INTUITIONISTIC AND CLASSICAL SATISFIABILITY IN KRIPKE MODELS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. A class $P^*$ of formulas was defined in [4] which whenever satisfied in a classical structure associated with a node of a Kripke model must also be forced at that node. Here we define a dual class $R$ of formulas which whenever forced at a node of a Kripke model must be satisfied in the classical structure associated with that node.

1. Introduction

A Kripke model for intuitionistic logic (or for some theory based on intuitionistic logic) may be regarded as a partially ordered collection of classical structures for the same non-logical language, where the partial ordering is the relation positive submodel. For such structures, a notion of forcing at a node ($t \models \varphi$), one point in that partial order, is defined by induction on the complexity of formulas, starting with identifying forcing for atomic formulas with (classical) satisfaction in the corresponding classical structure $A_t \models \varphi$.

The inductive clauses for $\lor$, $\land$ and $\exists$ appear the same as in the classical case (e.g., $t \models \varphi \lor \psi$ iff $t \models \varphi$ or $t \models \psi$), while the definitions for $\rightarrow$, $\neg$ and $\forall$ require the knowledge of what happens at the nodes above (e.g., $t \models \neg \varphi$ iff for all $t'$ such that $t \leq t'$, $t' \not\models \varphi$). A natural question arises then of the relation between forcing at a node ($t \models \varphi$) and satisfaction in the classical structure associated with that node ($A_t \models \varphi$). The general question of the relation between classical and intuitionistic theoremhood and derivability has been discussed extensively, mostly by proof-theoretical methods, from the earliest days (for survey see [6], section 2.3. or [1], section 81.).
an intuitionistic theory (i.e. the set of its consequences) is in an obvious way a subtheory of its classical counterpart, it was shown, using translations defined by Gödel and others, that a classical theory can be embedded into the "negative fragment" of the corresponding intuitionistic theory (i.e., the fragment consisting of formulas without \( \lor \) and \( \exists \), with each of atomic subformulas occurring only in a negative context). For particular theories a number of stronger results was proved (e.g., HA and PA have the same \( \Pi^0_2 \) theorems). For the question at hand, some results were proved in \([3]\) and \([4]\). It was shown that forcing and (local) satisfiability coincide exactly for the formulas which are intuitionistically equivalent to positive formulas (i.e., formulas containing only \( \lor \), \( \land \) and \( \exists \)). It was also shown that one implication \((A_t \models \varphi \Rightarrow t \models \varphi)\) holds for formulas \(\varphi\) for which there is some positive formula \(\psi\), classically equivalent to it but intuitionistically implying it. In this paper we describe a class of formulas for which the opposite implication holds \((t \models \varphi \Rightarrow A_t \models \varphi)\).

2. Preliminaries

We define a Kripke model for a language \(L\) to be a structure

\[\mathcal{M} = ((T, 0, \leq); A_t : t \in T)\]

where \((T, 0, \leq)\) is a partially ordered set with the least element 0 and \(A_t\) for \(t \in T\) are classical structures for the language \(L\) satisfying the condition, for \(s, t \in T\):

\[s \leq t \quad \text{implies} \quad A_s \subseteq^+ A_t\]

where \(\subseteq^+\) denotes the relation of being a positive submodel: the universe \(A_s\) of \(A_s\) is a subset of the universe \(A_t\) of \(A_t\) and the interpretation of some relation symbol in \(A_s\) is a subset of its interpretation in \(A_t\). The forcing relation is defined for \(t \in T\), \(\varphi, \psi\) formulas of \(L\) and \(a_1, a_2, \ldots, a_n \in A_t\) by:

1. \(t \models \varphi[a_1, a_2, \ldots, a_n]\) iff \(A_t \models \varphi[a_1, a_2, \ldots, a_n]\), for atomic \(\varphi\).
2. \(t \models \varphi \land \psi\) iff \(t \models \varphi\) and \(t \models \psi\).
3. \(t \models \varphi \lor \psi\) iff \(t \models \varphi\) or \(t \models \psi\).
4. \(t \models \exists x \varphi(x)[a_1, a_2, \ldots, a_n]\) iff \(A_t \models \varphi[a, a_1, a_2, \ldots, a_n]\), for some \(a \in A_t\).
5. \(t \models \varphi \rightarrow \psi\) iff for every \(t' \in T\) such that \(t \leq t'\) \((t' \nvdash \varphi \lor t' \vdash \psi)\).
6. \(t \models \neg \varphi\) iff for every \(t' \in T\) such that \(t \leq t'\) \((t' \nvdash \varphi)\).
7. \(t \models \forall x \varphi(x)\) iff for every \(t' \in T\) such that \(t \leq t'\) and for every \(a \in A_{t'}\), \((t' \vdash \varphi[a, a_1, a_2, \ldots, a_n])\).
By \( \mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n] \) we denote the (classical) satisfiability in the (classical) structure \( \mathcal{A}_t \), assuming also that all free variables of \( \varphi \) are evaluated by the elements in square brackets.

Let \( P \) be the set be the set of all formulas of \( L \) built using only connectives \( \lor, \land \) and \( \exists \). We call the formulas in \( P \) positive.

Let \( P^* \) be the set of all formulas \( \varphi \) of \( L \) such that for some \( \psi \in P \) we have \( \models \psi \rightarrow \varphi \) and \( \vdash \psi \rightarrow \varphi \) (by \( \models \) we denote the provability in classical logic while \( \vdash \) is reserved for intuitionistic logic).

