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A UNIFIED GENERALIZED VARIATIONAL PRINCIPLE OF ELASTICITY

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Abstract. *In the paper, by the semi-inverse method of establishing generalized variational principles proposed by He, a unified generalized variational principle with free parameters has been established. By specially choosing the parameters, the well-known Hu-Washizu principle, Hellinger-Reissner principle and Chien principle can be readily obtained.*

Keywords: *variational technique, Hellinger-Reissner principle, Hu-Washizu principle, semi-inverse method, trial-functional*

1. INTRODUCTION

Generally speaking, there exist two basic ways to describe a physical problem: 1) by partial differential equations (PDEs) with boundary or initial conditions (BC or IC); 2) by variational principles (VPs). PDE model requires strong local differentiability (smoothness) of the physical field, while its VP partner requires weaker local smoothness or only local integrability. For discontinuos field, the VP model will be powerfully applied. Moreover the VP model has many advantages over its PDE partner: simple and compact in form, while comprehensive in content (encompassing implicitly almost all information characterizing the problem under consideration - PDEs and natural BC/IC; capable of hinting naturally how the boundary/initial value problem should be properly posed. It is a sound theoretical foundation of the finite element method (FEM) and other direct variational methods such as Ritz's, Trefftz's and Kantorovitch's methods.

It is well known that in general, it is extremely difficult to deduce a generalized variational principle directly from its governing equations and boundary conditions or initial conditions. In the present paper, we will apply the semi-inverse method [1-9] to

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search for a unified generalized variational principle of elasticity.

2. MATHEMATICAL FORMULATION OF SMALL DISPLACEMENT [10]

1) Equilibrium conditions

$$\sigma_{ij,j} + f_i = 0 \quad (in\ \tau) \tag{1}$$

in which σ_{ij} are components of the stress tensor, $\sigma_{ij,k} = \partial\sigma_{ij} / \partial x_{k,}$ f_i represents body force, and τ is the volume of an elastic body.

2) Stress-strain relations. For nonlinear elasticity, we have

$$\frac{\partial A}{\partial e_{ij}} = \sigma_{ij} \qquad (in \quad \tau) \tag{2a}$$

or

$$\frac{\partial B}{\partial \sigma_{ij}} = e_{ij} \qquad (in \quad \tau) \tag{2b}$$

where A and B are the strain energy density and complementary energy density respectively.

3) Strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (in \quad \tau) \tag{3}$$

4) Boundary conditions for given surface displacement

$$u_i = \overline{u}_i \qquad (in \quad \Gamma_u) \tag{4}$$

5) Boundary conditions for given external force on boundary surface

$$\sigma_{ij} n_j = \overline{p}_i \qquad (in \quad \Gamma_\sigma) \tag{5}$$

where $\Gamma_u + \Gamma_\sigma = \Gamma$ covers the total boundary.

3. THE SEMI-INVERSE METHOD

The basic idea of the semi-inverse method [1–9] is to establish an energy trial-functional with an unknown function F like this (in this section the boundary integral term will not be taken into consideration)

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint L dV \tag{6a}$$

in which L is a trial-Lagrange function, and can be freely constructed, for example, we can write the following one

$$L = \sigma_{ij} e_{ij} + F \tag{6b}$$

Hereby σ_{ij}, e_{ij}, u_i are considered as independent variations, F is an unknown function.

There exist several ways to construct energy trial-functionals, details have been discussed in Refs. [1–9]. Here we will identify the unknown F step by step.

Step 1

Making the above trial-functional (6) stationary with respective to σ_{ij}

$$\delta_\sigma J = \iiint \left\{ e_{ij} + \frac{\delta F}{\delta \sigma_{ij}} \right\} \delta\sigma_{ij} dV = 0 \tag{7}$$

yields following equations

$$\delta\sigma_{ij} : \ e_{ij} + \frac{\delta F}{\delta \sigma_{ij}} = 0 \tag{8}$$

where $\delta F / \delta\sigma_{ij} = \partial F/\partial\sigma_{ij} - (\partial F/\partial\sigma_{ij,j})_{,j}$ is called functional derivative.

