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Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: +381 18 547-095, Fax: +381 18 547-950
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STABILITY ANALYSIS OF BARS WITH ASYMMETRIC OPEN THIN WALLED CROSS-SECTIONS UNDER ECCENTRIC AXIAL THRUST

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G.I. Ioannidis, J.H. Ermopoulos, A.N. Kounadis

Departement of Structural Analysis and Steel Bridges,
National Technical University of Athens, 42 Patission Str., Athens 10682, Greece

Abstract. *The critical instability load of pin-ended bars of open asymmetric thin-walled cross-sections subjected to a slightly eccentric thrust is discussed in detail using both linear and non-linear stability analyses.The critical instability load for this case is associated with simultaneous bending and torsion, been always smaller than the classical Euler buckling loads for flexural and torsional buckling. Comparisons of the above instability load with the Euler buckling loads are presented for various bars slenderness ratios in conjunction with various thickness of the cross-sections elements. Then, important findings are obtained regarding the instability failure load. The stability study is supplemented by a postbuckling analysis leading to the conclusion that the margins of the postbuckling strength are rather limited. The proposed method is applied in the case of bars with unequal-leg angle cross-section and illustrated by a numerical example.*

1. INTRODUCTION

The use of thin-walled elements from steel or other composite materials in structural member cross-sections is steadily increased in our days. This is due to the fact that structural members with such a cross-section may have a high load-carrying capacity (compared to their self weight) combined with adequate stiffness. However, the behaviour of these bars with asymmetric or singly-symmetric cross-sections, require particular attention in their stability analysis. This topic has been the field of extensive research within the framework of the linear classical analysis. Reviewing the present state-of-the-art it is worthmentioning the classical book by Vlassov [13] as well as a large amount of studies based on linear analyses [2, 11, 3, 12]. One should also mention pertinent work concerning the lateral buckling and postbuckling behaviour of beams and

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beam-columns [10, 14, 6, 4]. Nevertheless, to the knowledge of the authors, there is a lack of references in the area of postbuckling analyses of bars with asymmetric or singly symmetric thin-walled open cross-sections under centrally or eccentrically applied thrust. A rather complex formulation based on numerical simulation and non-linear kinematic relations is presented by Chen and Atsuta [3]. In a rather recent publication [1] a numerical solution based on a modified FEM scheme deals essentially with the interaction of local, flexural and flexural-torsional modes of buckling for columns with singly-symmetric thin-walled cross-sections. In a recent work [7] Kounadis presented a comprenhesive analysis concerning bars with asymmetric thin-walled cross-sections under simultaneous bending and torsion due to central thrust. This is based on a simple postbuckling analysis, (regarding the formulation of the non-linear differential equations and the associated solution), using an efficient and simple approximate technique for solving non-linear initial and boundary-value problems [8] An application of the above method, concerning the case of an equal-leg angle is, also, recently presented [5].

2. BASIC EQUATIONS

Consider the case of a bar with arbitrary constant thin-walled cross-section of length ℓ having pinned supports at both ends, and subjected at one of its ends to a compressive eccentically applied load *P*. Instability of this bar with such a cross-section (in which the centroid *G* does not coincide with the shear center *S*), occurs through a combination of bending and torsion in case in which there is no axis of symmetrie of the cross-section. It is assumed that the bar, after buckling, can be in equilibrium in a slightly deformed configuration consisting of a translation and a rotation of the cross-section.

Let *G* the origin of a cartesian coordinates system, *x* and *y* the principal centroidal axes of the cross-section, and x_0, y_0 the coordinates of the shear center *S*. The aforementioned translation is defined by the deflections *u* (along the *x* axis) and υ (along the *y* axis) of the shear center *S* as well as the centroid *G*. The new positions of *G* and *S* are denoted by G' and S' respectively (Fig. 1). The rotation of the cross-section about this new position S' of the shear center is denoted by φ and the final position of the centroid by G''.The coordinates of G'' with respect to the initial centroidal axes *x* and y are $u + y_o\varphi$ and $\upsilon - x_0\varphi$, respectively.The axial thrust *P* is applied at a point *A* of the cross-section determined by the small eccentricities e_x and e_y (Fig. 1). The deflections corresponding to the above point *A* in the *x* and *y* directions are $u - e_x + (y_0 - e_y)\varphi$ and $\upsilon - e_y + (x_0 - e_x)\varphi$ hence the external moments, along the longitudinal axis *z* of the bar, due to the axial thurst *P*, are:

$$\left.\begin{aligned} M_x &= P[u - e_x + (y_0 - e_y)\varphi] \\ M_y &= P[v - e_y + (x_0 - e_x)\varphi] \end{aligned}\right\} \quad (1)$$

or

$$\left.\begin{aligned} M_x &= P(u - e_x + y_0^*\varphi) \\ M_y &= P(\overline{\upsilon} - e_y + x_0^*\varphi) \end{aligned}\right\} \quad (2)$$

where

$$\left.\begin{aligned} y_0^* &= y_0 - e_y \\ x_0^* &= x_0 - e_x \end{aligned}\right\} \tag{3}$$

