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GENERALIZED GOURSAT PROBLEM AND THE QUASIASYMPTOTICS OF A SOLUTION

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. A generalized solution to a nonlinear Goursat problem when its nonlinear term has unbounded gradient is given by using a special regularization. The result is related to the known ones. The quasiasymptotics of a generalized solution to a nonlinear Goursat problem is analyzed.

1. Introduction

The aim of this paper is to present a new method of solving a non-linear Goursat problem with assumptions which do not allow their solving in the frame of classical spaces or in the space of Schwartz distributions. The paper is closely connected to results of [6] which will be also presented.

Goursat problem

$$\begin{aligned} \partial_x\partial_y U(x,y) &= F(x,y,U(x,y),\partial_x U(x,y),\partial_y U(x,y)),\\ U(x,0) &= \Phi(x),\\ U_{|\Gamma} &= \Theta_{|\Gamma}, \end{aligned}$$

has its origin in the mixed quasilinear hyperbolic problem of order two in two dimensions with data on $y=0$ and on a curve Γ. It has been studied from different aspects in many papers (cf. [4], [3]). In [6] are considered generalized solutions to the following special case of Goursat problem

$$\begin{aligned} \partial_x\partial_y U(x,y) &= F(\cdot,U(x,y)),\\ U_{|\{y=0\}} &= \Phi(x),\\ U_{|\{x=0\}} &= \Psi(y), \end{aligned} \tag{1}$$

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where Φ and Ψ are generalized functions of one variable (for instance, distributions), F is a given function nonlinear on its arguments, Γ is the line $\{x = 0\}$. When the data are singular, this problem is not possible to be considered in $\mathcal{D}'(\mathbb{R}^2)$. The nonlinearity of F with respect to U increases this impossibility. Because of that this problem is considered in the space of generalized functions which is a good frame for solving nonlinear equations with singular initial data. This space allows the multiplication of generalized functions and the association of generalized functions to Schwartz distributions. But, the transfer from a generalized function to a distribution which is its image or which is associated with it implies the loss of information about the topological structure of $\mathcal{D}'$. The restriction of an element on a manifold in the space of generalized functions is simply defined while for distributions it is not so. Moreover, the operation of restriction in $\mathcal{D}'(\mathbb{R}^2)$ requires additional assumptions on the regularity which is incompatible with the problem when the data are for example, the Dirac measure or its derivatives. For problem (1) in [6] is proposed a special algebraic construction: triplet of Goursat, which consists of a ring and two algebras which are defined by properties connected with the structure of the problem. A nature of the problem explains impossibility of solving it in $\mathcal{D}'(\mathbb{R}^2)$ and confirms this approach, since this algebra admits a generalized solution. Goursat algebra contains many classical Algebras of generalized functions (cf. [1], [2]).

It is proved in [6] that in a case of triplet of Goursat there exists a generalized solution to problem (1) with a help of method of successive approximations. By suitable assumptions, the impossibility of solving this problem in $\mathcal{D}'(\mathbb{R}^2)$ is proved which confirms the generalized function approach.

Seeking a solution to (1) in the algebra of generalized function begins with considering classes of regular problems which impose additional hypotheses on a function F :

$$\begin{aligned} &F \in C^\infty(\mathbb{R}^3, \mathbb{R}), \\ &\forall K \subset\subset \mathbb{R}^2, \quad \kappa = 1 + \sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |\partial_z F(x, y, z)| < +\infty. \end{aligned} \tag{2}$$

In this paper we relax this condition by a using special regularization adopting for this problem, in Colombeau algebra. We substitute the therm F with the one F_h which has a bounded partial derivative with respect to z, in a given domain. We prove the coherence between the solutions of so-called regularized system and those considered in [6] (when condition (2) is fulfilled). We consider the quasiasymptotics of such regularized Goursat problem and prove that it depends on the quasiasymptotics of the initial data.

Paper is organized in the following way. In the second section we give the description and the basic properties of the space of generalized functions and the triplet of Goursat by following [6]. In section 3 we adopt the special regularization from [7] for such a kind of problem. Section 4 is supplied by the quasiasymptotics for Goursat problem.

2. Basic Spaces

We shall recall from [6] the notions concerning Colombeau algebra of generalized functions and its generalization, Goursat algebra, which corresponds to the nature of the problem (1). Note, Egorov algebra, Colombeau and Delcroix one are the special cases of Goursat algebra (cf. [6]). Egorov algebra has not restriction on the growth, in Colombeau case polynomial type restrictions appear, while elements of Delcroix algebra have exponential growth (cf. [6]). We restrict our consideration to Colombeau algebra.