The above equation (8) with an unknown F is called as a trial-Euler equation, which should satisfy one of its field equations, saying the equation (1), accordingly we can set

$$e_{ij} + \frac{\delta F}{\delta \sigma_{ij}} = \eta \left\{ e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right\} \tag{9}$$

where η is a nonzero parameter

In view of (2b), the unknown F can be identified as follows

$$F = -B + \eta\sigma_{ij} \left\{ e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right\} + F_1 \tag{10a}$$

or

$$F = -B + \eta\sigma_{ij} e_{ij} + \eta\sigma_{ij,j} u_i + F_1 \tag{10b}$$

or in more general form

$$F = -B + \eta\sigma_{ij} e_{ij} - \frac{1}{2}\alpha\eta(u_{i,j} + u_{j,i})\sigma_{ij} + \beta\eta\sigma_{ij,j} u_i + F_1 \tag{10c}$$

where F_1 is a newly introduced unknown function free from σ_{ij}, α and β are constants, and $\alpha + \beta = 1$.

Substituting (10c) into the trial-Lagrange function (6b) results in a renewed one:

$$L = \sigma_{ij} e_{ij} - B + \eta\sigma_{ij} e_{ij} - \frac{1}{2}\alpha\eta(u_{i,j} + u_{j,i})\sigma_{ij} + \beta\eta\sigma_{ij,j} u_i + F_1 \tag{11}$$

Step 2

The stationary condition with respect to u_i reads

$$\delta u_i : \ (\alpha + \beta)\eta\sigma_{ij,j} + \frac{\delta F_1}{\delta u_i} = 0 \tag{12}$$

Supposing the above trial-Euler equations satisfy (1), we, therefore, can set

$$(\alpha+\beta)\eta\sigma_{ij,j}+\frac{\delta F_1}{\delta u_i}=\mu(\sigma_{ij,j}+f_i) \tag{13}$$

As mentioned before, the unknown F_1 should be free of σ_{ij} and their derivatives, so the parameter μ should be equal to η, i.e. $\mu=\eta$. The equation (13), therefore, can be rewritten down as follows

$$\frac{\delta F_1}{\delta u_i}=\eta f_i \tag{14}$$

From (14) unknown F_1 can be identified as follows

$$F_2=\eta f_i u_i+F_3 \tag{15}$$

where F_3 is newly introduced unknown function, which should be free from σ_{ij} and u_i.

Substituting (15) into the trial-Lagrange function (11) results in a renewed one which reads:

$$L=\sigma_{ij}e_{ij}-B+\eta\sigma_{ij}e_{ij}-\frac{1}{2}\alpha\eta(u_{i,j}+u_{j,i})\sigma_{ij}+\beta\eta\sigma_{ij,j}u_i+\eta f_i u_i+F_2 \tag{16}$$

Step 3

The trial-Euler equation for δe_{ij} reads

$$\delta e_{ij}:\quad \sigma_{ij}+\eta\sigma_{ij}+\frac{\delta F_2}{\delta e_{ij}}=0 \tag{17}$$

We set

$$F_2=-(1+\eta)A\,,\quad (\eta\neq-1) \tag{18}$$

so that the trial-Euler equation (17) satisfies the field equation (3).

Notice: $\eta=-1$ denotes that the constriants of stress-strain relations have not been eliminated, see the equation (31).

Finally we have following functional

$$J(\sigma_{ij},e_{ij},u_i)=\iiint L dV \tag{19a}$$

where

$$L=\sigma_{ij}e_{ij}-B-A-\eta\left\{A-f_i u_i-\sigma_{ij}e_{ij}+\frac{1}{2}\alpha(u_{i,j}+u_{j,i})\sigma_{ij}-\beta\sigma_{ij,j}u_i\right\} \tag{19b}$$

It is easy to prove that the stationary conditions of the above functional (19) satisfy the field equations (1)~(3)

[*Proof*] The stationary conditions with respect to σ_{ij}, e_{ij} and u_i can write down as follows

$$\delta\sigma_{ij}:\ e_{ij}-\frac{\partial B}{\partial\sigma_{ij}}-\eta\left\{-e_{ij}+\frac{1}{2}(\alpha+\beta)(u_{i,j}+u_{j,i})\sigma_{ij}\right\}=0 \tag{20}$$

$$\delta e_{ij}:\ \sigma_{ij}-\frac{\partial A}{\partial e_{ij}}-\eta\left\{\frac{\partial A}{\partial e_{ij}}-\sigma_{ij}\right\}=0,\quad (\eta\neq-1) \tag{21}$$

$$\delta u_i:\ -\eta\{-f_i+(\alpha+\beta)\sigma_{ij,j}\}=0 \tag{22}$$

It is obvious that the equation (22) is the field equation (1), and the equations (20) and (21) lead to the field equations (3) and (2).

