{n-1})$ are two sequences of nonnegative parameters. The choice $b_k^n(x) = d_k^n(x)$, where for $k = 0, ..., n$

$$d_k^n(x) = \lambda_{n,k}(x + \mu_0)...(x + \mu_{k-1})(1 - x + \nu_0)...(1 - x + \nu_{n-k-1}) \ ,$$

and the constants $\lambda_{n,k}, k = 0, ..., n$, are given recursively by

$$\lambda_{0,k} = \delta_{k,0} \ ,$$

$$\lambda_{n,k} = \frac{\lambda_{n-1,k-1}}{1+\mu_{k-1}+\nu_{n-k}} + \frac{\lambda_{n-1,k}}{1+\mu_k+\nu_{n-k-1}} \ ,$$

together with selecting nodes $x_0^n, \dots, x_n^n$, so that $\sum_{k=0}^n x_k^n d_k^n(x) = x$ is satisfied. Then,

$$D_n^{\mu,\nu}(f;x) = \sum_{k=0}^n f(x_k^n) d_k^n(x), \ x \in [0,1] \ ,$$

defines the sequence of PLO. These operators have been studied by Barry and Goldman [3]. They used the generalized Pólya urn model to generate $D_n^{\mu,\nu}$. The degree raising formula is

$$d_k^n(x) = \frac{\lambda_{n,k}}{1+\mu_k+\nu_{n-k}} \left(\frac{d_k^{n+1}(x)}{\lambda_{n+1,k}} + \frac{d_{k+1}^{n+1}(x)}{\lambda_{n+1,k+1}} \right) \ ,$$

and, by Theorem 3, $\mathcal{D}(D_n^{\mu,\nu}) \supseteq \Pi_n$, with x_k^n as nodes.

4) Let $\nu \in N, m \in N_0$ and $n = \nu + m$ and $\tau^n = (t_{-\nu}, ..., t_{n+1})$ be the knot sequence $0 = t_{-\nu} = ... = t_0 < t_1 \leq t_2 \leq ... \leq t_m < t_{m+1} = ... = t_{n+1} = 1$, such that $t_{i-\nu} < t_{i+1} (i = 0, ..., n)$. The sequence of nodes x^n, induced by τ^n is given by

$$x_k^n = \frac{1}{\nu}(t_{k-\nu+1} + ... + t_k), k = 0, ..., n. \tag{15}$$

In [22] Schoenberg introduced the operator T_n by

$$T_n(f;x) = \sum_{k=0}^n f(x_k^n) N_{\nu,k}(x), \ x \in [0,1] \ ,$$

where $N_{n,k}(x) = N(x/t_{k-\nu}, ..., t_{k+1}), k = 0, ..., n$, are the B-splines of degree ν, with respect to the knots $t_{i-n}, ..., t_{i+1}$. The operator T_n, now known as Schoenberg's variation diminishing spline operator, maps C[0,1] into S_n the space of splines of degree ν with the knots τ^n. It is known that $\{N_{\nu,k}\}, k = 0, ..., n$, is a basis for S_n. To obey our previous notation, we shall write $b_k^n(x) = N_{\nu,k}(x)(n = m + \nu)$. Thus, the operator T_n degenerates over the set Π_n with nodes at the points x_k^n, given by (15).

3. Multidimensional case

Let $a_0, a_1, \dots, a_s$ $(s \geq 1)$ be the set of affinely independent points from $\mathbf{R}^s$. Then, the s-dimensional simplex is defined by $\sigma_s = span\{a_0, \dots, a_s\}$. By

Caratheodory's theorem, for each $t \in \sigma_s$, there exist real numbers $t_0, \dots, t_s$, $t_i \geq 0$ and $\sum_{i=0}^{s} t_i = 1$ such that, for each $t \in \mathbf{R}^s$, $t = \sum_{i=0}^{s} t_i a_i$. The numbers t_i are called *barycentric* coordinates of t with respect to the simplex σ_s. Let $\mathbf{i} = (i_0, \dots, i_s)$ be the set of multiindices and $|\mathbf{i}| = \sum_j i_j$. For any function $f : \sigma_s \to \mathbf{R}$, the Bernstein operator over σ_s is defined by

$$B_n^{\sigma}(f;t) = \sum_{|\mathbf{i}|=n} f\left(\frac{\mathbf{i}}{n}\right) b_{\mathbf{i}}^n(t) \ , \tag{16}$$

where for $t = (t_0, \dots, t_s) \in \sigma_s$, the basis polynomials are

$$b_{\mathbf{i}}^n(t) = \frac{n!}{i_0! \cdots i_s!} t_0^{i_0} \cdots t_s^{i_s} \ ,$$

and $f(\mathbf{i}/n)$ is the value of f in the point $x_{\mathbf{i}}^n = (i_0/n, \dots, i_s/n) \in \sigma_s$.