Simplified version of Colombeau algebra. We define a subalgebra $\mathcal{X}_M(\mathbb{R}^d)$, of $\mathcal{X}(\mathbb{R}^d)$ which consists of families $(f_\varepsilon)_\varepsilon$, of smooth functions on $\mathbb{R}^d$, by setting:

$$\mathcal{X}_M(\mathbb{R}^d) = \begin{cases} (f_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^d);\ (\forall\ K \subset\subset \mathbb{R}^d)\ (\forall \alpha \in \mathbb{N}^d)\ (\exists p \in \bar{\mathbb{R}}^+), \\ (\exists \mu \in (0,1])\ (\forall \varepsilon \in (0,\mu))\ (\sup\limits_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq \varepsilon^{-p}), \end{cases}$$

where $\bar{\mathbb{R}}^+ = [0, -\infty)$. An ideal $\mathcal{I}_C(\mathbb{R}^d)$ for $\mathcal{X}_M(\mathbb{R}^d)$ is given by

$$\mathcal{I}_C(\mathbb{R}^d) = \begin{cases} (f_\varepsilon)_\varepsilon \in \mathcal{X}_M(\mathbb{R}^d);\ (\forall\ K \subset\subset \mathbb{R}^d)\ (\forall \alpha \in \mathbb{N}^d)\ (\forall q \in \bar{\mathbb{R}}^+), \\ (\exists \mu \in (0,1])\ (\forall \varepsilon \in (0,\mu))\ (\sup\limits_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq \varepsilon^{q}). \end{cases}$$

The simplified version of Colombeau algebra is given by the quotient $\mathcal{A}_C(\mathbb{R}^d) = \mathcal{X}_M(\mathbb{R}^d)\ /\mathcal{I}_C(\mathbb{R}^d)$. Analogously we define a ring of Colombeau moderate generalized numbers $\mathbb{C}_C$, $\varepsilon \in (0,1]$, as a quotient $\mathbb{C}_C = \mathbb{R}^{(0,1]}/\mathcal{I}_C$, where

$$\mathbb{R}_M^{(0,1]} = \{(m_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}; (\exists p \in \bar{\mathbb{R}}^+)(\exists \mu \in (0,1])(\forall \varepsilon \in (0,\mu))(|m_\varepsilon| \leq \varepsilon^{-p})\},$$
$$\mathcal{I}_C = \{(m_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}; (\forall q \in \bar{\mathbb{R}}^+)(\exists \mu \in (0,1])(\forall \varepsilon \in (0,\mu))(|m_\varepsilon| \leq \varepsilon^{q})\}.$$

Recall, a generalized constant $A \in \mathbb{C}_C$ is associated to a constant $a \in \mathbb{C}$, $A \approx a$, if it has a representative A_ε such that $\lim_{\varepsilon\to 0} A_\varepsilon = a$. Generalized functions G and H are associated in $\mathcal{A}_C(\Omega)$, where Ω is an open set in $\mathbb{R}^d$, $G \approx H$, if $< G - H, \psi > \approx 0$ for every $\psi \in C_0^\infty(\Omega)$. Elements $G, H \in \mathcal{A}_C(\Omega)$

are L^∞-associated on Ω if for every $K \subset\subset \Omega$ $\sup_{x\in K}|G_\varepsilon(x) - H_\varepsilon(x)| \to 0$ as $\varepsilon \to 0$. L^∞-association implies the ordinary association.

Algebraic structure adopted to Goursat problem. Let A be a subring of the ring $\mathbb{R}^{(0,1]}$ and I be an ideal of A. We say that a subring A in $\mathbb{R}^{(0,1]}$ is *stable for mayorizing* if for all family $(s_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}$ there exists $(r_\varepsilon)_\varepsilon \in A$ such that for all ε, $|s_\varepsilon| < r_\varepsilon$ implies $(s_\varepsilon)_\varepsilon \in A$. We assume that the sets A and I are stable for mayorizing. Then, $\mathcal{H}_{d,A}$ and $\mathcal{I}_{d,I}$ be an algebra and its ideal is defined by

$$\mathcal{H}_{d,A} = \{(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^d); (\forall K \subset\subset \mathbb{R}^d)(\forall \alpha \in \mathbb{N}^d)(||\partial^\alpha u_\varepsilon||_{L^\infty(K)})_\varepsilon \in A\},$$
$$\mathcal{I}_{d,I} = \{(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^d); (\forall K \subset\subset \mathbb{R}^d)(\forall \alpha \in \mathbb{N}^d)(||\partial^\alpha u_\varepsilon||_{L^\infty(K)})_\varepsilon \in I\}.$$