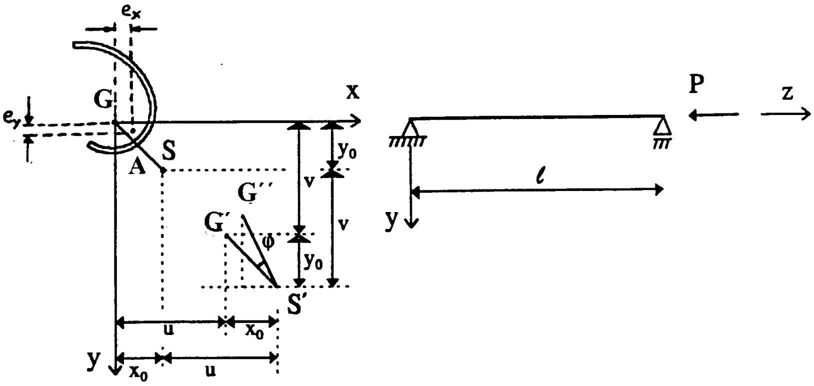

Fig. 1. Displacements of the shear center and the centroid of the cross-section

Equating the above external moments, given by eqs. (2) with the corresponding internal moments $-EI_x\upsilon''$, $-EI_y u''$ at an arbitrary point of the *z-axis* yield the following differential equations governing the displacements of the shear center axis

$$\left.\begin{aligned} EI_y u'' + P(u\text{-}e_x) &= -Py_0^*\varphi \\ \text{ÅÉ}_x\upsilon'' + P(\upsilon - e_y) &= -Px_0^*\varphi \end{aligned}\right\} \tag{4}$$

where EI_y , EI_x are the bending rigidities with respect to the principal centroidal axes x and y of the cross-section.

The third differential equation will be established by employing the condition of equality of the external torque $M_t^{\,ex}$ (for nonuniform torsion) with the internal torque

$$M_t^{\,in} = GJ\frac{d\varphi}{dz} - EC_\omega \cdot \frac{d^3\varphi}{dz^3} \tag{5}$$

where GJ and EC_ω are the torsional rigidity and the warping rigitity of the cross-section respectively. Taking into account that the effect of P on the bending stress at the initial postbuckling equilibrium path due to small eccentricities can be neglected compared to the total normal stress, the resulting differential equation is [11]

$$EC_\omega\varphi''''\text{-}\left(GJ\text{-}\frac{I_o^*}{A}P\right)\varphi'' = Px_0^*\upsilon'' - Py_0^*u'' \tag{6}$$

where

$$I_0^* = I_0 + A(\hat{a}_1 e_y + \hat{a}_2 e_x), \quad I_0 = I_x + I_y + (x_0^2 + y_0^2)A \tag{7}$$

with

$$\left.\begin{aligned} \hat{a}_1 &= \frac{1}{I_x}\left(\int_A y^3 dA + \int_A x^2 y dA\right) - 2y_0 \\ \hat{a}_2 &= \frac{1}{Iy}\left(\int_A x^3 dA + \int_A xy^2 dA\right) - 2x_0 \end{aligned}\right\} \qquad (8)$$

in which A is the cross-sectional area. Equations (4) and (6) govern the elastic instability of the bar for which Euler classical buckling is no longer valid since in such a case of bar with asymmetric cross-section instability occurs through a combination of bending and torsion. For the above pinned supports with simple rotationally restraint ends (both are free to warp and to rotate about the x and y axes but cannot deflect in the x and y directions and rotate about the z axis) the boundary conditions are

$$\left.\begin{aligned} &u(0) = \upsilon(0) = \varphi(0) = 0 \\ &u(\ell) = \upsilon(\ell) = \varphi(\ell) = 0 \\ &u''(0) = \upsilon''(0) = \varphi''(0) = 0 \\ &u''(\ell) = \upsilon''(\ell) = \varphi''(\ell) = 0 \end{aligned}\right\} \qquad (9)$$

The integration of the system of the above differential eqs (4) and (6), subjected to the boundary conditions (9) in the case $e_x = e_y = 0$, is presented in detail in ref [11]. The procedure employed is based on the Galerkin technique using the shape functions

$$u = u_0 \sin\frac{\partial z}{\ell}, \quad \upsilon = \upsilon_0 \sin\frac{\partial z}{\ell}, \quad \varphi = \varphi_0 \sin\frac{\partial z}{\ell}. \qquad (10)$$