Let $x_{\mathbf{i}}^{n+1} = \mathbf{i}/(n+1)$, $x_{\mathbf{i}-\mathbf{e}_k}^n = (\mathbf{i} - \mathbf{e}_k)/n$, $k = 0, \dots, s$, where $\mathbf{e}_k = (\delta_{j,k})_{j=0}^s$, be the points from $\mathbf{R}^s$. The functional $D_{\mathbf{i}}^n : f \mapsto \mathbf{R}$, defined by

$$D_{\mathbf{i}}^n f = f(x_{\mathbf{i}}^{n+1}) - \sum_{k=0}^{s} \frac{i_k}{n+1} f\big(x_{\mathbf{i}-\mathbf{e}_k}^n\big) \ , \quad |\mathbf{i}| = n+1 \ , \tag{17}$$

will be called *Jensen functional*.

In [7], the following results are proved.

Theorem 4. *If $D_{\mathbf{i}}^n$ is the Jensen functional given by* (17), *then*

$$\Delta B_n^{\sigma}(f;t) = \sum_{|\mathbf{i}|=n+1} b_{\mathbf{i}}^{n+1}(t) D_{\mathbf{i}}^n f \ .$$

Theorem 5. *Let $\pi_n(\sigma_s)$ be the set of continuous piecewise affine functions with the knots $\{\frac{\mathbf{i}}{n}\}$, $|\mathbf{i}| = n$, $\mathbf{i} \neq n\mathbf{e}_k$, $k = 0, \dots, s$. Then*

$$f \in \pi_n(\sigma_s) \Rightarrow B_n^{\sigma} f = B_{n+1}^{\sigma} f \ .$$

Proof. Note that every point $x_{\mathbf{i}}^{n+1} = \mathbf{i}/(n+1)$ is surrounded by $s+1$ points $x_{\mathbf{i}-\mathbf{e}_k}$, $k = 0, \dots, s$, which form a subsimplex $\sigma_s^{\mathbf{i}}$ =span$\{x_{\mathbf{i}-\mathbf{e}_k}, k = 0, \dots, s\}$. On the other hand, the function $f \in \pi_n(\sigma_s)$ is affine in $\sigma_s^{\mathbf{i}}$ for every $\mathbf{i}$. Thus, by (16), $D_{\mathbf{i}}^n f = 0$ and by Theorem 4, it follows that $\Delta B_n^{\sigma} f = 0$. $\square$

In [5], Chui, Hong and S. Wu obtained the result stated in Theorem 5.

Taking into account that $\pi_n(\sigma_s) \subseteq \pi_{pn}(\sigma_s)$ for every $p \in N$, one has the following

Corollary 1. *If $f \in \pi_n(\sigma_s)$, then $B^\sigma_{pn} f = B^\sigma_{pn+1} f$.*

Theorem 5 generalizes the result by Chang and Davis [4] about degeneracy of Bernstein polynomial operators defined on triangles, i.e. when $s = 2$. Note, also that Theorem 1 is a special case of Theorem 5 for $s = 1$. Indeed, σ_1 is a segment of the real axis, say $[a, b]$, while σ_2 is a triangle in the plane. An example of degeneracy of Bernstein operators, defined over triangle σ_2, by

$$B^\sigma_n(f;t) = \sum_{i+j+k=n} f\Big(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\Big) \frac{n!}{i!j!k!} u^i v^j w^k \ , \ t = (u, v, w) \in \sigma_2 \ ,$$

is shown in Figure 2. The function f belongs to the set $\pi_3(\sigma_2)$. By Corollary 1, $B^\sigma_3 f = B^\sigma_4 f$, $B^\sigma_6 f = B^\sigma_7 f$ and so on. The triangular patches, being graphs of the polynomials $B^\sigma_n f$, $n = 2, 3, 4, 6, 7$, are shown in Figure 2.

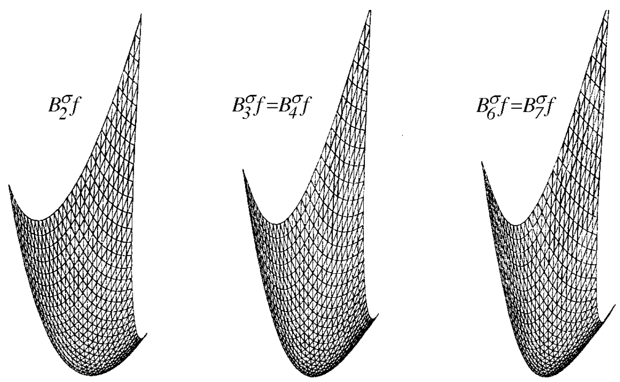

FIG. 2. Degeneracy of Bernstein polynomials on triangle

The following conjecture is the direct analogue of Passow's Conjecture 1 (see [7]):