Note, $\mathcal{H}_{d,A}$ is a subalgebra of $\mathcal{X}(\mathbb{R}^d)$. The corresponding algebra is $\mathcal{A}_G = \mathcal{H}_{d,A}/\mathcal{I}_{d,I}$. The ring of generalized constants which corresponds to $\mathcal{A}_G$ is $\mathbb{C}_G = A/I$.

The algebra $\mathcal{A}_G(\mathbb{R}^d)$ is called the *Goursat algebra.* A ring of generalized constants is $\mathbb{C}_G = A/I$.

Since $\mathcal{H}_{d,\mathbb{R}_M^{(0,1]}} = \mathcal{X}_M(\mathbb{R}^d)$, $\mathcal{I}_{d,\mathcal{I}_C} = \mathcal{I}_C(\mathbb{R}^d)$, Colombeau algebra is a special case of Goursat algebra.

Let $F : (x,z) \to F(x,z)$ is C^∞ mapping from $\mathbb{R}^d \times \mathbb{R}$ to $\mathbb{R}$. Then, it is said that $\mathcal{A}_G(\mathbb{R}^d)$ is stable for F if for a class $u = [u_\varepsilon] \in \mathcal{A}_G(\mathbb{R}^d)$, $F(\cdot, U) = [x \to F(x, u_\varepsilon(x))] \in \mathcal{A}_G(\mathbb{R}^d)$.

Recall (cf. [6]),

$$\mathcal{F}_C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}) = \begin{cases} f \in C^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}); (\forall K \subset\subset \mathbb{R}^d), \\ (\forall \beta \in \mathbb{N}^{d+1})(\exists C > 0)(\exists m \in \mathbb{N}), \\ (\forall (x,z) \in K \times \mathbb{R})(|\partial^\beta f(x,z)|) \le C(1+|z|)^m). \end{cases}$$

We will assume in this paper that $F \in \mathcal{F}_C$ such that $\mathcal{A}_G(\mathbb{R}^2)$ is stable for F..

Triplet of Goursat for problem (1) is $(\mathbb{C}_G, \mathcal{A}_G(\mathbb{R}), \mathcal{A}_G(\mathbb{R}^2))$.

Let $\mathcal{K}$ be a set of positive measurable functions defined on $(0,1)$ with the property

$$A^{-1}\epsilon^p \le c(\epsilon) \le A\epsilon^{-p},\ \epsilon \in (0,1),$$

for some $A > 0$ and $p > 0$.

In the sequel ω will denote an open set in $\mathbb{R}^d$ which contains 0.

Definition 1. Let $B \in \mathcal{G}(\omega)$. It is said that B has the quasiasymptotics at zero with respect to $c(\epsilon) \in \mathcal{K}$ if there is B_ϵ, a representative of B, such that for every $\psi \in \mathcal{D}(\omega)$ and some $s > 0$ there is $C_{\psi,s} \in \mathbb{C}$ such that

$$\lim_{\epsilon \to 0} \left\langle \frac{B_\epsilon(\epsilon s x)}{c(\epsilon)}, \ \psi(x) \right\rangle = C_{\psi,s} \tag{3}$$

and $C_{\psi,s} \neq 0$ for some ψ and s.

It follows from (3) that this limit exists for every $s > 0$ and that for every $s > 0$ there exists ψ such that $C_{\psi,s} \neq 0$. Note, in general, $B_{s\epsilon}$, $s \neq 1$, is not a representative of B.

It is proved in ([8]) that there exists $g \in \mathcal{D}'(\omega)$ such that for $s = 1$

$$C_{\psi,1} = C_\psi = \langle g, \psi \rangle\,, \ \psi \in \mathcal{D}(\omega).$$

Throughout the paper we denote by C a generic constant.

3. Regularized System

In order to relax condition (2) we adopt the method from [7] given for nonlinear Volterra system of integral equations with a non-Lipschitz nonlinearity for this problem in Colombeau algebra. We shall consider a case when $\partial_z F$ is not bounded by using special regularization for F.