The system of the above equations along the boundary conditions (9) and usage of the shape functions (10) leads, to an homogeneous system with respect to u_0, υ_0, φ_0 whose determinant, for a non trivial solution, must be zero. However, in the examined case, the presence of the terms Pe_x and Pe_y in eqs (4) leads to a non homogeneous system. An exact solution, if it is possible to be obtained, consists of two parts: the complementary solution and the particular one. In such a solution the evaluation of the integration constants, quite cumbersome, could be obtained only numerically [9]. Approximate solutions have been also reported [3]. In the present work, focusing attention on the initial postbuckling path, we can adopt the approximation presented in ref [11], regarding the smallness of the initial deflections due to Pe_x and Pe_y (assosiated with small eccentricities e_x and e_y). Thus, one can neglect e_x compared to u and e_y compared to υ. In this case the results associated with the central thrust, presented in ref [7] are also valid for the eccentrically applied load if x_0 and y_0 are replaced by x_0^* and y_0^* respectively, The corresponding critical load, producing instability according to a combined flexural and torsional mode, can be obtained as the smallest root of the following cubic equation

$$\frac{I_c}{I_0^*}P^3 + \left[\frac{A}{I_0^*}(P_x y_0^{*2} + P_y x_0^{*2}) - (P_x + P_y + P_t)\right]P^2 + (P_x P_y + P_y P_t + P_t P_x)P - P_x P_y P_t = 0 \qquad (11)$$

where the quantities

$$P_x = \frac{\partial^2 EI_x}{\ell^2}, \quad P_y = \frac{\partial^2 EI_y}{\ell^2}, \quad P_t = \frac{A}{I_0}\left(GJ + \frac{\partial^2}{\ell^2}EC_\omega\right) \qquad (12)$$

denote the flexural buckling (critical) loads about the x and y axes and the torsional buckling (critical) load respectively and

$$I_c = I_x + I_y \tag{13}$$

3. POSTBUCKLING ANALYSIS

In this section, attention is focused on the discussion of the bar response in the vicinity of the critical (bifurcational) state. This will allow to establish the stability or instability of the critical state [6, 4, 5]. To this end we adopt the more accurate relationship between bending moment and curvature [6] due to which

$$\left.\begin{aligned} M_y(z) &= -EI_y \frac{u''}{(1-u'^2)^{1/2}} \cong -EI_y u''\left(1+\frac{1}{2}u'^2\right) \\ M_x(z) &= -EI_x \frac{v''}{(1-\upsilon'^2)^{1/2}} \cong -EI_x \upsilon''\left(1+\frac{1}{2}\upsilon'^2\right) \end{aligned}\right\} \tag{14}$$

Subsequently, for the sake of notation simplification, only the case of the centrally applied thrust will be considered although the pertinent analysis holds also for a slightly eccentric thrust under the above assumptions. Using relations (14), eqs (4), become

$$\left.\begin{aligned} u'' + k_y^2 u &= -k_y^2 y_o \varphi - \frac{1}{2}u''u'^2 \\ \upsilon'' + k_x^2 \upsilon &= k_x^2 x_o \varphi - \frac{1}{2}\upsilon''\upsilon'^2 \end{aligned}\right\} \tag{15}$$

where

$$k_x^2 = P/EI_x\,, \quad k_y^2 = P/EI_y \tag{16}$$

Following the approximate analytic technique developed by Kounadis [8] for solving nonlinear boundary-value problems, we introduce into the Right Hand Side of eqs (15) the expressions for φ, u and v given in eqs (10). Then, it follows that

$$\left.\begin{aligned} \bar{u}'' + \bar{k}_y^2 \bar{u} &= -\bar{k}_y^2 \bar{y}_0 \varphi_0 \sin \pi\xi + \frac{\bar{u}_0^3 \pi^4}{2}\cos^2 \pi\xi \cdot \sin \pi\xi \\ \bar{\upsilon}'' + \bar{k}_x^2 \bar{\upsilon} &= \bar{k}_x^2 \bar{x}_0 \varphi_0 \sin \pi\xi + \frac{\bar{\upsilon}_0^3 \pi^4}{2}\cos^2 \pi\xi \cdot \sin \pi\xi \end{aligned}\right\} \tag{17}$$

where

$$\left.\begin{aligned} \xi = z/\ell\,,\ \bar{u} = u/\ell\,,\ \bar{\upsilon} = \upsilon/\ell\,,\ \bar{y}_0 = y_0/\ell\,,\ \bar{x}_0 = x_0/\ell \\ \bar{k}_y^2 = k_y^2\ell^2\,,\ \bar{k}_x^2 = k_x^2\ell^2\,,\ \bar{u}_0 = u_0/\ell\,,\ \bar{\upsilon}_0 = \upsilon_0/\ell \end{aligned}\right\} \tag{18}$$