Conjecture 2.

$$B^\sigma_n f = B^\sigma_{n+1} f \ , \forall p \in N \Rightarrow f \in \pi_n(\sigma_s) \ .$$

Concerning deficiency on the triangular domain σ_2, H.-Y. Wu [27] proved that if $f \in C^1(\sigma_2)$ is a piecewise quadratic function defined on the second subdivision of σ_2, then

$$(n+1)B^\sigma_{2n+2}(f;t) - (2n+1)B^\sigma_{2n+1}(f;t) + nB^\sigma_{2n}(f;t) = 0 \ .$$

4. Some applications

It is well known fact that (univariate) Bernstein polynomials preserve convexity (of any order) of generating function ([21]), and even some classes of generalized convexity [13]. Also, if f is convex then $\Delta B_n f \leq 0$, which is the result of Temple [26]. These facts are illustrated in Figure 1, where f is the special convex function $\varphi_c(x) = |x-c|$. Distribution of the members of the sequence $\{B_n(\varphi_c;x)\}_{n=2}^{+\infty}$ for various c is fairly complicated. Displaying the locus of minima for the polynomials $B_n(\varphi_c)$ for $2 \leq n \leq 14$ and $0 \leq c \leq 0.5$, on $[0,1/2]$ (Figure 3, left) can help us to see this complexity. It turns out to be paradoxal, but the approximation of the minima of φ_c, by polynomials $B_{pn}(\varphi_c)$ and $B_{pn+1}(\varphi_c)$, near the point $c = 1/n$ is equally good, as it is shown on the separated trajectories for the vicinity of $c = 1/3$, Figure 3 (right). This is the consequence of the degeneracy property of Bernstein operator.

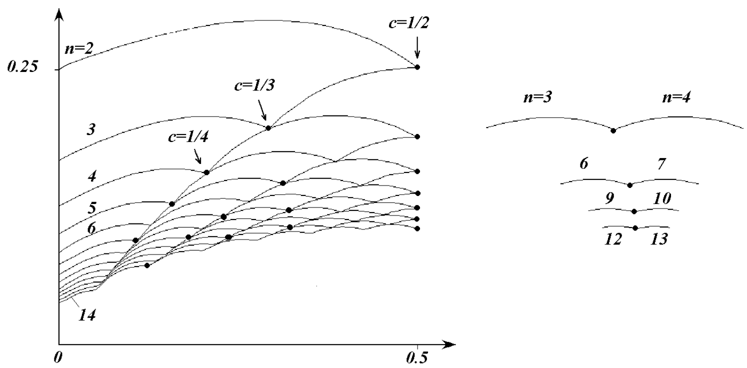

FIG. 3. Minima of $B_n\varphi_c$ on [0,1/2] for $2 \leq n \leq 14$ and for $c \in [0,1/2]$

In connection with approximation of convex functions by Bernstein polynomials, Aramă and Ripianu [2] proved that if $f \in C^\infty[0,1]$ and if for $\nu \geq 2$, $f^{(\nu)}(x) \geq 0$, $x \in [0,1]$, then $\Delta^2 B_n f \geq 0$, $n \in N$, where $\Delta^2 B_n = B_{n+2} - 2B_{n+1} + B_n$. In [11] Horová weakened conditions of Aramă and Ripianu in the following theorem:

Theorem 6. (Horová) *Let* $[x_1, x_2, \ldots, x_{k+2}; f] \geq 0$ *for* $k = 1, 2, 3$ *and* 4, *where* x_i *are arbitrary points from* $[0,1]$. *Then* $\Delta^2 B_n f \geq 0$.

The condition $[x_1, x_2, \ldots, x_{k+2}; f] \geq 0$, $x_i \in [0,1]$ means that f is convex of order k. So, convexity of orders $1, 2, 3$ and 4 of a generating function is the necessary condition for convexity of the sequence of Bernstein polynomials.

