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AGING CREEP DYNAMIC STABILITY - MATRIX EQUATION

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Abstract. *The paper deals with the matrix equation of dynamic stability of a curved rod of an aging linear viscoelastic material. The creep function is assumed to be any test function. A known method of integral equations is used as a method that produces the results under very general assumptions about curved rods and external loads As a special case the known matrix equation of dynamic stability of elastic systems is obtained.*

INTRODUCTION

Consider a curved rod under very general assumptions with respect to the form of the axis, the distribution of the masses, stiffness, and loads, and with respect to the end boundary conditions. An aging linear viscoelastic material is assumed.

The operator form of the uniaxial creep law may be formally written in the form

$$\sigma = (t, t_0) = E_0 \widetilde{R}'(t,\tau)\varepsilon(\tau, t_0)\,,$$

or (1)

$$\varepsilon = (t, t_0) = \frac{l}{E_0}\widetilde{F}'(t,\tau)\sigma(\tau, t_0)\,,$$

where $\sigma = (t,t_0)$ = stress; $\varepsilon(t,\tau)$ = strain; E_0 = instantaneous elastic modulus; t, τ and θ = time; t_0 = time of the first load application.

The known operator expressions are

$$\widetilde{R}'(t,\theta)\widetilde{F}'(\theta,\tau) = \widetilde{F}'(\tau,\theta)\widetilde{R}'(\theta,\tau) = l'(t,\tau)\,, \tag{2}$$

(see Appendix).

For an investigation of dynamic stability the following distributed external load is introduced

$$q_\lambda(s,t,t_0) = \alpha^*(t,t_0)q_{\lambda_0}(s) + \beta^*(t,t_0)\pi(t,t_0)q_{\lambda_t}(s)\,,\ \ \lambda = u, v, \tag{3}$$

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having the radial ($\lambda = u$) and the tangential ($\lambda = v$) direction, $\alpha^*(t,t_0)$ and $\beta^*(t,t_0)$ = dimensionaless load parameters, $\pi(t,t_0)$ = periodic function, s, S = coordinate along the curved rod axis. It is assumed that the load (*i*) does not change, (*ii*) changes its direction during the curved rod deformation, i.e., it is assumed that the load is of a "gravitational" or a "hydrostatical" kind.

The additional bending moments which arise during the deviation of the axis from the initial position can be taken into account by introducing and additional load. For a "gravitational" kind it is

$$q^g_{a\varphi}(s,t,t_0) = N_0(s)\tilde{\alpha}'(t,\tau)\varphi(s,\tau,t_0) + \tilde{\beta}'(t,\tau)[N_t(s,\tau,t_0)\varphi(s,\tau,t_0)], \tag{4}$$

representing distributed moments, while for a "hydrostatical" it is

$$\begin{aligned} q^h_{au}(s,t,t_0) &= N_0(s)\tilde{\alpha}'(t,\tau)\kappa(s,\tau,t_0) + \tilde{\beta}'(t,\tau)[N_t(s,\tau,t_0)\kappa(s,\tau,t_0)], \\ q^h_{av}(s,t,t_0) &= T_0(s)\tilde{\alpha}'(t,\tau)\kappa(s,\tau,t_0) + \tilde{\beta}'(t,\tau)[T_t(s,\tau,t_0)\kappa(s,\tau,t_0)], \end{aligned} \tag{5}$$

representing distributed forces in the radial and tangental direction, respectively $N_o(s)$ and $N_t(s,\tau,t_0)$ = axial forces; $T_0(s)$ and $T_t(s,\tau,t_0)$ = shear forces; $\varphi(s,\tau,t_0)$ = angle of rotation of the tangent with respect to the axis; $k(s,\tau,t_0)$ = change in the curvature of the axis. The known expressions are

$$\varphi(s,t,t_0) = \frac{\partial u(s,t,t_0)}{\partial s} + \frac{v(s,t,t_0)}{r(s)}, \tag{6}$$

and

$$\kappa(s,t,t_0) = -\frac{\partial \varphi(s,t,t_0)}{\partial s}, \tag{7}$$

where $u(s,\tau,t_0)$ and $v(s,\tau,t_0)$ = radial and tangential displacements of an arbitrary point of the axis, $r(s)$ = radius of the axis curvature [1][2].

