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ON CONVEX FAMILIES OF SUMMABILITY METHODS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. In this paper *the convex families of summability methods* defined by sequence-to-sequence transformations are consired. The paper introduces and extends the authors ideas on convexity of fammilies of summability methods started in papers [9–16]. The necessery and sufficient conditions for convexity are considered (section 3). A method for studying some families on convexity arises from these conditions (section 4). The method mentioned is applied to the family of generalized Nörlund methods (section 5). The results for Nörlund methods include the convexity theorems known from papers [6], [8] and the well-known convexity theorem for Cesàro methods. The convex families can be applied to characterization of speed of summability (Section 1).

1. On Notion of Convex Family of Summability Methods

Let us consider sequences $x = (\xi_n)$ with $\xi_n \in K$ ($K = \mathbb{C}$ or $K = \mathbb{R}$) for $n = 0, 1, 2, \dots$. Let $\{A_\alpha\}$ be a family of summability methods A_α given by sequence-to-sequence transformations of $x \in \omega A_\alpha$ into $A_\alpha x = (\eta_n^\alpha)$, where $\eta_n^\alpha \in K$ and α is a continuous parameter with values $\alpha > \alpha_0$. We note that the methods A_α can be, in particular, matrix methods $A_\alpha = (a_{nk}^\alpha)$.

We denote by ωA_α the set of all sequences x where the transformation A_α is applied, by mA_α the boundedness domain of method A_α, by cA_α the convergence domain of A_α and by c_0A_α the 0-convergence domain of A_α. Thus we have:

$$mA_\alpha = \Big\{x \in \omega A_\alpha \mid A_\alpha x \in m\Big\}, \quad cA_\alpha = \Big\{x \in \omega A_\alpha \mid A_\alpha x \in c\Big\},$$

$$c_0A_\alpha = \Big\{x \in \omega A_\alpha \mid A_\alpha x \in c_0\Big\}.$$

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Suppose λ_α=(λ_n^α) are monotonically increasing positive sequences, i.e. $0<\lambda_n^\alpha\uparrow$. Further we need the following notations:

$$\begin{aligned}
m^{\lambda_\alpha} &= \Big\{x=(\xi_n)\in c|\big(\beta_n^\alpha\big)=\big(\lambda_n^\alpha\big(\xi_n-\lim\xi_n\big)\big)\in m\Big\},\\
c^{\lambda_\alpha} &= \Big\{x\in m^{\lambda_\alpha}|\big(\beta_n^\alpha\big)\in c\Big\},\quad c^{*\lambda_\alpha}=\Big\{x\in c^{\lambda_\alpha}|\big(\beta_n^\alpha\big)\in c_0\Big\},\\
m_0^{\lambda_\alpha} &= \Big\{x\in m^{\lambda_\alpha}\,|\lim\xi_n=0\Big\},\\
c_0^{\lambda_\alpha} &= \Big\{x\in c^{\lambda_\alpha}\,|\lim\xi_n=0\Big\},\quad c_0^{*\lambda_\alpha}=\Big\{x\in c_0^{\lambda_\alpha}|\big(\beta_n^\alpha\big)\in c_0\Big\}.
\end{aligned}$$

The sequence x is said to be summable by method A_α with speed λ_α (shortly $A_\alpha^{\lambda_\alpha}$- summable) if $A_\alpha x\in c^{\lambda_\alpha}$. The sequence x is said to be $A_\alpha^{\lambda_\alpha}$-bounded if $A_\alpha x\in m^{\lambda_\alpha}$.

Definition. The family $\{A_\alpha\}$ is said to be convex if for every $\alpha<\beta$ and for every $\alpha<\gamma<\beta$ the conditions

$$mA_\alpha\subset mA_\beta,\quad cA_\alpha\subset cA_\beta \tag{1}$$

and

$$cA_\gamma\supset mA_\alpha\cap cA_\beta \tag{2}$$

hold.

The family $\{A_\alpha\}$ is said to be zero-convex (0-convex) if the conditions (1)-(2) hold with c_0 instead of c in them (see [10]).

We can give an idea for application of convex families.

