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SOME RECURRENCE FORMULAS RELATED TO THE DIFFERENTIAL OPERATOR θD

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. Some recurrence formulas related to the differential operator $\delta = \theta \cdot D$ are considered. Some examples, including Kurepa's function, are included.

1. Introduction

In the spring of 1971, I had an inspiriting talk with Professor D. Mitrinović about Stirling numbers of the second kind. This article is a reflection on this time and event.

The starting point of our consideration is the differential operator $\delta = \theta \cdot \mathrm{D}$. As the most of our analysis is of combinatorial and arithmetical nature, we shall put it in the frame of differential fields. Let us remind about this notion. The language of the differential fields is the language of fields augmented by a one-place function symbol D. The theory of differential fields of characteristic zero is the theory of fields increased by two axioms that relate to the derivative D:

$$\mathrm{D}(x+y) = \mathrm{D}x + \mathrm{D}y, \qquad \mathrm{D}(xy) = x\mathrm{D}y + y\mathrm{D}x.$$

It is easy to see that $\mathrm{D}x = 0$ for all integers x, namely $\mathrm{D}1 = \mathrm{D}(1 \cdot 1) = 1\mathrm{D}1 + 1\mathrm{D}1$, so $\mathrm{D}1 = 0$. By the additivity of D, it follows $\mathrm{D}x = 0$ for all integers x. So, every differential field is a structure of the form $\mathbf{F} = (F, +, \cdot, \mathrm{D}, 0, 1)$ satisfying the stated axioms. Let θ be an element of $\mathbf{F}$. Now we can introduce a new differential operator $\delta = \theta \cdot \mathrm{D}$, i.e. by $\delta(x) = \theta \cdot \mathrm{D}(x)$, $x \in F$.

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For notation in this paper, we use $\mathbb{N} = \{0, 1, 2, \dots\}$ to denote the set of natural numbers, $\mathbb{N}^+$ the set of positive integers, $\mathbb{Z}$ the set of integers, while $\mathbb{R}$ denotes the set of real numbers, if not otherwise stated. The symbol $x^{(k)}$ denotes the product $x(x-1)\cdots(x-k+1)$. As usual, if $\mathbf{A}$ denotes an (algebraic) structure, then A denotes domain of this structure. If $\mathbf{F}$ is a field, or commutative ring where applicable, then $\mathbf{F}[x_1, x_2, \dots, x_n]$ denotes the ring of all polynomials in variables $x_1, x_2, \dots, x_n$, while $\mathbf{F}(x_1, x_2, \dots, x_n)$ denotes the field of rational expressions over $\mathbf{F}$. Finally, by $\mathcal{R}$ we shall denote the field of real rational functions in infinite number of variables $x_0, x_1, x_2, \dots$.

2. Operator δ^n

Our goal in this section is to study the power δ^n of the operator δ introduced in the previous section. A lot of efforts were spent in studying of this operator, for history see, for example, [1]. Let us denote by $\theta^{(i)}$ the i-th derivative $\mathrm{D}^i\theta$. Then it is obvious that δ^n is a polynomial in $\theta, \theta^{(1)}, \dots, \theta^{(n-1)}$ and D, that is, we have:

$$\delta^n = \sum_{i=1}^{n} P_i^n(\theta, \theta', \dots, \theta^{(n-1)})\mathrm{D}^i, \tag{2.1}$$

where $P_i^n(x_0, x_1, \dots, x_{n-1}) \in \mathbf{Z}[x_0, x_1, \dots, x_{n-1}]$. Therefore,

$$\delta^n = \sum_{i=1}^{n} \bar{P}_i^n \mathrm{D}^i, \tag{2.2}$$

where $\bar{P}_i^n = P_i^n(\theta, \theta', \dots, \theta^{(n-1)})$.