Following the nature of the material accepted the load parameters in Eq (3) are time functions and in Eqs (4) and (5) Boltzmann's principle is applied.

The inertia forces are

$$\begin{aligned} q_{iu}(s,t,t_0) &= -m(s)\frac{\partial^2 u(s,t,t_0)}{\partial t^2} = -m(s)\ddot{u}(s,t,t_0), \\ q_{iv}(s,t,t_0) &= -m(s)\frac{\partial^2 v(s,t,t_0)}{\partial t^2} = -m(s)\ddot{v}(s,t,t_0), \end{aligned} \tag{8}$$

where $m(s)$ = mass per unit length.

For a more extensive investigation of dynamic stability of a curved rod it is expedient to use a method of integral equations.

The influence function $K_{\lambda_w}(s,S,t,\tau)$ refers to an aging linear viscoelastic curved rod. The first index denotes the displacement sought ($\lambda = u,v$) and the second denotes the unit load ($w = u,v,\varphi$). It can be shown that

$$K_{\lambda_w}(s,S,t,\tau) = F^*(t,\tau)K^e_{\lambda_w}(s,S), \tag{9}$$

where $K^e_{\lambda_w}(s,S)$ = influence function of a corresponding elastic curved rod.

On the bassis of the Boltzmann's principle the displacements $u(s,t,t_0)$ and $v(s,t,t_0)$, denoted by $Z_\lambda(s,t,t_0)$ for $\lambda = u$, v, due to distributed loads $q_w(s,\tau,t_0)$, for $w = u$, v, φ, can be obtained from

$$Z_\lambda(s,t,t_0) = \widetilde{F}'(t,\tau)\sum_w \int_L K^e_{\lambda_w}(s,S)q_w(S,\tau,t_0)dS\,. \tag{10}$$

Finally, the integral equations of dynamic stability can be created when the expressions for the additional load and the inertia forces are substituted into Eq (10), containing the usual assumptions applied when the dynamic stability of an elastic rod is investigated [1].

MATRIX EQUATION OF A "GRAVITATIONAL" LOAD

Consider a "gravitational" load Eq (4) and form the system of homogeneous integro-differential equations

$$\begin{aligned} &u(s,t,t_0) + \widetilde{F}'(t,\tau)\int_L K^e_{uu}(s,S)m(S)\ddot{u}(S,\tau,t_0)dS + \\ &\quad - \widetilde{F}'(t,\tau)\int_L K^e_{uv}(s,S)m(S)\ddot{v}(S,\tau,t_0)dS - \\ &\quad - \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{u\varphi}(s,S)N_0(S)\varphi(S,\tau,t_0)dS - \\ &\quad - \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{u\varphi}(s,S)N_t(S,\tau,t_0)\varphi(S,\tau,t_0)dS = 0\,, \end{aligned} \tag{11a}$$

$$\begin{aligned} &v(s,t,t_0) + \widetilde{F}'(t,\tau)\int_L K^e_{vu}(s,S)m(S)\ddot{u}(S,\tau,t_0)dS + \\ &\quad - \widetilde{F}'(t,\tau)\int_L K^e_{vv}(s,S)m(S)\ddot{v}(S,\tau,t_0)dS - \\ &\quad - \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{v\varphi}(s,S)N_0(S)\varphi(S,\tau,t_0)dS - \\ &\quad - \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{v\varphi}(s,S)N_t(S,\tau,t_0)\varphi(S,\tau,t_0)dS = 0\,. \end{aligned} \tag{11b}$$

Keeping in mind Eq (6) we can see that the above equations are not independent. For creating the matrix equation of dynamic stability it is enough to consider one of them only. Let us keep our attention on the first equation.

Leaving out the detailed derivation which is basically analogous to that for an elastic curved rod, we cite only the main ideas.