Given that

$$\cos^2 \pi\xi \cdot \sin \pi\xi = \frac{1}{4}\left(\sin \pi\xi + \sin 3\pi\xi\right) \tag{19}$$

the general integrals of eqs (17) are given by

$$\left.\begin{aligned}
\bar{u}(\xi) &= C_1 \sin \bar{k}_y\xi + C_2 \cos \bar{k}_y\xi - \frac{[\bar{k}_y^2 \bar{y}_o \varphi_o - (\pi^4/8)\bar{u}_0^3]}{\bar{k}_y^2 - \pi^2}\sin \pi\xi + \\
&+ \frac{(\pi^4/8)\bar{u}_0^3}{\bar{k}_y^2 - 9\pi^2}\sin 3\pi\xi, \quad (\bar{k}_y^2 \neq \pi^2, \; 9\pi^2) \\
\bar{v}(\xi) &= D_1 \sin \bar{k}_x\xi + D_2 \cos \bar{k}_x\xi + \frac{[\bar{k}_x^2 \bar{x}_o \varphi_o + (\pi^4/8)\bar{\upsilon}_0^3]}{\bar{k}_x^2 - \pi^2}\sin \pi\xi + \\
&+ \frac{(\pi^4/8)\bar{\upsilon}_0^3}{\bar{k}_x^2 - 9\pi^2}\sin 3\pi\xi, \quad (\bar{k}_x^2 \neq \pi^2, \; 9\pi^2)
\end{aligned}\right\} \tag{20}$$

Due to the boundary conditions (9), $\bar{u}(0) = \bar{u}(1) = \bar{\upsilon}(0) = \bar{\upsilon}(1) = 0$ which yield

$$C_1 = C_2 = D_1 = D_2 = 0\,, \tag{21}$$

eqs (20) become

$$\left.\begin{aligned}
\bar{u}(\xi) &= -\left[\frac{\bar{k}_y^2 \bar{y}_0 \varphi_0 - (\pi^4/8)\bar{u}_0^3}{\bar{k}_y^2 - \pi^2}\right]\sin \pi\xi + \frac{(\pi^4/8)\bar{u}_0^3}{(\bar{k}_y^2 - 9\pi^2)}\sin 3\pi\xi, \quad (\bar{k}_y^2 \neq \pi^2, \; 9\pi^2) \\
\bar{\upsilon}(\xi) &= \left[\frac{\bar{k}_x^2 \bar{x}_0 \varphi_0 + (\pi^4/8)\bar{\upsilon}_0^3}{\bar{k}_x^2 - \pi^2}\right]\sin \pi\xi + \frac{(\pi^4/8)\bar{\upsilon}_0^3}{(\bar{k}_x^2 - 9\pi^2)}\sin 3\pi\xi, \quad (\bar{k}_x^2 \neq \pi^2, \; 9\pi^2)
\end{aligned}\right\} \tag{22}$$

Integration of eq.(6), taking into account that the integration constants due to the conditions (9) is zero, yields

$$\varphi''(\xi) + \bar{k}_t^2 \varphi(\xi) = \beta^2 \left(\bar{x}_0 \bar{\upsilon} - \bar{y}_0 \bar{u}\right) \tag{23}$$

where $$\bar{k}_t^2 = k_t^2 \ell^2 \, and \, \beta^2 = P\ell^4 / EC_\omega\,. \tag{24}$$

Introducing into the second member of eq. (23) the expressions of $\bar{u}(\xi)$ and $\bar{v}(\xi)$ from eqs(22), we obtain

$$\varphi''(\xi) + \bar{k}_t^2 \varphi(\xi) = \beta^2 [(A_1 \bar{y}_0 + A_2 \bar{x}_0)\sin \pi\xi + (A_4 \bar{x}_0 - A_3 \bar{y}_0)\sin 3\pi\xi] \tag{25}$$

where

$$\left.\begin{aligned}
A_1 &= \frac{\bar{k}_y^2 \bar{y}_0 \varphi_0 - (\pi^4/8)\bar{u}_0^3}{\bar{k}_y^2 - \pi^2}, \quad A_2 = \frac{\bar{k}_x^2 \bar{x}_0 \varphi_0 + (\pi^4/8)\bar{\upsilon}_0^3}{\bar{k}_x^2 - \pi^2} \\
A_3 &= \frac{(\pi^4/8)\bar{u}_0^3}{(\bar{k}_y^2 - 9\pi^2)}, \quad A_4 = \frac{(\pi^4/8)\bar{\upsilon}_0^3}{(\bar{k}_x^2 - 9\pi^2)}
\end{aligned}\right\} \tag{26}$$