Some basic properties of the polynomials P_i^n are:

a) The polynomial P_i^n depends only on the variables $x_0, x_1, \dots, x_{n-i}$, namely $P_i^n \in \mathbf{Z}[x_0, x_1, \dots, x_{n-i}]$. So $\bar{P}_i^n = P_i^n(\theta, \theta', \dots, \theta^{(n-i)})$.

b) P_i^n are homogeneous polynomials of degree n.

c) The variable x_0 divides every polynomial P_i^n.

Further,

$$\begin{aligned}
\delta^{n+1} &= \theta \cdot \mathrm{D} \sum_{i=1}^{n} \bar{P}_i^n \mathrm{D}^i = \theta \Bigg(\sum_{i=1}^{n} \mathrm{D}\bar{P}_i^n \cdot \mathrm{D}^i + \sum_{i=1}^{n} \bar{P}_i^n \mathrm{D}^{i+1} \Bigg) \\
&= \theta \Bigg(\mathrm{D}\bar{P}_1^n \mathrm{D} + \sum_{i=2}^{n} (\bar{P}_{i-1}^n + \mathrm{D}\bar{P}_i^n)\mathrm{D}^i + \bar{P}_n^n \mathrm{D}^{n+1} \Bigg).
\end{aligned}$$

Thus,

$$\bar{P}_1^{n+1} = \theta \mathrm{D} \bar{P}_1^n = \theta \sum_{i=0}^{n-1} \overline{\frac{\partial P_1^n}{\partial x_i}} \,\mathrm{D}\theta^{(i)} = \theta \sum_{i=0}^{n-1} \overline{\frac{\partial P_1^n}{\partial x_i}} \,\theta^{(i+1)}.$$

So,

$$P_1^{n+1} = x_0 \sum_{i=0}^{n-1} \frac{\partial P_1^n}{\partial x_i} x_{i+1}. \tag{2.3a}$$

As $\bar{P}_i^{n+1} = \theta(\bar{P}_{i-1}^n + \mathrm{D}\bar{P}_i^n)$, $2 \le i \le n$, we find in a similar way

$$P_i^{n+1} = x_0 \left(P_{i-1}^n + \sum_{j=0}^{n-i} \frac{\partial P_i^n}{\partial x_j} x_{j+1} \right) \tag{2.3b}$$

and

$$P_{n+1}^{n+1} = x_0 P_n^n. \tag{2.3c}$$

For example, noting that $P_1^1 = x_0,\ P_1^2 = x_0 x_1,\ P_2^2 = x_0^2$, we compute

$$P_1^3 i = x_0 \sum_{j=1}^{1} \frac{\partial P_j^1}{\partial x_j} x_{j+1} = x_0(x_1^2 + x_0 x_2) = x_0 x_1^2 + x_0^2 x_2,$$

$$P_2^3 = x_0 \left(P_1^2 + \sum_{j=0}^{0} \frac{\partial P_j^2}{\partial x_j} x_{j+1} \right) = x_0(x_0 x_1 + 2x_0 x_1) = 3x_0^2 x_1,$$

$$P_3^3 = x_0^3.$$

Hence, the first few polynomials are:
$P_1^1 = x_0,\ P_1^2 = x_0 x_1,\ P_2^2 = x_0^2,$
$P_1^3 = x_0 x_1^2 + x_0^2 x_2,\ P_2^3 = 3x_0^2 x_1,\ P_3^3 = x_0^3,$
$P_1^4 = x_0 x_1^3 + 4x_0^2 x_1 x_2 + x_0^3 x_3,\ P_2^4 = 7x_0^2 x_1^2 + 4x_0^3 x_2,\ P_3^4 = 6x_0^3 x_1,\ P_4^4 = x_0^4.$