Assume that the solution of a problem concerning the free vibrations of an elastic curved rod is known, i.e., that the system of eigenfunctions $U_l(s)$ and $V_l(s)$ are known and that they constitute an orthonormal system.

$$\int_L m(s)[U_l(s)U_k(s) + V_l(s)V_k(s)]ds = \delta_{lk}\,, \quad \delta_{lk} = \begin{cases} 0 & l \neq k \\ 1 & l = k \end{cases}. \tag{12}$$

The corresponding eigenvalues ω_l^2 are also known. The kernels, i.e., the deflection influence functions can be expressed in terms of eigenfunctions by

$$K_{uu}^e(s,S) = \sum_l \frac{1}{\omega_l^2} U_l(s) U_l(S)\,,$$
$$K_{uv}^e(s,S) = \sum_l \frac{1}{\omega_l^2} U_l(s) V_l(S)\,, \tag{13}$$

and

$$K_{u\varphi}^e(s,S) = \sum_l \frac{1}{\omega_l^2} U_l(s) \Phi_l(S)\,, \tag{14}$$

where

$$\Phi_l(s) = \frac{dU_l(s)}{ds} + \frac{V_l(s)}{r(s)}\,. \tag{15}$$

In the series

$$u(s,t,t_0) = \sum_l f_l(t,t_0) U_l(s)\,,$$
$$v(s,t,t_0) = \sum_l f_l(t,t_0) V_l(s)\,, \tag{16}$$

and

$$\varphi(s,t,t_0) = \sum_l f_l(t,t_0) \Phi_l(s)\,, \tag{17}$$

the time functions $f_l(t,t_0)$ are unknown functions.

Concerning the convergence of the series above see [1].

Following the well known procedure the matrix elements can be found

$$a_{lk}^g = \frac{1}{\omega_l^2} \int_L N_0(s) \Phi_l(s) \Phi_k(s) ds\,,$$
$$b_{lk}^g(\tau,t_0) = \frac{1}{\omega_l^2} \int_L N_t(s,\tau,t_0) \Phi_l(s) \Phi_k(s) ds\,, \tag{18}$$

and

$$c_{lk} = \frac{1}{\omega_l^2} \delta_{lk}\,, \tag{19}$$

Introduce the matrices A^g, $B^g(t,\ t_o)$ and C composed of the elements in Eqs (18) and (19)

$$A^g = \| a_{lk}^g \|_{n,n}\,, \quad B^g(\tau,t_0) = \| b_{lk}^g(\tau,t_0) \|_{n,n} \tag{20}$$

and

$$C = \| c_{lk} \|_{n,n} \tag{21}$$

representing a diagonal matric, a unit matrix E as well as the n-dimensional vector

$$f(t,t_0) = \| f_1(t,t_0), f_2(t,t_0), \ldots, f_n(t,t_0) \|\,. \tag{22}$$

Then we arrive at the matrix equation

$$
\begin{aligned}
&Ef(t,t_0) + C\widetilde{F}'(t,\tau)\ddot{f}(\tau,t_0) - A^g \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau) f(\tau,t_0) - \\
&\quad - \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau) B^g(\tau,t_0) f(\tau,t_0) = 0 .
\end{aligned}
\tag{23}
$$

MATRIX EQUATION OF A "HYDROSTATICAL" LOAD

For a "hydrostatical" load, Eq (5) is used to form the following system of homogeneous integrodifferential equations

$$
\begin{aligned}
u(s,t,t_0) &+ \widetilde{F}'(t,\tau)\int_L K^e_{uu}(s,S)m(S)\ddot{u}(S,\tau,t_0)dS \\
&+ \widetilde{F}'(t,\tau)\int_L K^e_{uv}(s,S)m(S)\ddot{v}(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{uu}(s,S)N_0(S)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{uv}(s,S)T_0(S)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{uu}(s,S)N_t(S,\tau,t_0)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{uv}(s,S)T_t(S,\tau,t_0)\kappa(S,\tau,t_0)dS = 0 ,
\end{aligned}
\tag{24a}
$$