Let $\{A_\alpha\}$ be a convex family and α and β $(\beta>\alpha)$ be two fixed values of parameter. If the sequence x is bounded by the method A_α and summed by the method A_β, then x is summable by every method A_γ with $\gamma>\alpha$. So the method A_γ optimal in some sense can be chosen for summing the sequence x (for example, the method A_γ which sums x with the optimal speed).

The applications of convex families to summability with speed are based on the following theorem (which is a modification of Theorem 3 formulated in [12]).

Theorem 1. *Let us consider together with the methods A_α $(\alpha>\alpha_0)$ the methods B_α where $B_\alpha x=(\lambda_n^\alpha\eta_n^\alpha)$ with $(\eta_n^\alpha)=A_\alpha x$, $x\in\omega A_\alpha$ and $0<\lambda_n^\alpha\uparrow\infty$.*

a) *If the family* $\{B_\alpha\}$ *is 0-convex then the following implications are true:*

$$A_\alpha x \in m_0^{\lambda_\alpha},\ \alpha < \beta \quad \Longrightarrow \quad A_\beta x \in m_0^{\lambda_\beta}, \tag{3}$$

$$A_\alpha x \in c_0^{*\lambda_\alpha},\ \alpha < \beta \quad \Longrightarrow \quad A_\beta x \in c_0^{*\lambda_\beta} \tag{4}$$

and

$$A_\alpha x \in m_0^{\lambda_\alpha},\ A_\beta x \in c_0^{*\lambda_\beta},\ \alpha < \gamma < \beta \quad \Longrightarrow \quad A_\gamma x \in c_0^{*\lambda_\gamma}. \tag{5}$$

b) *If the family* $\{B_\alpha\}$ *is convex then the implications* (3)-(5) *hold with* c_0 *instead of* c_0^*.

If in addition the transformations A_α *are linear and* $A_\alpha e{=}a_\alpha e$ *where* $0 \neq a_\alpha \in K$ *and* $e = (1, 1, 1, \dots)$ *then the statements* a) *and* b) *hold with* m *instead of* m_0 *and with* c^* *and* c *instead of* c_0^* *and* c_0*, respectively.*

In general case the statements a) and b) of Theorem 1 follow directly from 0-convexity and convexity of family $\{B_\alpha\}$, respectively. To prove these statements in additional restrictions on A_α it is sufficient to notice that the equivalences

$$\begin{aligned} A_\alpha x \in c^{*\lambda_\alpha} \ &\Leftrightarrow \ x - \big(\lim_n \eta_n^\alpha / a_\alpha\big)e \in c_0 B_\alpha,\ A_\alpha x \in m^{\lambda_\alpha} \\ &\Leftrightarrow \ x - \big(\lim_n \eta_n^\alpha / a_\alpha\big)e \in m B_\alpha \end{aligned}$$

are true and the last equivalence is true also with c instead of m in it.

2. Short Review of Results on Convex Families of Summability Methods

First were known *the convexity theorems*[1] *for some special matrix methods.*

1) The convexity of the family of Cesàro methods $A_\alpha = (C, \alpha)$ was first proved by G.H. Hardy and J.E. Littlewood in 1912. Afterwards the convexity of this family was proved in different ways by several authors. The references to these proofs can be found in [4].

2) The sufficient conditions for convexity of the family of Nörlund methods $A_\alpha = (N, p_n^\alpha)$ were proved by F.P. Cass in [6].

[1] We mean under convexity theorems the theorems which state the convexity of a family $\{A_\alpha\}(\alpha > \alpha_0)$.

3) The sufficient conditions for convexity of the family of more general Nörlund methods $A_\alpha = (N, p_n^\alpha, q_n)$ were found by R. Sinha in [8].

4) Some convexity theorems for certain semi-continuous summability methods and methods of strong summability have been published also. For example, the convexity theorems for Nörlund methods of strong summability and Riesz means are known. We don't refer to these results here because the methods of such type do not belong to the subject of this paper. The references to these publications and also to Tauberian theorems connected with convexity theorems can be found in [5,9,10,11,14].

5) The convexity of the family of Euler-Knopp methods $A_\alpha = (E, \alpha)$ was proved by J. Boos and H. Tietz in [5] recently.