Following the previous procedure and taking into account that the homogeneous part of eq. (25), due to φ(0) = φ(1) = 0, is zero, it follows that

$$\varphi(\xi) = \beta^2 \left[\left(\frac{A_1 \bar{y}_0 + A_2 \bar{x}_0}{(\bar{k}_t^2 - \pi^2)}\right)\sin \pi\xi + \left(\frac{A_4 \bar{x}_0 - A_3 \bar{y}_0}{(\bar{k}_t^2 - 9\pi^2)}\right)\sin 3\pi\xi\right] \tag{27}$$

Applying eqs (22) and (27) for $\hat{\imath} = 0.5$, we obtain the following system of non-linear equilibrium equations

$$\left.\begin{aligned} \bar{u}_0(\bar{k}_y^2-\pi^2)(\bar{k}_y^2-9\pi^2) &= -\bar{k}_y^2\bar{y}_0\varphi_0(\bar{k}_y^2-9\pi^2)-\pi^6\bar{u}_0^3 \\ \bar{\upsilon}_0(\bar{k}_x^2-\pi^2)(\bar{k}_x^2-9\pi^2) &= \bar{k}_x^2\bar{x}_0\varphi_0(\bar{k}_x^2-9\pi^2)-\pi^6\bar{\upsilon}_0^3 \\ \ddot{o}_0(\bar{k}_t^2-\pi^2)(\bar{k}_t^2-9\pi^2) &= \beta^2\{\bar{y}_0[A_1(\bar{k}_t^2-9\pi^2)+A_3(\bar{k}_t^2-\pi^2)]+ \\ &\quad +\bar{x}_0[A_2(\bar{k}_t^2-9\pi^2)-A_4(\bar{k}_t^2-\pi^2)]\} \end{aligned}\right\} \quad (28)$$

where A_i (i = 1, 2, 3, 4) are given in eqs (26).

Introducing the notation

$$\bar{k}_x^2 = \rho_x\beta^2\ ,\ \ \bar{k}_y^2 = \rho_y\beta^2\ ,\ \ \bar{k}_t^2 = \mu\beta^2-\upsilon\ , \quad (29)$$

where

$$\rho_x = C_\omega / I_x\ell^2\ ,\ \ \rho_y = C_\omega / I_y\ell^2\ ,\ \ \mu = I_0 / A\ell^2\ ,\ \ \upsilon = GJ\ell^2 / EC_\omega\ , \quad (30)$$

one can establish the initial postbuckling path for given values of the above parameters ρ_x, $\tilde{n}_y$, μ ,v depended on the geometrical characteristics of the cross-section and the material properties. This is achieved by solving numerically the system of eqs (28) with respect to $\bar{u}_0, \bar{\upsilon}_0, \varphi_0$ for various values of the external (dimensionless) load $\hat{a}^2$.

Clearly, the trivial solution $\bar{u}_0 = \bar{\upsilon}_0 = \varphi_0 = 0$, which defines the fundamental equilibrium path, satisfies eqs (28). The intersection of this path with the nonlinear postbuckling path associated with eqs (28) corresponds to the critical bifurcational state.

4. Application to an unequal-leg angle cross-section

(α) Linear analysis

The analysis presented above will be applied to the case of a pin-ended bar, with length ℓ and an unequal-leg angle cross-section of uniform thickness t, subjected to an eccentrically applied load P (Fig. 2). Let x, y, the principal axes of the cross-section, ξ and n the geometric axes (forming an angle ω with the principal axes), b_1 and b_2 the lengths of the legs ($b_1 < b_2$) and x_0, y_0 the coordinates of the shear center S. Assuming $t \ll b_1$ and introducing the dimensionless quantities

$$\bar{b}_1 = \frac{b_1}{\ell}\ ,\ \ \bar{b}_2 = \frac{b_2}{b_1}\ ,\ \ \bar{t} = \frac{t}{b_1} \quad (31)$$

the cross-sectional characteristics are the folowing:
cross-sectional area

$$A = (b_1+b_2)t = \ell^2\bar{b}_1^2(1+\bar{b}_2)\bar{t} \quad (32)$$

coordinates of the shear center

$$\left.\bar{x}_0 = \bar{b}_1\frac{\bar{b}_2^2\sin\omega+\cos\omega}{2(1+\bar{b}_2)}\ ,\ \ \bar{y}_0 = \bar{b}_1\frac{\bar{b}_2^2\cos\omega-\sin\omega}{2(1+\bar{b}_2)}\right\} \quad (33)$$

