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POLYNOMIALS RELATED TO GEGENBAUER POLYNOMIALS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. In this paper we consider the polynomials $f_{n,m}^{\lambda,\nu}(z)$, which are the generalization of the Gegenbauer polynomials $C_n^\nu(z)$. Also, we find some relations for their coefficients $p_{n,k}^{\lambda,\nu}$ and we prove some differential-difference relations for the polynomials $f_{n,m}^{\lambda,\nu}(z)$.

1. Introduction

K. Dilcher [1] considered the polynomials $f_n^{\lambda,\nu}(z)$, which are given by

$$G^{\lambda,\nu}(z,t) = \big(1 - \big(1 + z + z^2\big)\,t + \lambda z^2 t^2\big)^{-\nu} = \sum_{n\ge 0} f_n^{\lambda,\nu}(z)\,t^n,$$

where $\nu > 1/2$ and λ is a nonnegative real number. Comparing this with the Gegenbauer polynomials $C_n^\nu(z)$ (see [2], [3], [4]), he obtained

$$f_n^{\lambda,\nu}(z) = z^n \lambda^{n/2} C_n^\nu\left(\frac{1 + z + z^2}{2\sqrt{\lambda}z}\right).$$

In this paper we are going to consider the polynomials $f_{n,m}^{\lambda,\nu}(z)$. Also, we are going to give some properties of these polynomials. In Section 2, we find a recurrence relation for their coefficients $p_{n,k}^{\lambda,\nu}$. In Section 3, we prove some results for $p_{n,k}^{\lambda,\nu}$. Finally, in Section 4, we give some differential-difference relations for the polynomials $f_{n,m}^{\lambda,\nu}(z)$.

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2. Polynomials $\mathbf{f_{n,m}^{\lambda,\nu}(z)}$

At first, we are going to introduce the polynomials $f_{n,m}^{\lambda,\nu}(z)$.

Definition 2.1. *The polynomials $f_{n,m}^{\lambda,\nu}(z)$ are given by the following generating function*

$$F(z,t) = \left(1 - \left(1 + z + z^2\right)t + \lambda z^m t^m\right)^{-\nu} = \sum_{n\geq 0} f_{n,m}^{\lambda,\nu}(z)\, t^n, \tag{2.1}$$

where $\nu > 1/2$, λ is a nonnegative real number and m is a natural number.

Comparing (2.1) with the generating function for the generalized Gegenbauer polynomials $p_{n,m}^{\nu}(z)$ (see [2], [3]), we get

$$f_{n,m}^{\lambda,\nu}(z) = z^n \lambda^{n/m} p_{n,m}^{\nu}\left(\frac{1+z+z^2}{2\sqrt{\lambda}z}\right). \tag{2.2}$$

From the recurrence relation (see [2])

$$np_{n,m}^{\nu}(x) = 2x(\nu+n-1)p_{n-1,m}^{\nu}(x) - (n+m(\nu-1))p_{n-m,m}^{\nu}(x), \quad n \geq m,$$

with starting polynomials:

$$p_{n,m}^{\nu}(x) = \frac{(\nu)_n}{n!}(2x)^n, \qquad n = 0, 1, \dots, m-1,$$

and by (2.2), we get the following recurrence relation

$$\begin{aligned} f_{n,m}^{\lambda,\nu}(z) = {} & \left(1 + \frac{\nu-1}{n}\right)\left(1+z+z^2\right) f_{n-1,m}^{\lambda,\nu}(z) \\ & - \left(1 + \frac{m(\nu-1)}{n}\right)\lambda z^m f_{n-m,m}^{\lambda,\nu}(z), \end{aligned} \tag{2.3}$$

with starting polynomials:

$$f_{n,m}^{\lambda,\nu}(z) = \frac{(\nu)_n}{n!}\left(1+z+z^2\right)^n, \qquad 0 \leq n \leq m-1.$$

Let us put $1/z$ instead z in (2.2). Then, it follows:

$$f_{n,m}^{\lambda,\nu}(z) = z^{2n} f_{n,m}^{\lambda,\nu}(1/z).$$

So, we get that the polynomials $f_{n,m}^{\lambda,\nu}(z)$ are self-inverse, or in other words, the coefficients are "centrally symmetric." Now, the polynomials $f_{n,m}^{\lambda,\nu}(z)$ have the following form

$$f_{n,m}^{\lambda,\nu}(z) = p_{n,n}^{\lambda,\nu} + p_{n,n-1}^{\lambda,\nu} z + \cdots + p_{n,0}^{\lambda,\nu} z^n + p_{n,1}^{\lambda,\nu} z^{n+1} + \cdots + p_{n,n}^{\lambda,\nu} z^{2n}. \tag{2.4}$$