The proofs of above mentioned convexity theorems *base on matrices* $A_\alpha = (a_{nk}^\alpha)$.

The author has proved the convexity theorems in papers [9-16]. In those papers the convex familes of summability methods are considered from the general point of view. The techniques of *proofs* in papers [9-16] *base on the connection matrices* $D_{\alpha,\beta-\alpha}$ (not on methods A_α) *between methods* A_α *and* A_β $(\beta > \alpha)$, i.e. on the relations

$$A_\beta x = D_{\alpha,\beta-\alpha}(A_\alpha x) \quad (x \in \omega A_\alpha).$$

We note that different families $\{A_\alpha\}$ may have the same connection matrices $D_{\alpha,\beta-\alpha}$.

Let us characterize briefly *the contents of papers* [9-16].

1) The necessary and sufficient conditions for convexity of a family of normal matrix methods are proved [15,16].

2) The sufficient convexity conditions for certain linearly connected families of summability methods are proved (including strong summability). A method for studying the families on convexity and for constructing new convex families arises from the proved convexity theorems. The new convex families are found [9,10,11,14,15]. The convexity theorems for Nörlund methods (including Cesàro methods) and Riesz means published by other authors (and known to the author) can be inferred from proved theorems as immediate corollaries.

3) The convexity theorems proved for summability of number sequences are generalized to summability in locally convex spaces [12,13,14,16] and applied to the summability with speed and to Tauberian theorems [10,12,14].

3. Some General Convexity Theorems

Suppose at first that $A_\alpha = (a^\alpha_{nk})$ (where $\alpha > \alpha_0$) are normal matrix methods, i.e. $a^\alpha_{nk} = 0$ for all $k > n$ and $a^\alpha_{nn} \neq 0$ ($n = 0, 1, 2, \dots$). Then there exists the inverse matrix A^{-1}_α for every A_α. Let us denote by $D_{\alpha\delta}$ the product of matrices $A_{\alpha+\delta}$ ($\delta > 0$) and A^{-1}_α, i.e.

$$D_{\alpha\delta} = A_{\alpha+\delta}A^{-1}_\alpha.$$

Due to normality of methods A_α the conditions (1) and (2) are equivalent to the reqularity of methods $D_{\alpha,\beta-\alpha}$ and to the inclusion $cD_{\alpha\gamma} \supset m \cap cD_{\alpha\beta}$, respectively. Thus the following theorem holds (see [15], Theorem 1.3).

Theorem 2. *Let $A_\alpha = (a^\alpha_{nk})$ be normal matrix methods for all $\alpha > \alpha_0$. Then the family $\{A_\alpha\}$ is convex and the methods A_α are pairwise consistent if and only if the following conditions hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$.*

1) *The matrix method $D_{\alpha\delta}$ is regular.*
2) *$cD_{\alpha\delta} \supset m \cap cD_{\alpha 1}$ (with consistency).*

The restrictions on methods A_α can be weakened so that the conditions 1) and 2) of Theorem 2 remain sufficient for the convexity of family $\{A_\alpha\}$. The methods A_α need not be normal, even not matrix methods (see [15], Theorem 1.4).

Theorem 3. *Let the summability methods A_α and $A_{\alpha+\delta}$ for every $\alpha > \alpha_0$ and $0 < \delta < 1$ be connected by the row finite matrix $D_{\alpha\delta}$ so that $A_{\alpha+\delta}x = D_{\alpha\delta}(A_\alpha x)$ for each $x \in \omega A_\alpha$. If the matrix $D_{\alpha\delta}$ satisfies for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions* 1) *and* 2) *of Theorem* 2*, then the family $\{A_\alpha\}$ is convex and the methods A_α are pairwise consistent.*

We notice that the conditions 1) and 2) of Theorem 2 put restrictions only on connection matrices, not on methods A_α. As we have said already, the different families $\{A_\alpha\}$ may have the same connection matrices $D_{\alpha\delta}$.

In order to give to the condition 2) in last two theorems more constructive form we need the Quotient Theorems of H. Baumann ([1], Theorem 1) and J. Boos ([2], Theorem 4 and [3], Theorem) for matrix methods $A = (a_{nk})$ and $B = (b_{nk})$. The elements of matrices belong to K everywhere[2].