3. Operator δ for D $x = 1$

The most known operator δ is for θ such that D $\theta = 1$ (in the case of the field $\mathbf{R}[x]$ or $\mathcal{R}$, θ consists of the variable x). This operator we shall call Abelian, as Abel used it to compute the sum $1^k + 2^k + \cdots + n^k$ (cf. [6]). As $\theta' = 1$, $\theta'' = 0$, ..., in the evaluation of the formula (2.2) we may take $x_1 = 1$, and $x_2 = 0,\ x_3 = 0,\ \ldots$, so $P_i^n = \sigma_i^n x_0^i$, where σ_i^n is an integer sequence. Thus,

$$\delta^n = \sum_{i=1}^{n} \sigma_i^n x_0^i \mathrm{D}^i, \quad \sigma_1^1 = 1, \quad \sigma_n^n = 1. \tag{3.1}$$

By (2.3b) we have

$$P_i^{n+1} = x_0 \left(P_{i-1}^n + \frac{\partial P_i^n}{\partial x_0} \right) = \left(\sigma_{i-1}^n + i\sigma_i^n\right) x_0^n.$$

Hence, the sequence σ_i^n satisfies the following recurrence formula:

$$\sigma_i^{n+1} = \sigma_{i-1}^n + i\sigma_i^n, \qquad \sigma_1^n = 1, \quad \sigma_n^n = 1, \tag{3.2}$$

i.e. $\sigma_i^n = s_i^n$, where s_i^n are the Stirling numbers of the second kind. Thus, taking $x_0 = x$,

$$\delta^n = \sum_{i=1}^{n} s_i^n x^i \mathrm{D}^i. \tag{3.3}$$

Let us denote by S_i^n the Stirling numbers of the second kind. As the matrices $||s_i^n||_{n\times n}$ and $||S_i^n||_{n\times n}$ are mutually inverse, then we have for any sequences $\langle a_n | n \in \mathbb{N}^+\rangle$ and $\langle b_n | n \in \mathbb{N}^+\rangle$ the following inversion formula:

$$b_n = \sum_{i=1}^{n} s_i^n a_i, \quad n \in \mathbb{N}^+ \qquad \text{iff} \qquad a_n = \sum_{i=1}^{n} S_i^n b_i, \quad n \in \mathbb{N}^+. \tag{3.4}$$

So, the operator D^n is represented by δ^i as follows

$$\mathrm{D}^n = x^{-n} \sum_{i=1}^{n} S_i^n \delta^i = x^{-n} \delta(\delta - 1) \cdots (\delta - n + 1). \tag{3.5}$$

Remark 1. We can arrive to the same formula if we note that $\mathrm{D} = x^{-1}\delta$, $\delta x^{-1} = -x^{-1}$, and that in (2.1) (but with D and δ interchanged, as now δ is given and D is derived operator) P_i^n depends only on x_0.

If $\mathrm{D}x = 1$, it is easily seen that $\delta^k x^i = i^k x^i$. As an example we compute the sum $S(k, n, x) = \sum\limits_{i=1}^{n-1} i^k x^i$:

$$1 + (x+1) + (x+1)^2 + \ldots + (x+1)^{n-1} = \frac{(x+1)^n - 1}{x} = \sum_{i=1}^{n} \binom{n}{i} x^{i-1},$$

i.e.

$$1 + x + x^2 + \ldots + x^{n-1} = \sum_{i=1}^{n} \binom{n}{i} (x-1)^{i-1}.$$

By applying δ^k on this identity, we obtain

$$\begin{aligned} 1^k x + 2^k x^2 + \cdots + (n-1)^k x^{n-1} &= \sum_{i=1}^{n} \binom{n}{i} \delta^k (x-1)^{i-1} \\ &= \sum_{i=1}^{n} \binom{n}{i} \sum_{j=1}^{k} s_j^k x^j \mathrm{D}^j (x-1)^{i-1}. \end{aligned}$$

So,

$$\begin{aligned} &1^k x + 2^k x^2 + \cdots + (n-1)^k x^{n-1} \\ &= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \le i-1}}^{k} \binom{n}{i} (i-1)(i-2)\cdots(i-j) s_j^k x^j (x-1)^{i-j-1}. \end{aligned} \tag{3.6}$$