Fix a decreasing function $h : (0,1) \to (0,\infty)$ such that

$$h(\varepsilon) = \mathcal{O}(|\log \varepsilon|^{1/2}), \ \text{as } \varepsilon \to 0.$$

Let $B_i = \{(x,y); |x| \leq i, |y| \leq i\}$. We denote by ε_i a decreasing sequence, $\varepsilon_i \in (0,1)$ such that $h(\varepsilon_i) = i$, $i \in \mathbb{N}$. We have $h(\varepsilon) \geq i$ for $\varepsilon < \varepsilon_i$. Then, we put

$$\bar{F}(x,y,z) = \begin{cases} F(x,y,z), & \text{if } \ (x,y) \in B_i, \ z \in \mathbb{R}, \ \text{ and } |F(x,y,z)| \leq i, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_{h(\varepsilon)}(x,y,z) = \bar{F} * \phi_{h(\varepsilon)^{-1}}(x,y,z), \ (x,y,z) \in \mathbb{R}^3, \tag{4}$$

for $\varepsilon \in [\varepsilon_{i+1}, \varepsilon_i)$, $i \in \mathbb{N}$, where $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \phi(t)dt = 1$, $\operatorname{diam}(\operatorname{supp}\phi) = 1$. Let $(x,y,z) \in \mathbb{R}^3$ such that $(x,y) \in B_{i0}$. Then, we have

$$\begin{aligned}
&\Big|\frac{\partial F_{h(\varepsilon)}(x,y,z)}{\partial z_j}\Big| \\
=&\Big|\frac{\partial}{\partial z_j}\int_{\mathbb{R}^3} \bar{F}(\xi,\tau,\mu)\phi_{h(\varepsilon)^{-1}}(x-\xi, y-\tau, z-\mu)d\xi d\tau d\mu\Big| \\
\leq& h(\varepsilon)\int_{\mathbb{R}^3} \bar{F}(x - \frac{1}{h(\varepsilon)}u, y - \frac{1}{h(\varepsilon)}v, z - \frac{1}{h(\varepsilon)}w)\frac{\partial}{\partial z}\phi(u,v,w)dudvdw \\
\leq& Ci \max_{\substack{(u,v)\in B_i \\ w\in\mathbb{R}}} |\bar{F}(u,v,w)| \leq Ch(\varepsilon)^2,\ \varepsilon \in (0,1).
\end{aligned}$$

Thus,

$$\sup_{\mathbb{R}^3} |F_{h(\varepsilon)}|,\ |\partial_z F_{h(\varepsilon)}| < C|\log\varepsilon|,\ \varepsilon \in (0,1). \tag{5}$$

Let Φ, $\Psi \in \mathcal{A}_G(\mathbb{R})$. The following is so called the regularized system

$$\begin{aligned}
\partial_x\partial_y U_\varepsilon(x,y) &= F_{h(\varepsilon)}(x,y,U_\varepsilon(x,y)), \\
U_{\varepsilon|\{y=0\}} &= \Phi_\varepsilon(x), \\
U_{\varepsilon|\{x=0\}} &= \Psi_\varepsilon(y),
\end{aligned} \tag{6}$$

or in the form of classes

$$\begin{aligned}
\partial_x\partial_y [U_\varepsilon(x,y)] &= [F_{h(\varepsilon)}(x,y,[U_\varepsilon(x,y)])], \\
[U_{\varepsilon|\{y=0\}}] &= \phi(x), \\
[U_{\varepsilon|\{x=0\}}] &= \psi(y).
\end{aligned} \tag{7}$$

A class $U = [U_\varepsilon]$ is the solution to (7), it means that there exist $(\eta_\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ which belong to corresponding ideals such that

$$\begin{aligned}
\partial_x\partial_y U_\varepsilon(x,y) &= F_{h(\varepsilon)}(x,y,U_\varepsilon(x,y)) + \eta_\varepsilon(x,y), \\
U_\varepsilon(x,0) &= \varphi_\varepsilon(x) + r_\varepsilon(x), \\
U_\varepsilon(0,y) &= \phi_\varepsilon(y) + s_\varepsilon(y),
\end{aligned} \tag{8}$$

or in integral form,

$$\begin{aligned}
U_\varepsilon(x,y) = U_{0\varepsilon}(x,y) &+ \int_0^x\int_0^y F_{h(\varepsilon)}(\xi,\eta,U_\varepsilon(\xi,\eta))d\xi d\eta \\
&+ (r_\varepsilon(x) + s_\varepsilon(y) - r_\varepsilon(0) + \int_0^x\int_0^y \eta_\varepsilon(\xi,\eta)d\xi d\eta),
\end{aligned}$$

where $U_{0\varepsilon}(x,y) = \phi_\varepsilon(x) + \psi_\varepsilon(y) - \phi_\varepsilon(0)$.