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[2]Lemmas 1 and 2 and Theorems 2-4 as well are generalized for summability in locally convex spaces [16].

Lemma 1. ([1], Theorem 1.) *Let A and B be regular matrix methods. Then the following statements are equivalent.*

1) *$cB \supset m \cap cA$.*

2) *For every $\varepsilon > 0$ there exists a row finite and column finite regular matrix method $Q_\varepsilon = (q_{nk}^\varepsilon)$ and a matrix $R_\varepsilon = (r_{nk}^\varepsilon)$ satisfying*

$$B = Q_\varepsilon A + R_\varepsilon \tag{6}$$

and

$$\limsup_n \sum_k |r_{nk}^\varepsilon| < \varepsilon. \tag{7}$$

Lemma 2. ([2], Theorem 4 and [3], Theorem.) *Let A and B be regular matrix methods and let A be normal. Then the following statements are equivalent.*

1) *$cB \supset m \cap cA$.*

2) *For every $\varepsilon > 0$ there exist matrices Q_ε and R_ε satisfying* (6), *$\sup\limits_n \sum\limits_k |q_{nk}^\varepsilon| < \infty$ and*

$$\sup_n \sum_k |r_{nk}^\varepsilon| < \varepsilon. \tag{8}$$

3) *Statement* 2) *from Lemma* 1 *is fulfilled with* (8) *instead of* (7).

4) *For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying*

$$\sup_n \Big| \sum_k b_{nk}\xi_k \Big| \le \varepsilon \cdot \left(\frac{1}{\delta} \cdot \sup_n \Big| \sum_k a_{nk}\xi_k \Big| + \sup_n \mid \xi_n \mid \right)$$

for each finite sequence $x = (\xi_n)$.

We give now some additional remarks to Theorems 2 and 3.

Remarks. Let the condition 1) be fulfilled for every $\alpha > \alpha_0$ and $0 < \delta < 1$. Applying Lemmas 1 and 2 to regular matrix methods $A = D_{\alpha 1}$ and $B = D_{\alpha\delta}$ we get the following statements (each of what is) equivalent to 2) (see also [15], Theorems 1.5-1.7 and [16], Theorem 4).

2a) $cD_{\alpha\delta} \supset m \cap cD_{\alpha 1}$.

2b) For every $\varepsilon > 0$ there exists a row finite and column finite regular matrix $Q_{\alpha\delta\varepsilon} = (q_{nk}^{\alpha\delta\varepsilon})$ and a matrix $R_{\alpha\delta\varepsilon} = (r_{nk}^{\alpha\delta\varepsilon})$ satisfying

$$D_{\alpha\delta} = Q_{\alpha\delta\varepsilon} D_{\alpha 1} + R_{\alpha\delta\varepsilon} \tag{9}$$

and

$$\limsup_n \sum_k |r_{nk}^{\alpha\delta\varepsilon}| < \varepsilon. \tag{10}$$

2c) For every $\varepsilon > 0$ there exists a row finite $c_0 \to c_0$ matrix $Q_{\alpha\delta\varepsilon}$ and a matrix $R_{\alpha\delta\varepsilon}$ satisfying (9) and (10).

2d) For every $\varepsilon > 0$ there exists a $c_0 \to c_0$ matrix $Q_{\alpha\delta\varepsilon}$ and a matrix $R_{\alpha\delta\varepsilon}$ satisfying (10) and

$$\Big|\sum_k d_{nk}^{\alpha\delta}\xi_k\Big| \le \Big|\sum_k q_{nk}^{\alpha\delta\varepsilon}\sum_\nu d_{k\nu}^{\alpha 1}\xi_\nu\Big| + \Big|\sum_k r_{nk}^{\alpha\delta\varepsilon}\xi_k\Big|$$

for each $n = 0, 1, 2, \dots$ and $x \in m \cap c_0 D_{\alpha 1}$.

2e) $c_0 D_{\alpha\delta} \supset m \cap c_0 D_{\alpha 1}$.

If in addition the matrices $D_{\alpha\delta}$ $(\alpha > \alpha_0, 0 < \delta < 1)$ are normal (as it already is in Theorem 2) then we have further statements being equivalent to 2).

