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INTUITIONISTIC AND CLASSICAL SATISFIABILITY IN KRIPKE MODELS

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. A class P^* of formulas was defined in [**4**] which whenever satisfied in a classical structure associated with a node of a Kripke model must also be forced at that node. Here we define a dual class R of formulas which whenever forced at a node of a Kripke model must be satisfied in the classical structure associated with that node.

1. Introduction

A Kripke model for intuitionistic logic (or for some theory based on intuitionistic logic) may be regarded as a partially ordered collection of classical structures for the same non-logical language, where the partial ordering is the relation positive submodel. For such structures, a notion of forcing at a node ($t \Vdash \varphi$), one point in that partial order, is defined by induction on the complexity of formulas, starting with identifying forcing for atomic formulas with (classical) satisfaction in the corresponding classical structure $\mathcal{A}_t \models \varphi$. The inductive clauses for $\vee$, $\wedge$ and $\exists$ appear the same as in the classical case (e.g. $t \Vdash \varphi \vee \psi$ iff $t \Vdash \varphi$ or $t \Vdash \psi$), while the definitions for $\rightarrow$, $\neg$ and $\forall$ require the knowledge of what happens at the nodes above (e.g., $t \Vdash \neg\varphi$ iff for all t' such that $t \leq t'$, $t' \nVdash \varphi$). A natural question arises then of the relation between forcing at a node ($t \Vdash \varphi$) and satisfaction in the classical structure associated with that node ($\mathcal{A}_t \models \varphi$). The general question of the relation between classical and intuitionistic theoremhood and derivability has been discussed extensively, mostly by proof-theoretical methods, from the earliest days (for survey see [**6**], section 2.3. or [**1**], section 81.). While

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an intuitionistic theory (i.e. the set of its consequences) is in an obvious way a subtheory of its classical counterpart, it was shown, using translations defined by Gödel and others, that a classical theory can be embedded into the "negative fragment" of the corresponding intuitionistic theory (i.e., the fragment consisting of formulas without $\vee$ and $\exists$, with each of atomic subformulas occurring only in a negative context). For particular theories a number of stronger results was proved (e.g., HA and PA have the same Π^0_2 theorems). For the question at hand, some results were proved in [**3**] and [**4**]. It was shown that forcing and (local) satisfiability coincide exactly for the formulas which are intuitionistically equivalent to positive formulas (i.e., formulas containing only $\vee$, $\wedge$ and $\exists$). It was also shown that one implication ($\mathcal{A}_t \models \varphi \;\Rightarrow\; t \Vdash \varphi$) holds for formulas φ for which there is some positive formula ψ, classically equivalent to it but intuitionistically implying it. In this paper we describe a class of formulas for which the opposite implication holds ($t \Vdash \varphi \;\Rightarrow\; \mathcal{A}_t \models \varphi$).

2. Preliminaries

We define a Kripke model for a language L to be a structure

$$\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle$$

where $(T, 0, \leq)$ is a partially ordered set with the least element 0 and $\mathcal{A}_t$ for $t \in T$ are classical structures for the language L satisfying the condition, for $s, t \in T$:

$$s \leq t \quad \text{implies} \quad \mathcal{A}_s \subseteq^+ \mathcal{A}_t$$

where $\subseteq^+$ denotes the relation of being a positive submodel: the universe A_s of $\mathcal{A}_s$ is a subset of the universe A_t of $\mathcal{A}_t$ and the interpretation of some relation symbol in $\mathcal{A}_s$ is a subset of its interpretation in $\mathcal{A}_t$. The forcing relation is defined for $t \in T$, φ, ψ formulas of L and $a_1, a_2, \ldots, a_n \in A_t$ by:

1° $t \Vdash \varphi[a_1, a_2, \ldots, a_n]$ iff $\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n]$, for atomic φ.
2° $t \Vdash \varphi \wedge \psi$ iff $t \Vdash \varphi$ and $t \Vdash \psi$.
3° $t \Vdash \varphi \vee \psi$ iff $t \Vdash \varphi$ or $t \Vdash \psi$.
4° $t \Vdash \exists x \varphi(x)[a_1, a_2, \ldots, a_n]$ iff $\mathcal{A}_t \models \varphi[a, a_1, a_2, \ldots, a_n]$, for some $a \in A_t$.
5° $t \Vdash \varphi \rightarrow \psi$ iff for every $t' \in T$ such that $t \leq t'$ ($t' \nVdash \varphi$ or $t' \Vdash \psi$).
6° $t \Vdash \neg\varphi$ iff for every $t' \in T$ such that $t \leq t'$ ($t' \nVdash \varphi$).
7° $t \Vdash \forall x \varphi(x)$ iff for every $t' \in T$ such that $t \leq t'$ and for every $a \in A_{t'}$ ($t' \Vdash \varphi[a, a_1, a_2, \ldots, a_n]$).

By $\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n]$ we denote the (classical) satisfiability in the (classical) structure $\mathcal{A}_t$, assuming also that all free variables od φ are evaluated by the elements in square brackets.

Let P be the set be the set of all formulas of L built using only connectives $\vee$, $\wedge$ and $\exists$. We call the formulas in P positive.

Let P^* be the set of all formulas φ of L such that for some $\psi \in P$ we have $\underset{c}{\vdash} \psi \longleftrightarrow \varphi$ and $\vdash \psi \to \varphi$ (by $\underset{c}{\vdash}$ we denote the provability in classical logic while $\vdash$ is reserved for intuitionistic logic).