Putting $x = 1$ in the above identity, we obtain:

$$\begin{aligned} 1^k + 2^k + \cdots + (n-1)^k &= \sum_{i=1}^{n} \binom{n}{i} (i-1)!\, s_{i-1}^k \\ &= \sum_{i=1}^{n} \frac{n^{(i)}}{i} s_{i-1}^k = \sum_{i=1}^{n-1} \frac{s_i^k}{i+1} n^{(i+1)}. \end{aligned}$$

For large values of n ($n \ge k+1$), we have

$$\sum_{i=1}^{n-1} \frac{s_i^k}{i+1} n^{(i+1)} = \sum_{i=1}^{k} \frac{s_i^k}{i+1} n^{(i+1)},$$

as $s_i^k = 0$ for $i > k$. The same identity holds for small values of n ($n \le i \le k$), since then $n^{(i)} = 0$. So we obtain the famous identity

$$1^k + 2^k + \cdots + (n-1)^k = \sum_{i=1}^{k} \frac{s_i^k}{i+1} n^{(i+1)}. \tag{3.7}$$

One can obtain various identities starting from (3.3). Let $f(x) = x^\alpha$, $\alpha \in \mathbb{R}$. Then $\delta^n f = \alpha^n x^\alpha$, while by (3.3),

$$\delta^n f = x^\alpha \sum_{k=1}^{n} s_k^n k! \binom{\alpha}{k}.$$

Comparing these identities, and first canceling and then replacing α with x, we get

$$x^n = \sum_{k=1}^{n} s_k^n x^{(k)}, \tag{3.8}$$

and by inversion formula (3.4) we get

$$x^{(n)} = \sum_{k=1}^{n} S_k^n x^k. \tag{3.9}$$

Putting $x = -1$ into (3.9) and by use of inversion formula, we get

$$\sum_{k=1}^{n} (-1)^k s_k^n k! = (-1)^n, \qquad \sum_{k=1}^{n} (-1)^k S_k^n = (-1)^n n!. \tag{3.10}$$

Assume $f(x) = \ln x$. Then $\delta^n f = 0$ for all $n \geq 2$, and by (3.3) we have $\delta^n f = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! s_k^n$, so

$$\sum_{k=1}^{n} (-1)^k (k-1)! s_k^n = 0, \quad \text{for} \quad n \geq 2. \tag{3.11}$$

4. Operator δ for $\theta(x) = x^{\alpha+1}$

Assume $\theta(x) = x^{\alpha+1}$, $\alpha \in \mathbb{R}$. Then it is easy to see that for $\delta = \theta\,\mathrm{D}$:

$$\delta^n = \sum_{k=1}^{n} c_{nk} x^{n\alpha+k} \mathrm{D}^k, \tag{4.1}$$

where c_{nk} satisfies the difference equation $(k, n \in \mathbb{N}^+)$:

$$c_{n+1,k} = c_{nk} + (n\alpha + k)c_{nk}, \quad 1 \leq k \leq n; \quad c_{11} = 1, \ c_{nk} = 0, \quad k > n. \tag{4.2}$$

Let $f(x) = x^\lambda$, $\lambda \in \mathbb{R}$. Then by (4.1)

$$\delta^n f = x^{\lambda+n\alpha} \sum_{k=1}^{n} c_{nk} \lambda^{(k)},$$

while computing $\delta^n f$ directly we find $\delta^n f = \lambda(\lambda+\alpha)\cdots(\lambda+(n-1)\alpha)x^{\lambda+n\alpha}$, so

$$\lambda(\lambda+\alpha)\cdots(\lambda+(n-1)\alpha) = \sum_{k=1}^{n} c_{nk}\lambda^{(k)}. \tag{4.3}$$

Remark 2. Numbers c_{nk} correspond to Carlitz's "degenerated" Stirling numbers $S(n,k,-\alpha)$, see [1, p. 36] for details.