2f) For every $\varepsilon > 0$ there exist matrices $Q_{\alpha\delta\varepsilon}$ and $R_{\alpha\delta\varepsilon}$ satisfying the conditions (9), $\sup_n \sum_k |q_{nk}^{\alpha\delta\varepsilon}| < \infty$ and

$$\sup_n \sum_k |r_{nk}^{\alpha\delta\varepsilon}| < \varepsilon. \tag{11}$$

2g) Statement 2b) is fulfilled with (11) instead of (10).

2h) Statement 2c) is fulfilled with (11) instead of (10).

2i) Statement 2d) is fulfilled with (11) instead of (10).

2j) For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$\sup_n\Big|\sum_k d_{nk}^{\alpha\delta}\xi_k\Big| \le \varepsilon \cdot \left(\frac{1}{\delta} \cdot \sup_n\Big|\sum_k d_{nk}^{\alpha 1}\xi_k\Big| + \sup_n |\ \xi_n\ |\right)$$

for each finite sequence $x = (\xi_n)$.

The analogues of Theorems 2 and 3 hold for 0-convexity. Let us formulate the analogue of Theorem 3 (see [16], Theorem 5).

Theorem 4. *Let the summability methods A_α and $A_{\alpha+\delta}$ for every $\alpha > \alpha_0$ and $0 < \delta < 1$ be connected by the row finite matrix $D_{\alpha\delta}$. If $D_{\alpha\delta}$ is a $c_0 \to c_0$ matrix and satisfies the condition* 2) *of Theorem* 2 *with c_0 instead of c for every $\alpha > \alpha_0$ and $0 < \delta < 1$, then the family $\{A_\alpha\}$ is 0-convex.*

Remark. The condition 2) is satisfied if the statement 2d) from Additional Remark to Theorems 2 and 3 is fulfilled.

4. A Method for Studying Certain Families on Convexity

Let every two methods A_α and $A_{\alpha+\delta}$ ($\alpha > \alpha_0,\ \delta > 0$) from a family $\{A_\alpha\}$ be connected with the relation

$$A_{\alpha+\delta}x = D_{\alpha\delta}(A_\alpha x) \quad (x \in \omega A_\alpha)$$

where $D_{\alpha\delta} = (d_{nk}^{\alpha\delta})$ is the matrix with

$$d_{nk}^{\alpha\delta} = b_k^\alpha c_{nk}^{\alpha\delta} \diagup b_n^{\alpha+\delta}, \tag{12}$$

$c_{nk}^{\alpha\delta} \in K$, $c_{nk}^{\alpha\delta} = 0$ for $k > n$, $c_{nk}^{\alpha 1} = c_n^\alpha \neq 0$ for $k \leq n$ and $0 \neq b_n^\alpha \in K$ $(n = 0, 1, 2, \dots)$.

Further we will construct for every matrix $D_{\alpha\delta}$ with $\alpha > \alpha_0$ and $0 < \delta < 1$ the quotient representation

$$D_{\alpha\delta} = Q_{\alpha\delta\kappa} D_{\alpha 1} + R_{\alpha\delta\kappa} \tag{13}$$

where κ is any number from interval $\left(\frac{1}{2}, 1\right)$ and $Q_{\alpha\delta\kappa}$ and $R_{\alpha\delta\kappa}$ are certain matrices (depending on α, δ and κ). We need the representation (13) to get effective sufficient conditions for the convexity of family $\{A_\alpha\}$ with the help of general convexity theorems 2-4.

Let us fix any $\alpha > \alpha_0$, $0 < \delta < 1$ and sequence $y = (\eta_n)$ in K and denote $D_{\alpha\delta}y$ by $(\mu_n^{\alpha\delta})$. Let us fix also $\frac{1}{2} < \kappa < 1$ and denote by $N = [\kappa n]$ the integer part of κn. Now we have

$$\mu_n^{\alpha\delta} = \frac{1}{b_n^{\alpha+\delta}} \sum_{k=0}^{N} c_{nk}^{\alpha\delta} b_k^\alpha \eta_k + \frac{1}{b_n^{\alpha+\delta}} \sum_{k=N+1}^{n} c_{nk}^{\alpha\delta} b_k^\alpha \eta_k.$$