In [**4**] the following two results have been proved.

Lemma 1. *A formula $\varphi(x_1, x_2, \ldots, x_n)$ of L is intuitionistically equivalent to a positive formula if and only if for any Kripke model $\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle$, any $t \in T$ and any $a_1, a_2, \ldots, a_n \in A_t$ we have*

$$\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n] \quad \textit{iff} \quad t \Vdash \varphi[a_1, a_2, \ldots, a_n].$$

Lemma 2. *$\varphi \in P^*$ if and only if for any Kripke model $\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle$, any $t \in T$ and any $a_1, a_2, \ldots, a_n \in A_t$ we have*

$$\mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n] \quad \textit{implies} \quad t \Vdash \varphi[a_1, a_2, \ldots, a_n].$$

3. Results

Definition 1. Let $R_0 = P \cup \{\neg\varphi : \varphi \in P^*\}$. If R_n is already defined, let R_{n+1} be the smallest set of formulas satisfying the following conditions:

(1) $R_n \subseteq R_{n+1}$,
(2) if $\varphi \in P^*$ and $\psi \in R_n$ then $(\varphi \to \psi) \in R_{n+1}$,
(3) if $\varphi, \psi \in R_n$ then $(\varphi \vee \psi), (\varphi \wedge \psi), \forall x\varphi, \exists x\varphi$ are in R_{n+1}.

Finally, let $R_\omega = \cup_{n\in\omega} R_n$.

Theorem 1. *If $\varphi(x_1, x_2, \ldots, x_n)$ is a formula in R_ω then for any Kripke model in the appropriate language $\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T \rangle$, any $t \in T$ and any $a_1, a_2, \ldots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, a_2, \ldots, a_n] \quad \textit{implies} \quad \mathcal{A}_t \models \varphi[a_1, a_2, \ldots, a_n].$$

Proof. Proof is by induction on the construction of R_ω. If $\varphi \in R_0$ it means $\varphi \in P$ or $\varphi = \neg\psi$ for some $\psi \in P^*$. Assume $\varphi \in P$ and $t \Vdash \varphi$. By Lemma 1., we immediately get $\mathcal{A}_t \models \varphi$. Assume now $\varphi = \neg\psi$, $\psi \in P^*$

and $t \Vdash \neg\psi$. This implies $t \nVdash \psi$ and by Lemma 2. we have $\mathcal{A}_t \not\models \psi$ and thus $\mathcal{A}_t \models \neg\psi$. Suppose that the theorem holds for formulas in R_n, let $\varphi \in R_{n+1} \setminus R_n$ and let $t \Vdash \varphi$. There are five cases:

(i) $\varphi = \psi \to \chi$ where $\psi \in P^*$ and $\chi \in R_n$. $t \Vdash \psi \to \chi$ implies that $t \nVdash \psi$ or $t \Vdash \chi$. If $t \nVdash \psi$, by Lemma 1. we have $\mathcal{A}_t \not\models \psi$, and if $t \Vdash \chi$ we have $\mathcal{A}_t \models \chi$, by the induction hypothesis. In either case $\mathcal{A}_t \models \psi \to \chi$.

(ii) The other four cases follow from the definition of forcing and induction hypothesis.

Definition 2. $R = \{\varphi\colon$ for some $\psi \in R_\omega$, $\vdash_c \psi \longleftrightarrow \varphi$ and $\vdash \varphi \to \psi\}$.

Corollary 1. *If $\varphi(x_1, x_2, \dots, x_n) \in R$ then for any Kripke model in the appropriate language $\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T\rangle$, any $t \in T$ and any $a_1, a_2, \dots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, a_2, \dots, a_n] \quad \textit{implies} \quad \mathcal{A}_t \models \varphi[a_1, a_2, \dots, a_n].$$

Proof. Assume $t \Vdash \varphi$ and $\psi \in R_\omega$ be such that $\vdash_c \psi \longleftrightarrow \varphi$ and $\vdash \varphi \to \psi$. Then $t \Vdash \psi$ and by Theorem 1. we get $\mathcal{A}_t \models \psi$ which means $\mathcal{A}_t \models \varphi$ since $\vdash_c \psi \longleftrightarrow \varphi$.

Corollary 2. *Let Γ be an intuitionistic theory with a set of axioms from R and let φ be a sentence from P^*. Then $\Gamma \vdash_c \varphi$ implies $\Gamma \vdash \varphi$.*

Proof. Let $\mathcal{M} = \langle (T, 0, \leq); \mathcal{A}_t : t \in T\rangle$ be a Kripke model for Γ. This means that $0 \Vdash \psi$ for every axiom ψ of Γ. Since $\psi \in R$ we have $\mathcal{A}_0 \models \Gamma$ and by classical completeness theorem we get $\mathcal{A}_0 \models \varphi$. As $\varphi \in P^*$, by Lemma 2. we get $0 \Vdash \varphi$. Using the strong completeness theorem of intuitionistic logic for Kripke models, we obtain $\Gamma \vdash \varphi$.

Corollary 3. *If φ is a sentence from R and ψ is a sentence from P^* then $\vdash_c \varphi \to \psi$ implies $\vdash \varphi \to \psi$.*

Proof. Trivial consequence of Corollary 2. and deduction theorem.

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