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A GENERALIZATION OF THE JOINT SPECTRAL RADIUS: THE GEOMETRICAL APPROACH

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. The generalized joint spectral radius with exponent p is investigated.

1. Basic Notions

Definition 1. The joint spectral radius of a given set $\{A_1, A_2, \dots, A_k\} \subset \mathcal{L}(\mathbb{R}^d)$ of linear operators is

$$\hat{\rho}(A_1, A_2, \dots, A_k) = \lim_{m\to\infty} \max_{\sigma} \|A_{\sigma(1)} \cdots A_{\sigma(m)}\|^{1/m},$$

where σ runs the set of all the substitutions $\sigma\colon \{1, \dots, m\} \to \{1, \dots, k\}$.

The notion of joint spectral radius have appeared birthly by G.C. Rota and G. Strang in 1960 [1]. In 1988–90 its close relation to the regularity exponent of Daubeshies wavelets in the space $C[a, b]$ was established by Collela, Heil, Daubeshies, Lagarias etc. [2], [3], [4], [5]. The wavelets of Daubeshies is a system of functions $\{\psi_{ij}\}_{i,j\in\mathbb{Z}}$

1) $\psi_{ij} = 2^{-j/2}\psi_{00}(2^j x - i)$,
2) $\{\psi_{ij}\}$ – ortonormal basis in $L_2(\mathbb{R})$,
3) $\operatorname{supp}\psi_{00} \subset [0, N]$, $N \in \mathbb{N}$.

The function ψ_{00} can be constructed by a scaling function ϕ by formula $\psi_{00}(x) = \sum\limits_{k=0}^{N} (-1)^k C_{N-k}\phi(2x - k)$. The scaling function ϕ is a solution of dilation equation or two-scale difference equation:

$$\phi = \sum_{k=0}^{N} C_k\, \phi(2x - k)\,, \tag{1}$$

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where $C_k \in \mathbb{C}$, $C_k = 0$ if $k \neq \{0, 1, \ldots, \mathbb{N}\}$ and

$$(2) \qquad \sum C_{2i+1} = \sum C_{2i} = 1, \quad \sum C_k \overline{C_{k+2j}} = 2\delta_{0j} \quad \text{for any} \quad j \in \mathbb{Z}\,.$$

Any solution ϕ of the scaling equation such that $\operatorname{supp} \phi \subset [0, N]$ is a fixed solution multiplied by a constant.

Let T_0, T_1 be two $N \times N$ matrix $(T_0)_{ij} = C_{2i-j-1}$, $(T_1)_{ij} = C_{2i-j}$. One can define the subspace $W \subset \mathbb{R}$ depending on T_0 and T_1 [3].

Dilation equation (1) with condition (2) has a continuous solution ϕ, $\phi \in C(\mathbb{R})$, $\operatorname{supp} \phi \in [0, N]$, where $A = T_0|_W$, $B = T_1|_W$.

If $\hat{\rho}(A, B) \in (1/2, 1)$, then $\alpha = -\log_2 \hat{\rho}(A, B)$ is a Holder exponent of ϕ, if $\hat{\rho}(A, B) \leq 1/2$ then f is a Lipschitzian function (Collela, Heil, 1989) [3].

We investigate the two following problems:

1° To create an algorithm for the calculation of $\hat{\rho}(A_1, \ldots, A_k)$ interms of the matrix $A_1, \ldots, A_k$ coefficients up to a given accuracy rate;

2° To study properties of the function $\hat{\rho} : [\mathcal{L}(\mathbb{R}^d)]^k \to \mathbb{R}_+$, in particular, to estimate its smoothness.

2. Geometrical Approach

The geometrical approach to the problem under concideration is presented in details in [6]. Before the statement of the main result we introduce some notations.

Let $X \subset \mathbb{R}^d$ be a convex compact body with a nonempty interior, $A_1, \ldots, A_k$ be given linear operators in $\mathbb{R}^d$. Let us define operator $\bar{A}$ in the space of convex bodies by the following formula:

$$\bar{A}(X) = \operatorname{conv}\,(A_1 X, \ldots, A_k X)\,.$$

Say a set $\{A_1, \ldots, A_k\} \subset \mathcal{L}(\mathbb{R}^d)$ of k operators is generic, if they do not have a nontrivial common invariant subspace.

Theorem 1. *For a generic set of operators $A_1, \ldots, A_k \in \mathcal{L}(\mathbb{R}^d)$*

a) *there exist a compact K $(K \neq \{0\})$ and a $\lambda \in \mathbb{R}_+$ such that*

$$(3) \qquad K = \lambda \bigcup_{i=1}^{N} A_i K.$$

b) *There exist a central-symmetric convex body M and a $\lambda \in \mathbb{R}_+$ such that*

$$\bar{A}M = \lambda M. \tag{4}$$

c) *For any compact with property* (3) *and any body with property* (4) *the correspondent homothety coeffitient λ is equal to the joint spectral radius $\hat{\rho}(A_1, \dots, A_k)$.*

Remark 1. If the joint spectral rsdius of operators $A_1, \dots, A_k$ is equal to 1 then property (3) of the correspondent invariant compact K almost coincides with the basic definition of self-similar sets (fractals by Hutchinson [8]): the difference is that the operators that define the fractal are affine and contractive, and the operators under consideration are linear and have the joint spectral radius equal to 1.