The unicity means that the solutions to (8) $U_{1\varepsilon}$ and $U_{2\varepsilon}$ (with corresponding $(\eta_\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$) satisfy $U_{1\varepsilon} - U_{2\varepsilon} \in \mathcal{I}_C(\mathbb{R}^2)$.

According to a special regularization we have instead of (2) the following constraint

$$F_{h\varepsilon} \in C^\infty(\mathbb{R}^3, \mathbb{R}),$$
$$\forall K \subset\subset \mathbb{R}^2, \sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |\partial_z F_{h(\varepsilon)}(x,y,z)| < C|\log\varepsilon|,\ \varepsilon \in (0,1).$$

Let $\kappa_{h\varepsilon} = 1 + \sup_{\substack{(x,y)\in K \\ t\in\mathbb{R}}} |\partial_z F_{h(\varepsilon)}(\xi,\eta,t)|$. For all $(\xi,\eta) \in K$ the following holds

$$|F_{h(\varepsilon)}(\xi,\eta,U_\varepsilon(\xi,\eta))| \le |F_{h(\varepsilon)}(\xi,\eta,0)| + \kappa_{h\varepsilon}|U_\varepsilon(\xi,\eta)|. \tag{9}$$

Recall,

Theorem 1. [6] *Let* $(\mathbb{C}_G, \mathcal{A}_G(\mathbb{R}), \mathcal{A}_G(\mathbb{R}^2))$ *be the Goursat triplet for the problem* (1) *where* $\mathbb{C}_G$, $\mathcal{A}_G(\mathbb{R}^d)$, $d = 1,2$ *are defined by Colombeau algebra. Assume* (2) *for* F *and let* $\Phi \in \mathcal{A}_G(\mathbb{R})$, $\Psi \in \mathcal{A}_G(\mathbb{R})$ *such that* $\Phi(0) = \Psi(0)$. *Then, problem* (1) *admits a unique solution* $[U_\varepsilon] \in \mathcal{A}_G(\mathbb{R}^2)$.

Note that $\Phi(0) = \Psi(0)$ means that the representatives $(\phi_\varepsilon)_\varepsilon$ and $(\psi_\varepsilon)_\varepsilon$, of Φ and Ψ respectively, satisfy $(\phi_\varepsilon(0) - \psi_\varepsilon(0))_\varepsilon \in \mathcal{I}_C$. We can choose (and we do it) representatives of Φ and Ψ such that $\phi_\varepsilon(0) = \psi_\varepsilon(0)$ for every $\varepsilon \in (0,1)$.

Assume that F is $C^\infty(\mathbb{R}^3, \mathbb{R})$ function and that condition (2) is not fulfilled. Then, the following Theorem holds.

Theorem 2. *Let* $(\mathbb{C}_G, \mathcal{A}_G(\mathbb{R}), \mathcal{A}_G(\mathbb{R}^2))$ *be the Goursat triplet for problem* (6) *where* $\mathbb{C}_G$, $\mathcal{A}_G(\mathbb{R}^d)$, $d = 1,2$ *are defined by Colombeau algebra. Let* $\Phi \in \mathcal{A}_G(\mathbb{R})$, $\Psi \in \mathcal{A}_G(\mathbb{R})$ *and* $(\phi_\varepsilon(0) - \psi_\varepsilon(0)) \in \mathcal{I}_C$. *Then, problem* (7) *admits a unique solution* $[U_{h\varepsilon}] \in \mathcal{A}_G(\mathbb{R}^2)$.

Proof. The proof can be obtained by following precisely the approach from [6] by using condition (5) and special regularization for F.