Sufficiency of these conditions has never been proved. Accordingly, one may think to weaken conditions of Theorem 6. It may seem that only the first order convexity of f will cause convexity of the sequence $\{B_n f\}$. But, the degeneracy of Bernstein operator can be used to disprove this conjecture. Namely, one has

Theorem 7. *Let $f(x) = |x - 0.5|$. Then the sequence $\{B_n f\}$ is not convex.*

Proof. It is obvious that $f \in \pi_2$, and f is convex (of the first order). By Theorem 1 (i.e. by (4)), $B_{2m} f = B_{2m+1}$, for $m \in N$. Then, for any $m \in N$,

$$\Delta^2 B_{2m} f = B_{2m+2} f - 2B_{2m+1} f + B_{2m} f = B_{2m+2} f - B_{2m+1} f\ .$$

Now, it can be proved that the difference

$$B_{2m+2} f - B_{2m+1} f \neq 0\ . \tag{18}$$

Contrary, suppose that $B_{2m+2} f - B_{2m+1} f = 0$. Following (5), it must be $\sum_{k=0}^{2m} p_{2m,k}(x) \delta_k^{2m+1} = 0$, where

$$\delta_k^{2m+1} f = \left[\frac{k}{2m+1}, \frac{k+1}{2m+2}, \frac{k+1}{2m+1}; f \right]\ .$$

Using the fact that polynomials $\{p_{2m,k},\ k = 0, \ldots, 2m\}$ form a basis in $\mathcal{Q}_{2m}$, it must be $\delta_k^{2m+1} f = 0$ for all $k = 0, \ldots, 2m$. But, the direct calculation shows that $\delta_k^{2m+1} f = 2(2m+1)$ which is different from zero for all $m \in N$. This contradiction shows that (18) is true. On the other hand, $B_n f \leq B_{n+1}$ due to convexity of f, which, together with (18) gives $\Delta^2 B_{2m} f < 0$, i.e. the sequence $\{B_n f\}$ can not be convex. $\square$

Another application of degeneracy is to derive so called *degree elevation* formula which is useful for CAGD in the theory of Bézier curves and surfaces, and similar free-form curve (surface) models generated by other positive linear operators.

Lemma 1. *For every $f \in \pi_n$,*

$$f\left(\frac{k+1}{n+1}\right) = \frac{n-k}{n+1} f\left(\frac{k+1}{n}\right) + \frac{k+1}{n+1} f\left(\frac{k}{n}\right)\ , \tag{19}$$

is valid.

Proof. By the Theorem 1, $f \in \pi_n$ implies $\Delta B_n f = 0$. It follows by (5) and the basis property of polynomials p_k^n that $\delta_k^n f = 0$. Now, from

$$\begin{aligned} 0 = \delta_k^n f &= \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f \right] \\ &= n^2(n+1) \left[\frac{1}{k+1} f\left(\frac{k+1}{n}\right) - \frac{n+1}{(k+1)(n-k)} f\left(\frac{k+1}{n+1}\right) + \frac{1}{n-k} f\left(\frac{k}{n}\right) \right]\ , \end{aligned}$$

(19) easily follows. □

Note that the Bernstein polynomial B_nf, for fixed n does not need the function f to be defined over the whole interval $[0,1]$. It is enough to know just the sequence $P = \{P_k = f(k/n)\}_{k=0}^n$. So, one may write $B_n(P;x)$ rather than $B_n(f;x)$.

The following statement is given in [6]:

Theorem 8. *Let $B_n(P;x)$ and $B_{n+1}(Q;x)$ be two Bernstein polynomials generated by the sequences $\{P_k\}_{k=0}^n$ and $\{Q_k\}_{k=0}^{n+1}$. Then $B_n(P;x) = B_{n+1}(Q;x)$ identically, for $x \in [0,1]$ if and only if*

$$\left\{\begin{array}{ll} Q_0 & = P_0 \ , \\ Q_{k+1} & = \frac{n-k}{n+1}P_{k+1} + \frac{k+1}{n+1}P_k \ , \ k = 0,\dots,n-1 \ , \\ Q_{n+1} & = P_n \ . \end{array}\right. \tag{20}$$

Proof. The analogy of the difference formula (5) is

$$B_n(P;x) - B_{n+1}(Q;x) =$$
$$nx(1-x)\sum_{k=0}^{n-1} p_{n-1,k}(x)\left(\frac{P_{k+1}}{k+1} - \frac{(n+1)Q_{k+1}}{(k+1)(n-k)} + \frac{P_k}{n-k}\right) \ .$$

Therefore from $B_n(P;x) = B_{n+1}(Q;x)$, it follows that $\frac{P_{k+1}}{k+1} - \frac{(n+1)Q_{k+1}}{(k+1)(n-k)} + \frac{P_k}{n-k} = 0$, $(k = 0,\dots,n-1)$. Besides, for $x = 0$, $B_n(P;0) = P_0$ and $B_{n+1}(Q;0) = Q_0$; for $x = 1$, $B_n(P;1) = P_n$ and $B_{n+1}(Q;1) = Q_{n+1}$, i.e. (20) is valid.

Suppose that (20) is true. Then there exists a function $f \in \pi_n$, such that $P_k = f(k/n)$ and $Q_k = f\big(k/(n+1)\big)$, and therefore $B_n(P;x) = B_{n+1}(Q;x)$. □

One generalization of the algorithm (20) is given in [14].

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