Thus, we have the following triangle

$$\begin{array}{ccccc} & & p_{0,0}^{\lambda,\nu} & & \\ & p_{1,1}^{\lambda,\nu} & p_{1,0}^{\lambda,\nu} & p_{1,1}^{\lambda,\nu} & \\ p_{2,2}^{\lambda,\nu} & p_{2,1}^{\lambda,\nu} & p_{2,0}^{\lambda,\nu} & p_{2,1}^{\lambda,\nu} & p_{2,2}^{\lambda,\nu} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array} \tag{2.5}$$

From (2.3) and (2.4), we get the following recurrence relation

$$\begin{aligned} p_{n,k}^{\lambda,\nu} = &\left(1 + \frac{\nu-1}{n}\right)\left(p_{n-1,k-1}^{\lambda,\nu} + p_{n-1,k}^{\lambda,\nu} + p_{n-1,k+1}^{\lambda,\nu}\right) \\ &- \left(1 + m\frac{\nu-1}{n}\right)\lambda p_{n-m,k}^{\lambda,\nu}, \qquad n \geq m, \end{aligned} \tag{2.6}$$

where $p_{n,k}^{\lambda,\nu} = p_{n,-k}^{\lambda,\nu}$

3. Coefficients $\mathbf{p_{n,k}^{\lambda,\nu}}$

The main purpose of this paper is to study the coefficients $p_{n,k}^{\lambda,\nu}$. We are going to derive the following explicit expressions.

Theorem 3.1. *The coefficients $p_{n,k}^{\lambda,\nu}$ are given by*

$$\begin{aligned} p_{n,k}^{\lambda,\nu} = &\frac{1}{\Gamma(\nu)} \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \frac{\Gamma(\nu+n-(m-1)s)}{s!(n-ms)!} \times \\ &\times \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{2j+k}\binom{2j+k}{j}. \end{aligned} \tag{3.1}$$

Proof. Using the explicit representation (see [2])

$$p_{n,m}^{\nu}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\nu)_{n-(m-1)k}}{k!(n-mk)!}(2x)^{n-mk},$$

and from (2.2), we get

$$\begin{aligned} f_{n,m}^{\lambda,\nu}(z) &= z^n \lambda^{n/m} p_{n,m}^{\nu}\left(\frac{1+z+z^2}{2\lambda^{1/m} z}\right) \\ &= \sum_{s=0}^{[n/m]} (-1)^s \frac{(\nu)_{n-(m-1)s}}{s!(n-ms)!} \left(1+z+z^2\right)^{n-ms} z^{ms}\lambda^s. \end{aligned}$$

Now, from the last equalities and by formula (see [1])

$$\left(1+z+z^2\right)^r = \sum_{m=0}^{2r} z^m \sum_{j=0}^{[m/2]} \binom{r}{m-j}\binom{m-j}{m-2j},$$

where r is a positive integer, we get

$$\begin{aligned} f_{n,m}^{\lambda,\nu}(z) = \frac{1}{\Gamma(\nu)} \sum_{k=-n}^{n} z^{n-k} \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \frac{(\nu)_{n-(m-1)s}}{s!(n-ms)!} \times \\ \times \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{2j+k}\binom{2j+k}{j}. \end{aligned}$$

The statement (3.1) follows immediately from the last equalities. $\square$

The next two results are related to coefficients $p_{n,k}^{\lambda,\nu}$.

Theorem 3.2. *The coefficients* $p_{n,k}^{\lambda,\nu}$ *have the following representation*

(3.2)

$$p_{n,k}^{\lambda,\nu} = \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \binom{n-k-(m-1)s}{s} \frac{(\nu)_{n-(m-1)s}}{k!(n-k-(m-1)s)!} B_k^{(n-k-ms)},$$

where

$$B_k^{(r)} = \sum_{j=0}^{[r/2]} \binom{2j}{j}\binom{r}{2j}\left(\binom{k+j}{j}\right)^{-1}. \tag{3.3}$$

Theorem 3.3. *The coefficients* $p_{n,k}^{\lambda,\nu}$ *can be expressed as*

$$p_{n,k}^{\lambda,\nu}=\frac{1}{k+1}\sum_{j=0}^{[(n-k)/m]}\frac{(-\lambda)^s}{s!(k!)^2(n-k-ms)!}\sum_{j=0}^{[r/2]}\frac{\left(-\frac{r}{2}\right)_j\left(\frac{1-r}{2}\right)_j}{j!}\frac{2^{2j}}{\Gamma(k+j)},$$

where $r=n-k-ms$.