Theorem 3. *Let the conditions in Theorem 1 are fulfilled. Moreover, assume that for every* $K \subset\subset \mathbb{R}^2$ *there exists* $C > 0$ *such that*

1. $\sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |F(x,y,z)| \le C,$
2. $\sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |F(x+\varepsilon, y+\varepsilon, z+\varepsilon) - F(x,y,z)| = o(1),$

where $o(1)$ *means that the left side tends to zero as* $\varepsilon \to 0$. *Then, the solution* U_ε *to* (1) *(with* ϕ_ε *and* ψ_ε *instead of* Φ *and* Ψ), *and the solution* $U_{h\varepsilon}$ *to* (6) *are* L^∞*-associated on every compact set* $K \subset\subset \mathbb{R}^2$.

Proof. Let $K \subset\subset \mathbb{R}^2$ and $(x,y) \in K$. Then,

$$
\begin{aligned}
(10)\qquad & |U_{h\varepsilon}(x,y) - U_\varepsilon(x,y)| \\
&= \int_0^x \int_0^y |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) - F(\xi,\eta,U_\varepsilon(\xi,\eta))| d\xi d\eta \\
&\le \left| \int_0^x \int_0^y |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) - F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))| d\xi d\eta \right| \\
&+ \left| \int_0^x \int_0^y |F(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) - F(\xi,\eta,U_\varepsilon(\xi,\eta))| d\xi d\eta \right| \\
&= J_1 + J_2.
\end{aligned}
$$

By the mean value Theorem, there is $C > 0$ such that

$$
(11)\qquad J_2 \le C \sup_{\substack{(x,y)\in K \\ z \in \mathbb{R}}} |\frac{\partial F}{\partial z}(x,y,z)||U_{h\varepsilon}(x,y) - U_\varepsilon(x,y)|.
$$

By (4), we have

$$
\begin{aligned}
(12)\qquad & |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) - F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))| \\
&\le \int_{\mathbb{R}^3} |\bar{F}(\xi - \frac{1}{h(\varepsilon)}u, \eta - \frac{1}{h(\varepsilon)}v, U_{h\varepsilon}(\xi,\eta) - \frac{1}{h(\varepsilon)}w) \\
&-F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|\phi(u,v,w) du dv dw.
\end{aligned}
$$

Let $(u,v,w) \in \operatorname{supp}\phi$. By assumption 1., for $(\xi,\eta) \in K$ we have

$$
\begin{aligned}
&\bar{F}(\xi - \frac{1}{h(\varepsilon)}u, \eta - \frac{1}{h(\varepsilon)}v, U_{h\varepsilon}(\xi,\eta) - \frac{1}{h(\varepsilon)}w) - F(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) \\
=&F(\xi - \frac{1}{h(\varepsilon)}u, \eta - \frac{1}{h(\varepsilon)}v, U_{h\varepsilon}(\xi,\eta) - \frac{1}{h(\varepsilon)}w) - F(\xi,\eta,U_{h\varepsilon}(\xi,\eta)).
\end{aligned}
$$

Thus, by continuing (12) and by assumption 2., we have

$$
(13)\quad ||F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta)) - F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))] d\xi d\eta||_{L^\infty(K)} = p(\varepsilon) = o(1),
$$

and

$$
(14)\qquad J_1 \le C p(\varepsilon),\ \varepsilon \in (0,1) \text{ for suitable } C > 0.
$$

Now, by (10), (11) and (14)

$$
\begin{aligned}
&||U_{h\varepsilon}(x,y) - U_\varepsilon(x,y)||_{L^\infty(K)} \\
\leq &C(p(\varepsilon) + \int_0^x \int_0^y ||U_{h\varepsilon}(\xi,\eta) - U_\varepsilon(\xi,\eta)||_{L^\infty(K)} d\xi d\eta)
\end{aligned}
$$

which gives, by Gronwall inequality (cf. [5], Theorem 1.2.2)

$$
||U_{h\varepsilon}(x,y) - U_\varepsilon(x,y)||_{L^\infty(K)} \leq Cp(\varepsilon).
$$

This proves the Theorem.

By similar techniques we can prove the following Theorem.