4. NATURAL BOUNDARY CONDITIONS

In this section we will remove the boundary constraints by the semi-inverse method. Supposing a generalized variational principle without any constraints can be expressed as

$$J(\sigma_{ij},e_{ij},u_i) = \iiint L dV + \mathrm{IB} \tag{23a}$$

in which L is defined by (19b), IB is the boundary integral term, which can be in general expressed as

$$IB = \iint_{\Gamma_u} G_1 dS + \iint_{\Gamma_\sigma} G_2 dS \tag{23b}$$

where G_1 and G_2 are unknowns.

Making the above trial-functional (20) stationary, and applying the Green's theory, we obtain following trial-Euler equations on the boundary.

$$\delta u_i: \quad -\alpha\eta\sigma_{ij}n_j + \frac{\delta G_m}{\delta u_i} = 0 \quad (m = 1 \sim 2) \tag{24}$$

$$\delta\sigma_{ij}: \quad \beta\eta u_i n_j + \frac{\delta G_m}{\delta \sigma_{ij}} = 0 \quad (m = 1 \sim 2) \tag{25}$$

On Γ_u, we set

$$G_1 = \alpha\eta\sigma_{ij}n_j(u_i - \bar{u}_i) - \beta\eta\sigma_{ij}n_j\bar{u}_i \tag{26}$$

so that the Euler equations for δu_i and $\delta\sigma_{ij}$ satisfy and the boundary condition (7A) respectively.

By the same manipulation, on Γ_σ, we can identify the unknown G_2 as follows

$$G_2 = \alpha\eta\bar{p}_i u_i - \beta\eta u_i(\sigma_{ij}n_j - \bar{p}_i) \tag{27}$$

The subestitution of the identified unknowns $G_i\,(i = 1\sim2)$ into the equation (23) results in a required generalized variational principles

$$\begin{aligned} J(\sigma_{ij},e_{ij},u_i) &= \iiint(\sigma_{ij}e_{ij} - B - A)dV - \\ &-\eta\iiint\{A - f_i u_i - \sigma_{ij}e_{ij} + \frac{1}{2}\alpha(u_{i,j} + u_{j,i})\sigma_{ij} - \beta\sigma_{ij}u_i\}dV + \\ &+\eta\iint_{\Gamma_u}\{\alpha\sigma_{ij}n_j(u_i - \bar{u}_i) - \beta n_j\sigma_{ij}\bar{u}_i\}dS + \eta\iint_{\Gamma_\sigma}\{\alpha\bar{p}_i u_i - \beta u_i(\sigma_{ij}n_j - \bar{p}_i)\}dS \end{aligned} \tag{28}$$

The parameters α, β and η can be chosen arbitrarily ($\alpha + \beta = 1$). The presence of the free parameters offers an opportunity for the systematic derivation of energy-balanced finite elements that combine displacement and stress assumptions, details can be found in Felippa's Ref. [10].

We can obtain some famous generalized functionals by prescribing the free parameters. For example, the Chien principle [11] can be obtained by setting $\alpha = 1$, $\beta = 0$ and $\eta = 1/\lambda$.

$$J_{Chien}(\sigma_{ij}, e_{ij}, u_i) = \lambda \iiint (\sigma_{ij} e_{ij} - B - A) dV -$$
$$- \iiint \{A - f_i u_i - \sigma_{ij} e_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij}\} dV + \iint_{\Gamma_u} \{\sigma_{ij} n_j (u_i - \overline{u}_i)\} dS + \iint_{\Gamma_\sigma} \overline{p}_i u_i dS \tag{29}$$