Transforming the first sum by the Abelian transformation we get for the sequence $D_{\alpha\delta}y = (\mu_n^{\alpha\delta})$ the representation[3]

$$\begin{aligned} \mu_n^{\alpha\delta} = {} & \sum_{k=0}^{N-1} \Delta_k c_{nk}^{\alpha\delta} b_k^{\alpha+1} \mu_k^{\alpha 1} \diagup c_k^\alpha b_n^{\alpha+\delta} + c_{nN}^{\alpha\delta} b_N^{\alpha+1} \mu_N^{\alpha 1} \diagup c_N^\alpha b_n^{\alpha+\delta} \\ & + \sum_{k=N+1}^{n} c_{nk}^{\alpha\delta} b_k^\alpha \eta_k \diagup b_n^{\alpha+\delta} \quad (n = 0, 1, 2, \dots). \end{aligned} \tag{14}$$

[3] $\Delta_k c_{nk}^{\alpha\delta} = c_{nk}^{\alpha\delta} - c_{n,k+1}^{\alpha\delta}$.

Denoting by $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$ and $R_{\alpha\delta\kappa} = (r_{nk}^{\alpha\delta\kappa})$ the matrices defined by

$$q_{nk}^{\alpha\delta\kappa} = \begin{cases} \Delta_k c_{nk}^{\alpha\delta} b_k^{\alpha+1} \big/ c_k^{\alpha} b_n^{\alpha+\delta} & \text{if} \quad k < N, \\ c_{nN}^{\alpha\delta} b_N^{\alpha+1} \big/ c_N^{\alpha} b_n^{\alpha+\delta} & \text{if} \quad k = N, \\ 0 & \text{if} \quad k > N \end{cases} \tag{15}$$

and

$$r_{nk}^{\alpha\delta\kappa} = \begin{cases} 0 & \text{if} \quad k \le N, \\ c_{nk}^{\alpha\delta} b_k^{\alpha} \big/ b_n^{\alpha+\delta} & \text{if} \quad k > N, \end{cases} \tag{16}$$

we can present the equality (14) in the following form:

$$D_{\alpha\delta} y = Q_{\alpha\delta\kappa}(D_{\alpha 1} y) + R_{\alpha\delta\kappa} y.$$

Thus we have constructed for the matrix $D_{\alpha\delta}$ the quotient representation (13) where the matrices $Q_{\alpha\delta\kappa}$ and $R_{\alpha\delta\kappa}$ are defined by (15) and (16), respectively. The next theorem follows now immediately from Theorems 4 and 3 with the help of statement 2c) from additional remark to them (see [15], Theorems 2.1 and 2.2).

Theorem 5. *Let the methods A_α and $A_{\alpha+\delta}$ be connected by the relation $A_{\alpha+\delta} x = D_{\alpha\delta}(A_\alpha x)$ for each $\alpha > \alpha_0$ and $0 < \delta < 1$ and $x \in \omega A_\alpha$ where the matrix $D_{\alpha\delta}$ is defined by* (12).

Suppose the following conditions hold for every $\alpha > \alpha_0, 0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$.

1) *The matrix $D_{\alpha\delta}$ is a $c_0 \to c_0$ matrix.*
2) *The matrix $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$ defined by* (15) *is a $c_0 \to c_0$ matrix.*
3) *The matrix $R_{\alpha\delta\kappa} = (r_{nk}^{\alpha\delta\kappa})$ defined by* (16) *satisfies the condition*

$$\sum_{k=N+1}^{n} |r_{nk}^{\alpha\delta\kappa}| \le \varphi_{\alpha\delta}(\kappa)$$

where $N = [\kappa n]$ and $\lim\limits_{k \to 1-} \varphi_{\alpha\delta}(\kappa) = 0$.

Then the family $\{A_\alpha\}$ is 0-convex.

If in addition the method $D_{\alpha\delta}$ $(\alpha > \alpha_0,\ 0 < \delta < 1)$ is regular then the family $\{A_\alpha\}$ is convex and the methods A_α are pairwise consistent.