However the properties of the unvariant compact and the analogous ones of fractal are quite different.

Remark 2. In [6] a geometrical approach for calculation of the joint spactral radius based on properties of the invariant bodies is presented. For $d = 2$ the correspondent algorithm takes at most $C\varepsilon^{-3/2}$ arithmetic operations for the calculation of $\hat{\rho}(A_1, \dots, A_k)$ with the relative error ε where $C = C(A_1, \dots, A_k)$ is a constant that depends on $A_1, \dots, A_k$.

3. Generalization

For any $p \in [1, +\infty]$ we shall define p-joint spectral radius $\hat{\rho}_p$.

Definition 2. Let $p \in [1, +\infty]$. The p-joint spectral radius of a given set og operators $A_1, \dots, A_k \in \mathcal{L}(\mathbb{R}^d)$ is

$$\hat{\rho}_p = \lim_{m \to \infty} \left(\sum_{\sigma} \|A_{\sigma(1)} \cdots A_{\sigma(m)}\|^p \right)^{1/pm},$$

where the sum is taken through all the substitutions $\sigma : \{1, \dots, m\} \to \{1, \dots, k\}$.

The notion of p-radius appeared in 1995. It was related to the smoothness problem for Daubeshies wavelets in the space $L_p[a, b]$.

One can define the subspace $W^1 \subset \mathbb{R}^N$, depending on T_0 and T_1 [8]. Dilation equation (1) with condition (2) has a solution in $L_p[0, N]$ if and only if $2^{-1/p}\hat{\rho}_p(A', B') < 1$, $A' = T_0|_{W'}$, $B' = T_1|_{B'}$. If $2^{-1/p}\hat{\rho}_p(A', B') \in (1/2, 1)$, then $\alpha_p = -\log_2(2^{-1/p}\hat{\rho}_p(A', B'))$ is a Holder exponent of solution ϕ in the space $L_p[0, N]$ (cf. [7]).

Remark 3. The joint spectral radius is obviously a particular case of p-radius (with $p = \infty$).

To extend the geometrical approach to the p-radius, it is necessary to construct a new operation with convex bodies.

Definition 3. Let M be a convex body, and its interior contains 0. The support function ϕ_M correspondng to M is the function on the unit sphere defined by

$$\phi_M(x) = \sup_{y \in M} \langle x, y \rangle \qquad (\|x\| = 1).$$

Lemma 1. *For any convex bodies M_1, M_2 as in Definition 3 and any $p \in [1, +\infty]$ there exists a body $M_3 = M_1 \overset{p}{\oplus} M_2$ defined by the following relation:*

$$(\forall x)\ \|x\| = 1 : \quad [\phi_{M_3}(x)]^p = [\phi_{M_1}(x)]^p + [\phi_{M_2}(x)]^p.$$

We have defined p-addition of convex bodies by using Lemma 1. For $p = 1$ it coincides with Minkovsky addition of convex bodies: for $p = \infty$ it is the operation of taking the convex hull

$$M_1 \overset{1}{\oplus} M_2 = M_1 + M_2\,, \qquad M_1 \overset{\infty}{\oplus} M_2 = \operatorname{conv}(M_1, M_2).$$

Example 1. For $p = 2$ the sum $M_1 \overset{2}{\oplus} M_2$ of the segments

$$M_1 = [(-1, 0), (1, 0)]\,, \qquad M_2 = [(0, -1), (0, 1)]$$

is the unit circle.

Now, we are able to state an analogue of Theorem 1 for arbitrary p.

Theorem 2. *For any generic set of operators $A_1, \ldots, A_k$ and arbitrary $p \in [1, +\infty]$*

a) *there exists a central-symmetric body M such that*

$$\bar{A}M = \lambda M, \tag{5}$$

where $\bar{A}M = A_1 M \overset{p}{\oplus} A_2 M \overset{p}{\oplus} \cdots \overset{p}{\oplus} A_k M$.

b) *For any body with property* (5) *the homothety coefficient λ is equal to the p-radius.*

The case $p = \infty$ is analysed in [6]. Let us consider one more case,i.e. the case $p = 2$. In this case for any operators there exists an invariant ellipsoid. Namely, the following statement holds.

Statement. *For any generic set* $A_1, \dots, A_k$ *of operators there exists a non-degenerated ellipsoid* M *such that* $\bar{A}M = \lambda M$,

$$\lambda = \widehat{\rho}_2(A_1, \dots, A_k), \quad \bar{A}M = A_1 M \overset{2}{\oplus} A_2 M \overset{2}{\oplus} \cdots \overset{2}{\oplus} A_k M.$$

The number λ *and the matrix* B *of the quadratic form corresponding to* M *can be determined by the equation*

$$\sum_{i=1}^{k} A_i^* B A_i = \lambda^2 B.$$

Therefore, in the case when $p = 2$ the p-radius can be calculated explicitly.

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