If $\alpha = -1$, then c_{nk} satisfies

$$c_{n+1\,k} = c_{n\,k-1} + (k-n)c_{nk}, \tag{4.4}$$

while $c_{nk} = \delta_k^n$, where δ_k^n is the Kronecker δ-symbol. Now suppose $\alpha \in \mathbb{R}$ is arbitrary. If $f(x) = \ln x$, then computing $\delta^n f$ directly, and comparing so obtained identities, we get

$$\sum_{k=1}^{n}(-1)^k c_{nk} = (2\alpha-1)(3\alpha-1)\cdots((n-1)\alpha-1). \tag{4.5}$$

In a symilar way one can obtain

$$\sum_{k=1}^{n}(-1)^k c_{nk} k\alpha(k\alpha+1)\cdots(k\alpha+n-1) = 0, \quad \text{for} \quad n = 1,2,\dots,n-1.$$

By Newton's interpolation formula the sequence c_{nk} satisfies $c_{nk} = \Delta^k f(0)/k!$, where

$$f(x) = x(x+\alpha)\cdots(x+(n-1)\alpha) \quad \text{and} \quad \Delta f(x) = f(x+h) - f(x).$$

Further,

$$\Delta^k f(x) = \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i} f(x+ih). \tag{4.6}$$

Putting $x = 0$ and $h = 1$ into (4.6), we obtain:

$$c_{nk} = \frac{1}{k!}\sum_{k=1}^{n}(-1)^{k-i}\binom{k}{i} i(i+\alpha)(i+2\alpha)\cdots(i+(n-1)\alpha). \tag{4.7}$$

So we obtained the explicit solution of the difference equation (4.2). Observe that for $\alpha = 0$, $s_k^n = c_{nk}$, and so we have well-known representation of the Stirling numbers of the second kind:

$$s_k^n = \frac{\Delta^k 0^n}{k!} = \frac{1}{k!}\sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i} i^n.$$

5. Algebraic Theory of the Operator δ

A. Robinson proved that the theory DCF_0 of differential fields of characteristic 0 has a model completion, this is the theory of differential closed fields. For an axiomatization of these fields the reader may consult [7] for example, where are stated simple axioms by L. Blum. Thus, every differential field $\mathbf{F}$ is contained in a differential closed field $\bar{\mathbf{F}}$. This means that every differential equation with coefficients in $\bar{\mathbf{F}}$ if it has a solution in some extension of the field $\bar{\mathbf{F}}$ then it has solution already in $\bar{\mathbf{F}}$.

Our first example will concern Kurepa's function which is defined in complex domain (cf. [4])[1]:

$$K(z) = \int_0^\infty e^{-t} \frac{t^z - 1}{t - 1}\, dt, \quad \operatorname{Re} z > 0. \tag{5.1}$$

In what follows, we restrict our attention to $z \in \mathbb{R}^+$.

In the following we shall use Hölder's theorem which says that the gamma function is not a solution of any algebraic differential equation. Namely, according to this theorem, there is no differential equation of the form

$$F(x, y, \mathrm{D}y, \mathrm{D}^2 y, \ldots, \mathrm{D}^n y) = 0, \tag{5.2}$$

where $F(x, y_0, y_1, \ldots, y_n) \in \mathcal{R}$, which is satisfied by $\Gamma(x)$. So, $\Gamma(x)$ is differentially transcendental over the differential field $\mathcal{R}$ of real rational functions, i.e. $\Gamma(x) \notin \bar{\mathcal{R}}$ where $\bar{\mathcal{R}}$ is the differential closure of $\mathcal{R}$.

Theorem 1. *Kurepas function $K(x)$ is differentially transcendental over $\mathcal{R}$, namely, $K(x)$ is not a solution of any differential equation of the form* (5.2).