Remark. In particular if $c_{nk}^{\alpha\delta} = A_{n-k}^{\delta-1}$ $(k \le n)$ are the Cesàro numbers in (12) and the sequences (b_n^α) and $(b_n^{\alpha+\delta})$ satisfy for each $\alpha > \alpha_0$, $0 < \delta < 1$ the conditions

$$|b_n^\alpha \big/ b_{n+k}^\alpha| \le N_\alpha \quad (n, k = 0, 1, 2, \dots) \tag{17}$$

and

$$K_{\alpha\delta} n^{\delta} \leq |b_n^{\alpha+\delta} \diagup b_n^{\alpha}| \leq L_{\alpha\delta} n^{\delta} \quad (n = 1, 2, \dots) \tag{18}$$

then the conditions 1)-3) are fulfilled for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$. (N_α, $K_{\alpha\delta}$ and $L_{\alpha\delta}$ are real constants depending only on α or only on α and δ. See [14], Theorem 2.A and also [10], Theorem 4.)

The idea of the method described above for constructing the quotient representation (13) and studying the families on convexity was used by the author first in [9] and further also in papers [10,11,14,15]. We note that this method can be extended to the summability methods connected with the integral analogue of the transformation (12) (see [9,10]) and to certain methods of strong summability [11,13,14].

5. Convex Families of Generalized Nörlund Methods

We state that the sequence $(A_n^{\alpha\beta})$ is formally defined by the power series

$$f_{\alpha\beta}(\chi) = (1-\chi)^{-(\alpha+1)} \left(\log \frac{e}{1-\chi} \right)^{\beta} = \sum_{n=0}^{\infty} A_n^{\alpha\beta} \chi^n,$$

where $\alpha, \beta, \chi \in \mathbb{R}$ and e is the base of Naperian logarithm. In particular, if $\beta = 0$ then $A_n^{\alpha,0} = A_n^{\alpha}$ are the Cesàro numbers.

Let us consider the *generalized Nörlund summability methods* (see [14]) $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n) = (a_{nk}^{\alpha})$ where

$$a_{nk}^{\alpha} = \begin{cases} p_{n-k}^{\alpha\beta_0} q_k \diagup R_n^{\alpha\beta_0} & \text{if} \quad k \leq n, \\ 0 & \text{if} \quad k > n, \end{cases}$$

$R_n^{\alpha\beta_0} = \sum\limits_{k=0}^{n} p_{n-k}^{\alpha\beta_0} q_k$, $p_n^{\alpha\beta_0} = \sum\limits_{k=0}^{n} A_{n-k}^{\alpha-1,\beta_0} p_k$, $\beta_0 \in \mathbb{R}$, $p_n, q_n \in K$ and $R_n^{\alpha\beta_0} \neq 0$ $(n = 0, 1, 2, \dots)$.

Remark. We notice, in particular, if $\beta_0 = 0$ then the methods $(N, p_n^{\alpha\beta_0}, q_n)$ become the generalized Nörlund methods (N, p_n^{α}, q_n) (see [8,14]). If besides the condition $\beta_0 = 0$ there is $q_n = 1$ $(n = 0, 1, 2, \dots)$ then we get the Nörlund methods (N, p_n^{α}) (see [6,14]). In particular, if $q_n = A_n^{\gamma_0\sigma_0}$ (γ_0 and σ_0 are fixed real numbers) and $p_n = A_n^{-1,0} = A_n^{-1}$ then the methods $(N, p_n^{\alpha\beta_0}, q_n)$ become the quasi-Cesàro methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ (see [7]). If $\beta_0 = \sigma_0 = 0$ then the methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ become the generalized Cesàro methods $(C, \alpha, \gamma_0,)$; if we add to the previous conditions the presumption $\gamma_0 = 0$ then we get the Cesàro methods (C, α).