We mention now some special cases:

1° For $m=2$, we get the polynomials $f_n^{\lambda,\nu}(z)$ (see [1]). Then (3.1) becomes

$$p_{n,k}^{\lambda,\nu}=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{[(n-k)/2]}(-\lambda)^s\frac{\Gamma(\nu+n-s)}{s!(n-2s)!}\sum_{j=0}^{[(n-k-2s)/2]}\binom{n-2s}{2j+k}\binom{2j+k}{j}.$$

Hence

$$f_n^{\lambda,\nu}(1)=\sum_{k=-n}^{n}p_{n,k}^{\lambda,\nu}=\lambda^{n/2}p_n^{\nu}\left(\frac{3}{2\sqrt{\lambda}}\right).$$

2° If $m=2$ and $\nu=1$, we obtain

$$f_n^{\lambda,1}(1)=\lambda^{n/2}U_n\left(\frac{3}{2\sqrt{\lambda}}\right),$$

where

$$p_{n,k}^{\lambda,1}=\sum_{s=0}^{[(n-k)/2]}(-\lambda)^s\frac{(n-s)!}{s!(n-2s)!}\sum_{j=0}^{[(n-k-2s)/2]}\binom{n-2s}{2j+k}\binom{2j+k}{j},$$

and $U_n(x)$ is the Chebyshev polynomials of the second kind.

3° With $z=1$, from (2.2) and (2.4), we get

$$f_{n,m}^{\lambda,\nu}(1)=\sum_{k=-n}^{n}p_{n,k}^{\lambda,\nu}=\lambda^{n/m}p_{n,m}^{\nu}\left(\frac{3}{2\sqrt[m]{\lambda}}\right).$$

This is the sum of the coefficients of the n-th row in the triangle (2.5).

4. Differential-difference relations

Firstly, we are going to prove the following theorem.

Theorem 4.1. *The polynomials* $f_{n,m}^{\lambda,\nu}(z)$ *satisfy the following relations:*

$$Df_{n,m}^{\lambda,\nu}(z)=\frac{n}{z}f_{n,m}^{\lambda,\nu}(z); \tag{4.1}$$

$$D^k f_{n,m}^{\lambda,\nu}(z)=z^{-k}\frac{n!}{(n-k)!}f_{n,m}^{\lambda,\nu}(z); \tag{4.2}$$

$$(n+m\nu)f_{n,m}^{\lambda,\nu}(z)=m\nu f_{n,m}^{\lambda,\nu+1}(z)-\nu(m-1)\frac{1+z+z^2}{z\sqrt[m]{\lambda}}f_{n-1,m}^{\lambda,\nu+1}(z); \tag{4.3}$$

$$z^{2k}f_{n-k,m}^{\lambda,k+1/2}(z)=\frac{(-1)^k n!}{(2k-1)!!}\lambda^{-k/m}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(n)_{k-i}}{(n-i)!}f_{n,m}^{\lambda,1/2}(z); \tag{4.4}$$

$$z^2D^2f_{n,m}^{\lambda,\nu}(z)-nzDf_{n,m}^{\lambda,\nu}(z)+nf_{n,m}^{\lambda,\nu}(z)=0. \tag{4.5}$$

Proof. Differentiating (2.2), with respect to z, we get

$$\begin{aligned}Df_{n,m}^{\lambda,\nu}(z)=\frac{n}{z}f_{n,m}^{\lambda,\nu}(z)-\frac{n\lambda^{-1/m}\left(z^2-1\right)}{2z^3}f_{n,m}^{\lambda,\nu}(z)\\+\frac{z^{-2}}{2}\lambda^{-1/m}\left(z^2-1\right)Df_{n,m}^{\lambda,\nu}(z).\end{aligned}$$

So, we obtain

$$\left(1-\frac{z^2-1}{2z^2\sqrt[m]{\lambda}}\right)Df_{n,m}^{\lambda,\nu}(z)=\left(\frac{n}{z}-\frac{n\left(z^2-1\right)}{2z^3\sqrt[m]{\lambda}}\right)f_{n,m}^{\lambda,\nu}(z).$$

Now, from the last equalities, we get (4.1).