Theorem 4. *Let F be a smooth function and Φ and Ψ be continuous on $\mathbb{R}$, such that $\Phi(0) = \Psi(0)$. Then, there is a bounded interval I around 0 such that the classical solution to Goursat problem on $I \times I$ is L^∞-associated to the solution to (6) with $U_\varepsilon|_{y=0} = \Phi_\varepsilon(x)$, $U_\varepsilon|_{x=0} = \Psi_\varepsilon(x)$.*

Note that there exist representatives $(\phi_\varepsilon)_\varepsilon$ and $(\psi_\varepsilon)_\varepsilon$ of generalized functions determined by Φ and Ψ such that $\phi_\varepsilon(0) = \psi_\varepsilon(0)$ for every $\varepsilon \in (0,1]$. Theorem 4, as well as the results of [6] imply that our method for solving (8) and that of [6] give, for $\Phi \in C(\mathbb{R})$, $\Psi \in C(\mathbb{R})$ and $F \in C^\infty(\mathbb{R}^3)$, the solutions which are L^∞-associated to the classical solution to (1) on $I \times I$, where I is a finite interval around 0.

Theorem 5. *Let $F \in C^1(\mathbb{R}^3)$ and Φ and Ψ be continuous and $\Phi(0) = \Psi(0)$.*
(a) *If for every $K \subset\subset \mathbb{R}^2$*

$$
\sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |F(x,y,z)| < \infty,
$$

then the solution to (6) is L^∞-associated to the global solution to (1).
(b) *If for every $K \subset\subset \mathbb{R}^2$*

$$
\sup_{\substack{(x,y)\in K \\ z\in\mathbb{R}}} |\frac{\partial}{\partial z} F(x,y,z)| < \infty,
$$

then the solution to (6) is L^∞-associated to the global solution to (1).

Proof. (a) Assumptions imply that the solution $U_{h\varepsilon}$ to (6) is uniformly bounded with respect to ε on every compact set $K \subset\subset \mathbb{R}^2$. Denote by $U(x,y)$

the solution to (1). Let $(x,y)\in K$. We have

$$
\begin{aligned}
(15)\qquad & |U_{h\varepsilon}(x,y)-U(x,y)| \\
\le & |U_{0\varepsilon}(x,y)-U_0(x,y)| \\
& +\int_0^x\int_0^y |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))-F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|d\xi d\eta \\
& +\int_0^x\int_0^y |F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))-F(\xi,\eta,U(\xi,\eta))|d\xi d\eta.
\end{aligned}
$$

Since $\{U_{h\varepsilon}(\xi,\eta);(\xi,\eta)\in K,\varepsilon\in(0,1)\}$ is bounded, we have

$$\sup_{(\xi,\eta)\in K}|F(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|<\infty.$$

Thus, by (12) as in proof of Theorem 3 we obtain the assertion.

(b) By the mean value Theorem applied to (15) we conclude that the assumption of (b) implies the boundedness of $\{U_{h\varepsilon}(\xi,\eta);(\xi,\eta)\in K,\varepsilon\in(0,1)\}$. Then, the proof is the same as the previous one.

4. Quasiasymptotics and Goursat Problem

Let $(U_{h\varepsilon})_\varepsilon$ be a solution to (6). Then, we have

$$(16)\quad U_{h\varepsilon}(\varepsilon sx,\varepsilon sy)=U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)+\int_0^{\varepsilon sx}\int_0^{\varepsilon sy}F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))d\xi d\eta,$$

where

$$U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)=\phi_\varepsilon(\varepsilon sx)+\psi_\varepsilon(\varepsilon sy)-\phi_\varepsilon(0).$$

We shall prove the following Theorem.

Theorem 6. *Let $c(\epsilon)\in\mathcal{K}$, $\lim_{\varepsilon\to 0}\dfrac{\varepsilon^2|\log\varepsilon|}{c(\varepsilon)}=0$ and $\lim\limits_{\epsilon\to 0}\dfrac{U_{0\epsilon}}{c(\epsilon)}$ exists in $\mathcal{D}'(\omega)$ for some $s>0$, where ω is open in $\mathbb{R}^2$ and $(0,0)\in\omega$. Then, the solution $U_{h\varepsilon}(x,y)$ to (6) has the quasiasymptotics at zero with respect to $c(\epsilon)$, i. e.*

$$\lim_{\epsilon\to 0}<\frac{U_{h\epsilon}(\epsilon sx,\epsilon sy)}{c(\epsilon)},\psi(x,y)>=C_{\psi,s},\ C_{\psi,s}\in\mathbb{C},\ \psi\in\mathcal{D}(\omega),\ i=1,...,n.$$