with

$$\tan 2\omega = \frac{6\bar{b}_2^2}{(\bar{b}_2^2 - 1)(1 + 4\bar{b}_2 + \bar{b}_2^2)} \tag{34}$$

second orders moments

$$\left.\begin{aligned} I_{x,y} &= \ell^4 \frac{\bar{t}\bar{b}_1^4}{24(1+\bar{b}_2)}\left[(1+\bar{b}_2)^4 - 6\bar{b}_2^2 \pm \sqrt{(\bar{b}_2^2 - 1)^2(1+\bar{b}_2^2 + 4\bar{b}_2)^2 + 36\bar{b}_2^4}\right] \\ I_0 &= \ell^4 \bar{b}_1^4 \bar{t}\,\frac{1+\bar{b}_2^3}{3} \end{aligned}\right\} \tag{35}$$

torsional constant

$$J = \frac{1}{3}(b_1 + b_2)t^3 = \frac{1}{3}\ell^4 \bar{b}_1^4 (1+\bar{b}_2)\bar{t}^3 \tag{36}$$

and warping constant

$$C_\omega = \frac{1}{36}(b_1^3 + b_2^3)t^3 = \frac{1}{36}\ell^6 \bar{b}_1^6 (1+\bar{b}_2^3)\bar{t}^3 \,. \tag{37}$$

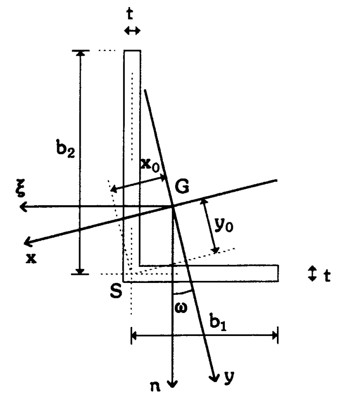

Fig. 2. Unequal-leg angle cross-section. Geometrical data.

Using eqs (35), (36) and (37) the dimensionless quantities introduced by eqs (30) can be written as follows

$$\left.\begin{aligned} &\rho_x = \bar{b}_1^2 \bar{t}^2 B_x \,, \quad \tilde{n}_y = \bar{b}_1^2 \bar{t}^2 B_y \\ &\mu = \bar{b}_1^2 \frac{1+\bar{b}_2^3}{3(1+b_2)} \,, \quad \upsilon = f\,\frac{12}{\bar{b}_1^2}\,\frac{1+\bar{b}_2}{1+\bar{b}_2^3} \end{aligned}\right\} \tag{38}$$

where

$$B_{x,y} = \frac{2}{3}\,\frac{(1+\bar{b}_2)(1+\bar{b}_2^3)}{(1+\bar{b}_2)^4 - 6\bar{b}_2^2 \pm \sqrt{(\bar{b}_2^2 - 1)^2(1+\bar{b}_2^2 + 4\bar{b}_2)^2 + 36\bar{b}_2^4}} \tag{39}$$

and

$$f = G/E \,. \tag{40}$$

When e_x, $e_y \neq 0$, I_0 in the second of eqs(35) should be replaced by I_0^*, (eqs (7), (8)) and the quantity μ in eqs(38) should be determined as $ì = É_0^* / Á\ell^2$. The integrals of eqs. (8) can be calculated as functions of the geometric characteristics of the cross-section. For example

$$\int y^3 dA = \frac{1}{4} y_0^4 t \frac{(\bar{b}_2^2 + 2\bar{b}_2 \tan\omega + \tan\omega)^4 - (\bar{b}_2^2 + 2\bar{b}_2 + \tan\omega)^4}{(\bar{b}_2^2 - \tan\omega)^4} \tag{41}$$

Using the dimensionless quantities

$$\bar{P}_x = \frac{P_x}{P} = \frac{\partial^2}{\hat{a}^2 \rho_x}, \quad \bar{P}_y = \frac{P_y}{P} = \frac{\partial^2}{\hat{a}^2 \rho_y}, \quad \bar{P}_t = \frac{P_t}{P} = \frac{\partial^2 + \upsilon}{\mu \hat{a}^2} \tag{42}$$