Proof. First let us observe that $K(x)$ satisfies

$$K(x+1) - K(x) = \Gamma(x). \tag{5.3}$$

If we assume that $K(x)$ is differential algebraic over $\mathcal{R}$, then $K(x) \in \bar{\mathcal{R}}$ and $K(x)$ satisfies an equation of the form (5.2) (for $y = K(x)$), and then $K(x+1)$ satisfies $F(x+1, y, \mathrm{D}y, \mathrm{D}^2 y, \ldots, \mathrm{D}^n y) = 0$, i.e. $K(x+1) \in \bar{\mathcal{R}}$ also. As $\bar{\mathcal{R}}$ is a field, it follows that $K(x+1) - K(x)$ belongs to $\bar{\mathcal{R}}$, and so, by (5-3),

[1] This function is connected with famous Kurepa's left-factorial hypothesis. For an overview it's properties see [3].

we would have $\Gamma(x) \in \mathcal{R}$. But this is a contradiction to Hölder's theorem, thus $K(x)$ is differentially transcendental over $\mathcal{R}$. $\square$

Related to this function we have the following questions:

Question 1. If $\mathcal{R}(\Gamma)$ is the differential field obtained adjoining $\Gamma(x)$ to $\mathcal{R}$, is $K(x)$ transcendental over $\mathcal{R}$, that is, is it true that $K(x) \notin \bar{\mathcal{R}}(\Gamma)$, where $\bar{\mathcal{R}}(\Gamma)$ is the differential closure of $\mathcal{R}$?

Observe that elementary functions, as $\sin(x), \cos(x), e^x, \ln(x)$ belong to $\bar{\mathcal{R}}$, as all these functions are solutions of algebraic differential equations. G. V. Milovanović introduced (cf. [5]) a sequence of functions satisfying:

$$K_0(x) = \Gamma(x), \;\; K_1(x) = K(x), \quad K_{n+1}(x+1) - K_{n+1}(x) = K_n(x), \quad n \in N.$$

As in the case of Kurepa's function $K(x)$, by use of a simple induction argument one can show that all functions $K_n(x)$ are transcendental over $\mathcal{R}$.

Question 2. What is the transcendental rank $\rho\mathcal{S}$ over differential field $\mathcal{R}$ where $\mathcal{S} = \mathcal{R}(K_0, K_1, \dots)$? Is $\rho\mathcal{S} = \infty$?

Now, let $\mathbf{F}_{\mathrm{D}} = (F, +, \cdot, \mathrm{D}, 0, 1)$ be a differential field, and $\mathbf{F}_\delta = (F, +, \cdot, \delta, 0, 1)$ where $\delta = \theta \cdot \mathrm{D}$, $\theta \in \mathbf{F}$, $\theta \neq 0$. Further, let $\bar{\mathbf{F}}_{\mathrm{D}}$ and $\bar{\mathbf{F}}_\delta$ be respectively their differential closures. Let us remind that $\tau\colon F_\delta \to F_{\mathrm{D}}$ is an isomorphism if τ satisfies:

$$\tau(x+y) = \tau x + \tau y, \;\; \tau(xy) = \tau x \tau y, \;\; \tau(\delta x) = \mathrm{D}\tau(x), \;\; \tau(0) = 0, \tau(1) = 1.$$

Theorem 2. *Fields $\bar{\mathbf{F}}_{\mathrm{D}}$ and $\bar{\mathbf{F}}_\delta$ are isomorphic, i.e. $\bar{\mathbf{F}}_{\mathrm{D}} \cong \bar{\mathbf{F}}_\delta$.*