In [4] the following two results have been proved.

Lemma 1. A formula \( \varphi(x_1, x_2, \ldots, x_n) \) of \( L \) is intuitionistically equivalent to a positive formula if and only if for any Kripke model \( M = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle \), any \( t \in T \) and any \( a_1, a_2, \ldots, a_n \in \mathcal{A}_t \) we have

\[
\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n] \iff t \vdash \varphi[a_1, a_2, \ldots, a_n].
\]

Lemma 2. \( \varphi \in P^* \) if and only if for any Kripke model \( M = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle \), any \( t \in T \) and any \( a_1, a_2, \ldots, a_n \in \mathcal{A}_t \) we have

\[
\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n] \implies t \vdash \varphi[a_1, a_2, \ldots, a_n].
\]

3. Results

Definition 1. Let \( R_0 = P \cup \{ \neg \varphi : \varphi \in P^* \} \). If \( R_n \) is already defined, let \( R_{n+1} \) be the smallest set of formulas satisfying the following conditions:

1. \( R_n \subseteq R_{n+1} \).
2. If \( \varphi \in P^* \) and \( \psi \in R_n \) then \( (\varphi \lor \psi) \in R_{n+1} \).
3. If \( \varphi, \psi \in R_n \) then \( (\varphi \land \psi), (\varphi \lor \psi), \forall x \varphi, \exists x \varphi \) are in \( R_{n+1} \).

Finally, let \( R_\omega = \bigcup_{n \in \omega} R_n \).

Theorem 1. If \( \varphi(x_1, x_2, \ldots, x_n) \) is a formula in \( R_\omega \) then for any Kripke model in the appropriate language \( M = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle \), any \( t \in T \) and any \( a_1, a_2, \ldots, a_n \in \mathcal{A}_t \)

\[
t \vdash \varphi[a_1, a_2, \ldots, a_n] \implies \mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n].
\]

Proof. Proof is by induction on the construction of \( R_\omega \). If \( \varphi \in R_0 \) it means \( \varphi \in P \) or \( \varphi = \neg \psi \) for some \( \psi \in P^* \). Assume \( \varphi \in P \) and \( t \vdash \varphi \). By Lemma 1., we immediately get \( \mathcal{A}_t \models \varphi \). Assume now \( \varphi = \neg \psi, \psi \in P^* \).
and \( t \vDash \neg \psi \). This implies \( t \not\vDash \psi \) and by Lemma 2. we have \( \mathcal{A}_t \not\vDash \psi \) and thus \( \mathcal{A}_t \models \neg \psi \). Suppose that the theorem holds for formulas in \( R_n \), let \( \varphi \in R_{n+1} \setminus R_n \) and let \( t \vDash \varphi \). There are five cases:

(i) \( \varphi = \psi \to \chi \) where \( \psi \in P^* \) and \( \chi \in R_n \). \( t \vDash \psi \to \chi \) implies \( t \not\vDash \psi \) or \( t \vDash \chi \). If \( t \not\vDash \psi \), by Lemma 1. we have \( \mathcal{A}_t \not\vDash \psi \), and if \( t \vDash \chi \) we have \( \mathcal{A}_t \models \chi \), by the induction hypothesis. In either case \( \mathcal{A}_t \models \psi \to \chi \).

(ii) The other four cases follow from the definition of forcing and induction hypothesis.

**Definition 2.** \( R = \{ \varphi: \text{for some } \psi \in R_\omega, \psi \leftarrow \varphi \text{ and } \vdash \varphi \rightarrow \psi \} \).

**Corollary 1.** If \( \varphi(x_1, x_2, \ldots, x_n) \in R \) then for any Kripke model in the appropriate language \( \mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle \), any \( t \in T \) and any \( a_1, a_2, \ldots, a_n \in \mathcal{A}_t \)

\[
\begin{align*}
\text{if } t \vDash \varphi[a_1, a_2, \ldots, a_n] & \text{ implies } \mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n].
\end{align*}
\]

**Proof.** Assume \( t \vDash \varphi \) and \( \psi \in R_\omega \) be such that \( \psi \leftarrow \varphi \) and \( \vdash \varphi \rightarrow \psi \). Then \( t \vDash \psi \) and by Theorem 1. we get \( \mathcal{A}_t \models \psi \) which means \( \mathcal{A}_t \models \varphi \) since \( \psi \leftarrow \varphi \).

**Corollary 2.** Let \( \Gamma \) be an intuitionistic theory with a set of axioms from \( R \) and let \( \varphi \) be a sentence from \( P^* \). Then \( \Gamma \models \varphi \) implies \( \Gamma \vdash \varphi \).

**Proof.** Let \( \mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle \) be a Kripke model for \( \Gamma \). This means that \( 0 \vDash \psi \) for every axiom \( \psi \) of \( \Gamma \). Since \( \psi \in R \) we have \( \mathcal{A}_0 \models \Gamma \) and by classical completeness theorem we get \( \mathcal{A}_0 \models \varphi \). As \( \varphi \in P^* \), by Lemma 2. we get \( 0 \vDash \varphi \). Using the strong completeness theorem of intuitionistic logic for Kripke models, we obtain \( \Gamma \vdash \varphi \).

**Corollary 3.** If \( \varphi \) is a sentence from \( R \) and \( \psi \) is a sentence from \( P^* \) then \( \models \varphi \rightarrow \psi \) implies \( \vdash \varphi \rightarrow \psi \).

**Proof.** Trivial consequence of Corollary 2. and deduction theorem.

**References**


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