eq(11) can be written in dimensionless form as follows

$$\mu\rho_x r_y \frac{I_c}{I_0^*}\beta^6 + \{\pi^2[\rho_y(\bar{y}_0^2 - \bar{e}_y\bar{b}_1)^2 + \rho_x(\bar{x}_0^2 - \bar{e}_x\bar{b}_1)^2] - \mu\pi^2(\rho_x + \rho_y) -$$
$$- \rho_x\rho_y(\pi^2 + \upsilon)\}\beta^4 + \pi^2[\mu\pi^2 + (\pi^2 + \upsilon)(\rho_x + \rho_y)]\beta^2 - \pi^4(\pi^2 + \upsilon) = 0 \tag{43}$$

where

$$\bar{x}_0^* = x_0^* / \ell, \quad \bar{y}_0^* = y_0^* / \ell, \bar{e}_x = e_x / b_1, \quad \bar{e}_y = e_y / b_1 \tag{44}$$

and therefore

$$\bar{x}_0^* = \bar{x}_0 - e_x\bar{b}_1, \quad \bar{y}_0^* = \bar{y}_0 - e_y\bar{b}_1 \tag{45}$$

From equation (43) one can determine the critical instability load associated with simultaneous bending and torsion.

In Fig. 3 the variation of the dimensionless critical load $\beta^2 = P\ell^4/EC_\omega$ versus $\bar{b}_1 = b_1 / \ell$ is presented for the specific case $\bar{e}_x = \bar{e}_y = 0$ in two plots for $f = 0.3846$, various values of $\bar{b}_2$ and $\bar{t} = 0.05$, 0.10. In Fig. 4 one can see the variation of the dimensionless loads $\bar{P}_y$ and $\bar{P}_t$ against $\bar{b}_1$ for two values of the dimensionless thickness $\bar{t}$ and three values of $\bar{b}_2$.

From the above plots it is worthobserving the following:

(a) the critical instability load P (associated with simultaneous bending and torsion), is until 1.6 times smaller than the smaller of P_y and P_t

(b) for thick cross-sections $(\bar{t} = 0.20)$ and for slender bars ($\bar{b}_1 < 0.02$) the critical load of flexural-torsional buckling practically coincide with the critical (Euler) buckling load

(c) for thick cross-sections $(\bar{t} = 0.20)$ and bars with small slenderness ratio, P practically coincides with the critical load of torsional buckling for $\bar{b}_1 > 0.08$ (when $\bar{b}_2 = 2.00$), for $\bar{b}_1 > 0.11$ (when $\bar{b}_2 = 1.50$), and for $\bar{b}_1 > 0.15$ (when $\bar{b}_2 = 1.00$).

(d) for thin cross-sections $(\bar{t} = 0.05)$ the above coincidence is observed when $\bar{b}_1 > 0.05$ (for $\bar{b}_2 \geq 1.5$) and when $\bar{b}_1 > 0.08$ (for $\bar{b}_2 = 1.00$).

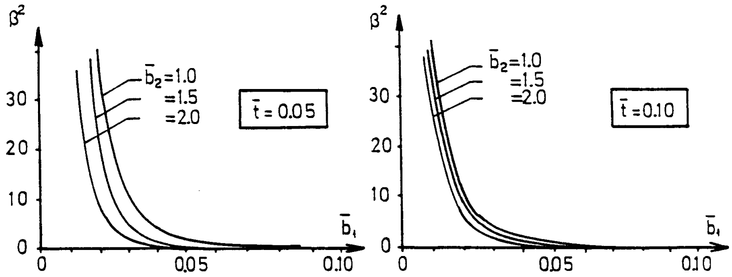

Fig. 3. Variation of the dimensionless instability load β^2 vs $\overline{b}_1$ for a bar with an unequal leg angle cross-section and various values of $\overline{b}_2$ and $\bar{t}$.

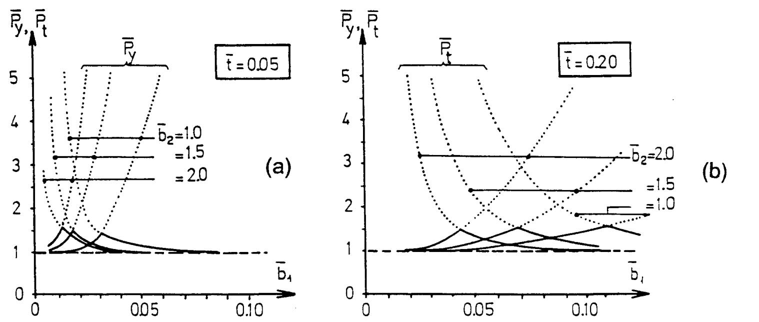

Fig. 4. Variation of $\overline{P}_1$ and $\overline{P}_t$ against $\overline{b}_1$ for a bar with an unequal leg angle cross-section and various values of $\overline{b}_2$ and $\bar{t}$.