Proof. Let $(x,y)\in K$. Setting (9) in (16) we obtain

$$
\begin{aligned}
|U_{h\varepsilon}(\varepsilon sx,\varepsilon sy)| &\le |U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)| \\
&\quad +\int_0^{\varepsilon sx}\int_0^{\varepsilon sy}(|F_{h(\varepsilon)}(\xi,\eta,0)|+\kappa_{h\varepsilon}\sup_{(\xi,\eta)\in K}|U_{h\varepsilon}(\xi,\eta)|)d\xi d\eta
\end{aligned}
$$

and

$$|U_{h\varepsilon}(\varepsilon sx,\varepsilon sy)| \le |U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)| + \sup_{(\xi,\eta)\in K}(|F_{h(\varepsilon)}(\xi,\eta,0)|)(\varepsilon^2 s^2)|xy| + \int_0^{\varepsilon sx}\int_0^{\varepsilon sy}\kappa_{h\varepsilon}|U_{h\varepsilon}(\xi,\eta)|d\xi d\eta.$$

Gronwall inequality (cf. [5], Theorem 1.2.2) and (5) yield

$$|U_{h\varepsilon}(\varepsilon sx,\varepsilon sy)| \le C(|U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)| + \varepsilon^2). \tag{17}$$

We shall estimate the integral part of (16). Because of (9) we have

$$\int_0^x\int_0^y |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|d\xi d\eta \le \int_0^x\int_0^y (|F_{h(\varepsilon)}(\xi,\eta,0)| + \kappa_{h\varepsilon}|U_{h\varepsilon}(\xi,\eta)|d\xi d\eta).$$

By using (5) and (17) we obtain

$$\begin{aligned}&\int_0^{\varepsilon sx}\int_0^{\varepsilon sy} |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|d\xi d\eta\\ &\le \sup_{(\xi,\eta)\in K}(|F_{h(\varepsilon)}(\xi,\eta,0)|)(\varepsilon s^2)|xy|\\ &+|\kappa_{h\varepsilon}|\sup_{(\xi,\eta)\in K}|U_{h\varepsilon}(\xi,\eta)|(\varepsilon s)^2|xy| \le C\varepsilon^2\left(1+|\log\varepsilon|(|U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)|+\varepsilon^2)\right)\\ &\le C\varepsilon^2(1+|U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)||\log\varepsilon| + \varepsilon^2|\log\varepsilon|).\end{aligned}$$

Thus, for $(x,y)\in K$,

$$\begin{aligned}&\int_0^{\varepsilon sx}\int_0^{\varepsilon sy} |F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))|d\xi d\eta\\ &\le C\varepsilon^2(1+|U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)||\log\varepsilon| + \varepsilon^2|\log\varepsilon|).\end{aligned} \tag{18}$$

Then, we have in (16)

$$\left\langle \frac{U_{h\varepsilon}(\varepsilon sx,\varepsilon sy)}{c(\varepsilon)},\psi(x,y)\right\rangle = \left\langle \frac{U_{0\varepsilon}(\varepsilon sx,\varepsilon sy)}{c(\varepsilon)},\psi(x,y)\right\rangle + \int_{(x,y)\in I\times I}\psi(x,y)\left(\frac{1}{c(\varepsilon)}\int_0^{\varepsilon sx}\int_0^{\varepsilon sy} F_{h(\varepsilon)}(\xi,\eta,U_{h\varepsilon}(\xi,\eta))d\xi d\eta\right)dxdy.$$

Since (18) and $\lim_{\varepsilon\to 0}\frac{\varepsilon^2|\log\varepsilon|}{c(\varepsilon)} = 0$ hold by assumption the integral part tends to zero and the assertion of Theorem follows.

By Theorem 1 and Theorem 3 we have

Theorem 7. *Assume conditions of* Theorem 3. *Let* $c(\epsilon) \in \mathcal{K}$, *and*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2 |\log \varepsilon|}{c(\varepsilon)} = 0,$$

$\lim\limits_{\epsilon \to 0} \dfrac{U_{0\epsilon}}{c(\epsilon)}$ *exists in* $\mathcal{D}'(\omega)$ *for some* $s > 0$. *Then, the solution* $U(x, y)$ *to* (1) *has the quasiasymptotics at zero with respect to* $c(\epsilon)$, *i. e.*

$$\lim_{\epsilon \to 0} \left\langle \frac{U(\epsilon s x, \epsilon s y)}{c(\epsilon)}, \psi(x, y) \right\rangle = C_{\psi,s},\ C_{\psi,s} \in \mathbb{C},\ \psi \in \mathcal{D}(\omega),\ i = 1, ..., n.$$

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