$$
\begin{aligned}
v(s,t,t_0) &+ \widetilde{F}'(t,\tau)\int_L K^e_{vu}(s,S)m(S)\ddot{u}(S,\tau,t_0)dS \\
&+ \widetilde{F}'(t,\tau)\int_L K^e_{vv}(s,S)m(S)\ddot{v}(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{vu}(s,S)N_0(S)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)\int_L K^e_{vv}(s,S)T_0(S)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{vu}(s,S)N_t(S,\tau,t_0)\kappa(S,\tau,t_0)dS \\
&- \widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)\int_L K^e_{vv}(s,S)T_t(S,\tau,t_0)\kappa(S,\tau,t_0)dS = 0 .
\end{aligned}
\tag{24b}
$$

Substituting Eqs (13), (16) and

$$
\kappa(s,\tau,t_0) = \sum_l f_l(\tau,t_0)K_l(s) ,
\tag{25}
$$

where

$$
K_l(s) = -\frac{d\Phi_l(s)}{ds} ,
\tag{26}
$$

see Eq (15), we obtain the matric elements

$$a_{lk}^h = \frac{1}{\omega_l^2} \int_L [N_0(s)U_l(s) + T_0(s)V_l(s)]K_k(s)ds\,,$$
$$b_{lk}^h = \frac{1}{\omega_l^2} \int_L [N_t(s,\tau,t_0)U_l(s) + T_t(s,\tau,t_0)V_l(s)]K_k(s)ds\,, \tag{27}$$

and C_{lk}, *Eq* (19), i.e. the matrices

$$A^h = \| a_{lk}^h \|_{n,n}\,, \quad B^h(\tau,t_0) = \| b_{lk}^h(\tau,t_0) \|_{n,n}\,, \tag{28}$$

and C, Eq (21), as well as the n-dimensional vector f, Eq (22). Finally, we develop the matrix equation

$$Ef(t,t_0) + C\widetilde{F}'(t,\tau)\ddot{f}(\tau,t_0) - A^h\widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)f(\tau,t_0)$$
$$-\widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)B^h(\tau,t_0)f(\tau,t_0) = 0\,. \tag{29}$$

UNIQUE FORM OF MATRIX EQUATIONS

The matrix equations concerning a "gravitational" and a "hydrostatical" kind of the external load, Eqs (23) and (29), differ only in elements of matrices A^g and A^h, $B^g(\tau,t_0)$ and $B^h(\tau,t_0)$. They can be written in a unique form

$$Ef(t,t_0) + C\widetilde{F}'(t,\tau)\ddot{f}(\tau,t_0) - A\widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\tau)f(\tau,t_0)$$
$$-\widetilde{F}'(t,\theta)\widetilde{\beta}'(\theta,\tau)B(\tau,t_0)f(\tau,t_0) = 0 \tag{30}$$

where $A = A^g, A^h$ and $B(\tau,t_0) = B^g(\tau,t_0), B^h(\tau,t_0)$. After simple transformations, using Eq (2), it can be written in the form.

$$C\ddot{f}(\tau,t_0) + [E\widetilde{I}'(t,\tau) - A\widetilde{\alpha}'(t,\tau) - \widetilde{\beta}'(\theta,\tau)B(\tau,t_0)]f(\tau,t_0)$$
$$-E[\widetilde{I}'(t,\tau) - \widetilde{R}'(t,\tau)]f(\tau,t_0) = 0\,, \tag{31}$$

keeping in mind that the elements of the matrix $B(\tau,t_0)$ are connected with a periodic component of the external load, Eq (3).