Proof. If $a \in \bar{F}_\delta$ then a is a solution of an algebraic differential equation $\mathcal{E}(\delta)$ of the form (5.2) in respect to the operator δ. By (2.1), we can substitute in this equation operator δ with D, and we shall obtain again an algebraic differential equation $\mathcal{E}'(\mathrm{D})$ but now in respect to D. Then a is a solution of this equation, hence $a \in \bar{F}_{\mathrm{D}}$. So we proved that $\bar{\mathbf{F}}_\delta \subseteq \bar{\mathbf{F}}_{\mathrm{D}}$. On the other hand, $\mathrm{D} = \theta^{-1}\delta$, so we may apply a symmetrical argument, hence $a \in \bar{F}_{\mathrm{D}}$ implies $a \in \bar{F}_\delta$, i.e. $\bar{\mathbf{F}}_{\mathrm{D}} \subseteq \bar{\mathbf{F}}_\delta$. Therefore, we proved $\bar{\mathbf{F}}_{\mathrm{D}} = \bar{\mathbf{F}}_\delta$, and so $\bar{\mathbf{F}}_{\mathrm{D}} \cong \bar{\mathbf{F}}_\delta$. $\square$

It should be observed that it is not necessary $\mathbf{F}_{\mathrm{D}} \cong \mathbf{F}_\delta$. For example, if $\mathbf{F} = \mathbf{R}[x]$, D is the ordinary differentiation operator and $\delta = x\mathrm{D}$, then the equation $\delta y = y$ has a solution in $\mathbf{F}_\delta$, $y = x$, while the equation $\mathrm{D}y = y$ has no solution in $\mathbf{F}_{\mathrm{D}}$. Hence $\mathbf{F}_{\mathrm{D}} \ncong \mathbf{F}_\delta$. By the previous theorem $\bar{\mathbf{F}}_{\mathrm{D}} \cong \bar{\mathbf{F}}_\delta$, and in fact we can produce an explicit isomorphism $\tau : \bar{\mathbf{F}}_\delta \cong \bar{\mathbf{F}}_{\mathrm{D}}$. We can define τ by $\tau : f \to f \circ g$, $f \in \bar{F}$, where $g(x) = e^x$ and $\circ$ is the composition of functions. Observe that this isomorphism corresponds to the transformation $x = e^z$ in the algorithm of solving of Cauchy linear differential equations. This observation give us a new, algebraic insight into the classical method of solving Cauchy and similar types (e.g. Legendre linear equation) of differential equations. We shall give an illustration by example:

Example. Consider $x^3y''' + 3x^2y'' - 2xy' + 2y = 0$.

By (3.5) this equation is equivalent to

$$(\delta(\delta-1)(\delta-2) + 3\delta(\delta-1) - 2\delta + 2)y = 0,$$

i.e. to the equation $(\delta^3 - 3\delta + 2)y = 0$ in $\bar{\mathbf{F}}_\delta$. The corresponding equation $(\mathrm{D}^3 - 3\mathrm{D} + 2)y = 0$ in $\bar{\mathbf{F}}_\mathrm{D}$ has general solution $c_1h_1 + c_2h_2 + c_3h_3$, where $h_1(x) = e^x,\ h_2(x) = xe^x,\ h_3(x) = e^{-2x}$. As $\tau\colon \bar{\mathbf{F}}_\delta \cong \bar{\mathbf{F}}_\mathrm{D}$, and τ^{-1} is given by $\tau^{-1}\colon f \to f \circ g^{-1}$ (here $g^{-1}(x) = \ln x$), it follows that

$$\tau^{-1}(c_1h_1 + c_2h_2 + c_3h_3) = c_1h_1 \circ g^{-1} + c_2h_2 \circ g^{-1} + c_3h_3 \circ g^{-1}$$

is the general solution of $(\delta^3 - 3\delta + 2)y = 0$ in $\bar{\mathbf{F}}_\delta$, and so the solution of the starting equation is $y = c_1x + c_2x\ln x + c_3x^{-2}$.

Accordingly, one should expect that to standard methods of solving differential equations which are done by "properly chosen transformations of the independent variable" correspond in fact constructions of an isomorphism between $\bar{\mathbf{R}}_\mathrm{D}[x]$ and $\bar{\mathbf{R}}_\delta[x]$ for properly chosen differential operators δ.

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