By differentiating (4.1), with respect to z, and using induction on n, we get (4.2).

Now, from (2.2) and by the following relation (see [2])

$$(n+m\nu)p_{n,m}^{\nu}(x)=m\nu p_{n,m}^{\nu+1}(x)-2(m-1)\nu x p_{n-1,m}^{\nu+1}(x),$$

we have the wanted relation (4.3).

To prove (4.4), we are going to use the relation (see [2])

$$p_{n-k,m}^{k+1/2}(x)=\frac{1}{(2k-1)!!}D^k p_{n,m}^{1/2}(x),$$

and from (2.2) and (4.2), we get

$$\begin{aligned} f_{n-k,m}^{\lambda,k+1/2}(z) &= \frac{z^{n-k}\lambda^{(n-k)/m}}{(2k-1)!!}\lambda^{-n/m}\sum_{i=0}^{k}\binom{k}{i}\left(z^{-n}\right)^{(k-i)}D^i f_{n,m}^{\lambda,1/2}(z) \\ &= \frac{z^{-2k}\lambda^{-k/m}n!}{(2k-1)!!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{(n)_{k-i}}{(n-i)!}f_{n,m}^{\lambda,1/2}(z). \end{aligned}$$

The statement (4.4) follows immediately, from the last equalities.

Finally, differentiating (4.1), with respect to z, we obtain

$$D^2 f_{n,m}^{\lambda,\nu}(z)=-\frac{n}{z}f_{n,m}^{\lambda,\nu}(z)+\frac{n}{z}Df_{n,m}^{\lambda,\nu}(z).$$

Multiplying the both sides of the last equality by z^2, we obtain (4.5). □

Finally, we prove the following theorem.

Theorem 4.2. *For the polynomials $f_{n,m}^{\lambda,\nu}(z)$ holds*

$$\begin{aligned} f_{n,m}^{\lambda,\nu}(z) = \frac{g^{-1}}{n}\big(&\left(nz+2z^2g^{-1}D\{g\}\right)-z^2D^2-nzD\{g\}g^{-1} \\ &+z^2D^2\{g\}g^{-1}+2z^2D\{g\}D\{g^{-1}\}\big)\{gf_{n,m}^{\lambda,\nu}(z)\}, \end{aligned} \tag{4.6}$$

where $g(z)$ is a differentiable function not identically zero.

Proof. From (4.5),we have

$$nf_{n,m}^{\lambda,\nu}(z)=\left(nzD-z^2D^2\right)f_{n,m}^{\lambda,\nu}(z).$$

Now, multiplying the both sides of the last equality by g, we get

$$\begin{aligned} gnf_{n,m}^{\lambda,\nu}(z) &= g\left(nzD - z^2D^2\right)\{f_{n,m}^{\lambda,\nu}(z)\} \\ &= \Big(\left(nz - 2z^2D\{g\}g^{-1}\right)D - z^2D^2 - nzg^{-1}D\{g\} + z^2g^{-1}D^2\{g\} \\ &\qquad + 2z^2D\{g\}D\{g^{-1}\}\Big)\{gf_{n,m}^{\lambda,\nu}(z)\}. \end{aligned}$$

Hence

$$\begin{aligned} f_{n,m}^{\lambda,\nu}(z) =& \frac{g^{-1}}{n}\big(\left(nz - 2z^2D\{g\}g^{-1}\right)D - z^2D^2 - nzg^{-1}D\{g\} \\ &+ z^2g^{-1}D^2\{g\} + 2z^2D\{g\}D\{g^{-1}\}\big)\{gf_{n,m}^{\lambda,\nu}(z)\}. \end{aligned}$$

This is the wanted equality (4.6). $\square$

Example. If $g(z) = e^z$, then $g^{-1}(z) = e^{-z}$ and we get

$$f_{n,m}^{\lambda,\nu}(z) = \frac{e^{-z}}{n}\left(\left(nz + 2z^2\right)D - z^2D^2 - nz - z^2\right)\{e^z f_{n,m}^{\lambda,\nu}(z)\}.$$

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