The methods $A_{\alpha+\delta} = (N, p_n^{\alpha+\delta,\beta_0}, q_n)$ and $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ for every $\alpha > \alpha_0$ and $\delta > 0$ are connected by the relation $A_{\alpha+\delta}x = D_{\alpha\delta}(A_\alpha x)$ where $D_{\alpha\delta} = (d_{nk}^{\alpha\delta})$ are matrices with

$$d_{nk}^{\alpha\delta} = \begin{cases} A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha+\delta,\beta_0} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n \end{cases} \tag{19}$$

(see [14]). The matrix $D_{\alpha\delta}$ defined by (19) satisfies the relation (12) with $c_{nk}^{\alpha\delta} = A_{n-k}^{\delta-1}$ $(k \le n)$ and $b_n^\alpha = R_n^{\alpha\beta_0}$. Thus the next result can be obtained as a direct application of Theorem 5 (see [15], Theorem 3.2).

Theorem 6. *Suppose the following conditions hold.*

1) $|R_n^{\alpha\beta_0} / R_{n+k}^{\alpha\beta_0}| \le N_\alpha \quad (n, k = 0, 1, 2, \dots)$ *for every* $\alpha > \alpha_0 + 1$.

2) $K_{\alpha\delta} n^\delta \le |R_n^{\alpha+\delta,\beta_0} / R_n^{\alpha\beta_0}| \le L_{\alpha\delta} n^\delta \ (n = 1, 2, \dots)$ *for every* $\alpha > \alpha_0$ *and* $0 < \delta < 1$.

3) *The matrix method* $D_{\alpha\delta}$ *defined by* (19) *is regular for every* $\alpha > \alpha_0$ *and* $0 < \delta < 1$.

Then the family $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ *is convex for* $\alpha > \alpha_0$ *and the methods* A_α *are pairwise consistent.* ($N_\alpha, K_{\alpha\delta}$ *and* $L_{\alpha\delta}$ *are real constants depending only on* α *or only on* α *and* δ.

Remark. If in addition the condition 1) is fulfilled for every $\alpha > \alpha_0$ (instead of $\alpha > \alpha_0 + 1$), then the right inequality in condition 2) and the condition 3) are satisfied (see [14], Theorem 2.1).

Corollary. ([15], Corollary 3.2.)

a) *If* $p_0 > 0, \ p_n \ge 0, q_n > 0$ *for every* $n = 0, 1, 2, \dots$ *and the condition* 2) *of Theorem 6 holds for* $\beta_0 = 0$ *and* $\alpha_0 = 0$ *then the family* $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n) = (N, p_n^\alpha, q_n)$ *is convex for* $\alpha > 0$ (see [8]).

b) *If* $p_0 > 0, \ p_n \ge 0, \ q_n = 1$ *for every* $n = 0, 1, 2, \dots$ *and the condition* 2) *of Theorem 6 holds for* $\beta_0 = 0$ *and* $\alpha_0 = -1$ *then the family* $A_\alpha = (N, p_n^\alpha)$ *is convex for* $\alpha > -1$ (see [6]).

c) *The family* $A_\alpha = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ *is convex for* $\alpha > -\gamma_0 - 1$ (see [14]). *In particular, the family* $A_\alpha = (C, \alpha, \gamma_0)$ *is convex for* $\alpha > -\gamma_0 - 1$ *and the family* $A_\alpha = (C, \alpha)$ *is convex for* $\alpha > -1$.

Theorem 7. *Let* $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ $(\alpha > \alpha_0)$, (b'^{α}_n) *be the sequences satisfying for every* $\alpha > \alpha_0$, $0 < \delta < 1$ *the conditions* (17), (18) *and* $\lambda_n^\alpha = |R_n^{\alpha\beta_0} / b'^{\alpha}_n| \uparrow \infty$. *Then the implications* (3)-(5) *hold with* m *and* c^* *instead of* m_0 *and* c_0^*, *respectively.*

The last theorem is a modification of Theorem 2.3 from [14]. To infer this result from Theorem 1 as a conclusion it is sufficient to see that the family $\{B_\alpha\}$ defined in it is 0-convex here. Indeed, $\{B_\alpha\}$ has the connection matrices (12) with $c_{nk}^{\alpha\delta} = A_{n-k}^{\delta-1}$ $(k \leq n)$ and $b_n^\alpha = R_n^{\alpha\beta_0} |b'^{\alpha}_n / R_n^{\alpha\beta_0}|$ which satisfy (17) and (18). Thus $\{B_\alpha\}$ is 0-convex by Theorem 5. We note that (17) and (18) involve the comparative estimates for speeds λ_α.

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