Consider special cases of the matrix equation of dynamic stability. When $\alpha^*(t,\tau) \equiv 0$, $\beta^*(t,\tau) \equiv 0$ it gives

$$C\ddot{f}(t,t_0) + E\widetilde{R}'(t,\tau)f(\tau,t_0) = 0\,, \tag{32}$$

i.e., the matric equation of free vibrations is obtained. When $\beta^*(t,\tau) \equiv 0$ and when the inertia forces are omitted it gives

$$[E\widetilde{I}'(t,\tau) - A\widetilde{\alpha}'(t,\tau)]f(\tau,t_0) - E[\widetilde{I}'(t,\tau) - \widetilde{R}'(t,\tau)]f(\tau,t_0) = 0\,, \tag{33}$$

i.e., the matric equation of static stability is obratined. The equations above refer to an aging linear viscoelastic curved rod.

Consider a linear elastic material as a special case of an aging linear viscoelastic material. Then

$$\tilde{R}'(t,\tau) = \tilde{l}\,'(t,\tau), \text{ i.e., } \tilde{F}'(t,\tau) = \tilde{l}\,'(t,t_0) = 0. \tag{34}$$

The load parameters are time-independent, i.e.,

$$\tilde{\alpha}'(t,\tau) = \alpha^* \tilde{l}\,'(t,\tau), \quad \tilde{\beta}'(t,\tau) = \beta^* \tilde{l}\,'(t,\tau), \tag{35}$$

and the coefficients in Eqs (16), (17) and (25) do not depend on the parameter to, i.e. $f_l = f_l(t)$. Then Eq (33) becomes

$$C\ddot{f}(t) + [E - \alpha^* A - \beta^* B(t)]f(t) = 0, \tag{36}$$

representing the well-known matric equation of dynamic stability of an elastic curved rod.

CONCLUSIONS

A method of integral equations produces the results under very general assumptions about the construction and the character of the external load. A matrix equation of dynamic stability depends on the rheological properties of material: for an elastic system it is a differential equation while for an aging linear viscoelastic system it is an integro-differential equation. A form of a rod axis and the other trait of a construction determine the matrix elements only.

APPENDIX

The linear integral operator $\tilde{G}'(t,\tau)$ is associated to a function $G(t,\tau)$ ($G(t,\tau) \equiv 0$ for $t < \tau$). It is defined for any function $U(t,\tau)$, $\tau \geq t_0$

$$I(t,\tau) = \int_\tau^t G(t,\theta) U(\theta,\tau) d\theta.$$

Introduce

$$\tilde{G}'(t,\tau) = \frac{\partial G(t,\tau)}{\partial \tau},$$

$$l^*(t,\tau) = H(t-\tau) = \begin{cases} 1 & \text{for } t > \tau \\ 0 & \text{for } t \leq \tau \end{cases},$$

$$\tilde{G}'(t,\tau) = \delta(t-\tau),$$

where $H(t-\tau)$ = Heaviside step function, $\delta(t-\tau)$ = Dirac function. A unit operator $\tilde{l}\,'(t,\tau)$ plays the role of unity in operator algebra

The integral of the function $G'(t,\tau)$ is defined by

$$G^*(t,\tau) = \tilde{G}' l^* = \int_\tau^t \frac{\partial G(t,\theta)}{\partial \theta} H(\theta - \tau) d\theta = G(t,t) - G(t,\tau), \quad G'(t,\tau) \neq l'(t,\tau), \quad \tau \geq t_0$$

$F^*(t,\tau)$ = creep function
$R^*(t,\tau)$ = relaxation function
$\widetilde{F}'(t,\tau)$ = creep operator
$\widetilde{R}'(t,\tau)$ = relaxation operator

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DINAMIČKA STABILNOST VISOKOELASTIČNIH SISTEMA SA STARENJEM - MATRIČNA JEDNAČINA

Vera B. Lazić

U radu je izvedena matrična jednačina dinamičke stabilnosti krivog štapa od linearno viskoelastičnog materijala s osobinom starenja. Pretpostavljena je opšta funkcija puzanja. Primenjen je poznati metod integralnih jednačina i tada rezultati obuhvataju krive štapove i spoljno opterećenje različitih karakteristika. Kao specijalan slučaj dobijena je poznata matrična jednačina dinamičke stabilnosti elastičnih sistema.