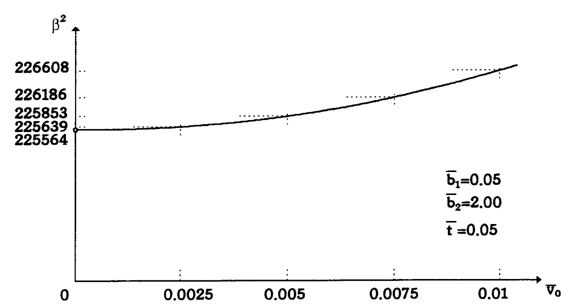

Fig. 5. Postbuckling equilibrium path for a bar with unequal-leg angle cross-section under centrally applied axial thtrust.

(b) Non linear analysis

The system of non-linear equilibrium equations for bars with asymmetric cross-sections, is associated with eqs(28).In the above system, for the specific case of a bar with an unequal-leg angle cross-section the dimensionless critical loads $\bar{k}_x^2, \bar{k}_y^2, \bar{k}_t^2$ are calculated from eqs (29) using the auxiliary quantities ρ_x, ρ_y, μ, υ obtained by eqs (38), (39). The dimensionless coordinates of the shear center $\bar{x}_0$ and $\bar{y}_0$ are, also, introduced as expressions of $\bar{b}_1, \bar{b}_2$, $\bar{t}$, using eqs (33), (34). Then, for given values of $\bar{b}_1, \bar{b}_2$, $\bar{t}$ the postbuckling path (β^2 vs $\bar{u}_0$) or (β^2 vs $\bar{\upsilon}_0$) or (β^2 vs φ_0) can be established by determining $\bar{u}_0, \bar{v}_0$ and φ_0 for several values of the postbuckling load.

Figure 5, shows an example of a postbuckling equilibrium path (β^2 vs $\bar{\upsilon}_0$) for the case of a bar with an unequal-leg angle cross-section having the following geometric characteristics: $\bar{b}_1 = 0.05, \bar{b}_2 = 2.0, \bar{t} = 0.05$. The corresponding dimensionless critical load is β^2_{cr}= 225564.

In view of the above numerical results and the established postbuckling equilibrium paths, one can conclude that the critical bifurcation state is related to a stable and symmetric branching point. Thus, the bar develops postbuckling strength and it is not sensitive to initial imperfections. Nevertheless the postbuckling paths are rather shallow and therefore the margin of the above postbuckling strength is limited.

CONCLUSIONS

The most important conclusions of the present study are the following:

(a) A simple and efficient technique for establishing the initial part of the postbuckling equilibrium path is presented, for the case of axially compressed bars with asymmetric open thin-walled cross-sections.

(b) The critical instability state, corresponding to bars with unequal-leg angle cross-section, is related to a stable and symmetric bifurcation point.

(c) The postbuckling paths, corresponding to the above case, are very shallow and therefore the postbuckling strength is limited.

(d) For bars with unequal-leg angle cross-section, the critical instability load (associated with simultaneous bending and torsion) is until 1.6 times smaller than the smaller of the critical (Euler) loads corresponding to pure flexural and torsional buckling.

(e) Areas of the geometrical data of the above bars for which the critical instability load practically coincides with the critical loads of pure flexural or pure torsional buckling, are indicated.

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ANALIZA STABILNOSTI ŠTAPOVA SA ASIMETRIČNO OTVORENIM TANKOZIDNIM PRESECIMA POD EKSCENTRIČNIM AKSIJALNIM PRITISKOM

G.I. Ioannidis, J.H. Ermopoulos, A.N. Kounadis

Opterećenje kritične nestabilnosti za zglobno oslonjene štapove otvorenih asimetričnih tankozidnih poprečnih preseka izloženih blago ekscentričnom pritisku je detaljno razmatrano korišćenjem i linerane i nelinearne analize stabilnosti. Opterećenje kritične nestabilnosti za ovaj slučaj povezano sa istovremenim (simultanim) savijanjem i uvijanjem, je uvek bilo manje od klasičnih Euler-ovih opterećenja izvijanja za savojno i torziono izvijanje. Poređenja prethodnog opterećenja nestabilnosti sa Euler-ovim opterećenjima izvijanja su prikazana za različite vitkosti štapova u sprezi sa različitom debljinom elemenata poprečnog preseka. Važna otkrića su dobijena s obzirom na opterećenje kritične nestabilnosti. Studija stabilnosti je dopunjena analizom posle izvijanja koja vodi do zaključka da su rezerve čvrstoće posle izvijanja prilično ograničene. Predložena metoda je primenjena na slučaj raznokrakih ugaonih (ili L) poprečnih preseka i ilustrovana je numeričkim primerom.