By setting $\alpha = 1$, $\beta = 0$ and $\lambda = 0$, the well-known Hu-Washizu principle [12] can be deduced

$$J_{H-W}(\sigma_{ij}, e_{ij}, u_i) = -\iiint \{A - f_i u_i - \sigma_{ij} e_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij}\} dV +$$
$$+ \iint_{\Gamma_u} \{\sigma_{ij} n_j (u_i - \overline{u}_i) dS + \iint_{\Gamma_\sigma} \overline{p}_i u_i dS \tag{30}$$

By setting $\alpha = 0$, $\beta = 1$ and $\eta = -1$, the well-known Hellinger-Reissner principle [12] can be arrived at

$$J_{H-R}(\sigma_{ij}, u_i) = \iiint \{-B - (f_i + \sigma_{ij,j}) u_i\} dV + \iint_{\Gamma_u} \sigma_{ij} n_j \overline{u}_i dS + \iint_{\Gamma_\sigma} \{u_i (\sigma_{ij} n_j - \overline{p}_i)\} dS \tag{31}$$

in which the stress-strain relations are its variational constraints.

5. Conclusion

Hereby a unified generalized variational principle with three free parameters, without using the Lagrange multiplier method, has been successfully established by the semi-inverse method, by specially setting the parameters, we can naturally obtain the well-known Hu-Washizu principle and Hellinger-Reissner principle.

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References

1. He, J.H. (1997) *Semi-inverse Method of Establishing Generalized Variational Principles for Fluid mechanics with Emphasis on Turbomachinery Aerodynamics*, Int. J. Turbo & Jet-Engines, Vol. 14, No. 1, pp 23-28.
2. He, J. H. (1997) *Modified Lagrange multiplier Method and Generalized Variational principles in Fluid Mechanics*, J. Shanghai University (English Edition), Vol. 1, No. 2, pp. 117-122. (Find the paper in the Internet: http//www.shu.edu.cn/journal/)
3. He, J. H. (1997) *Equivalent theorem of Hellinger-Reissner and Hu-Washizu principle*, J. Shanghai University (English Edition), Vol. 1, No. 1, pp. 36-41.
4. He, J. H. (1998) *A variational theory for one-dimensional unsteady compressible flow: an image plane approach*. Applied Math. Modelling, Vol. 22, pp 395-403.
5. He, J. H. (1998) *A family of variational principles for compressible rotational blade-to-blade flow using semi-inverse method*, Interl. J, Turbo & Jet-Engines, Vol. 15, No. 2, 1998, 95-100.
6. He, J. H. (1998) *Generalized Variational Principle for compressible S2-flow in mixed-flow turbomachinery using semi-inverse method*, Interl. J, Turbo & Jet-Engines, Vol. 15, No. 2, pp 101-107.

7. He, J. H. (1998) *A Variational model for an asymptotic magnetohydrodynamic system*, Communications in Nonlinear Science & Numerical Simulation, Vol. 3, No. 3, pp 176-179 (find the paper in the Internet: http://www.nlspku.ac.cn)
8. He, J. H. (1998) *Variational approach to a strictly hyperbolic system of conservation laws with singularity*, Communication in Nonlinear Science & Numerical Simulation, Vol. 3, No. 3, 179-183.
9. He, H. H. *Generalized Hellinger-Reissner principle*, ASME J. Appl. Mech. (accepted), 1999.
10. Felippa, C. (1996) *Recent developments in parametrized variational principles for mechanics, Computational mechanics*, Vol. 18, pp. 159-174.
11. Chien, W. Z. (1983) *Method of High-Order Lagrange Multiplier and Generalized Variational Principles of Elasticity with More General Forms of Functionals*, Applied Mathematics and Mechanics, Vol. 4, No. 2, pp. 137-150.
12. Washizu, K (1982) *Variational Methods in Elasticity & Plasticity* (3rd edition), Pergamon Press

UNIFICIRANI GENERALISANI VARIJACIONI PRINCIP ELASTIČNOSTI

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U ovom radu poluinverznom metodom uspostavljanja generalisanih varijacionih principa koje je predložio He uspostavljen je unificirani generalizovani princip sa slobodnim parametrima. Posebnim biranjem parametara dobro poznati Hu-Washizu princip, Hellinger-Reissner princip i Chien princip mogu da budu lako dobijeni.

Ključne reči: *Varijaciona tehnika, Hellinger-Reissner princip, Hu-Washizu princip, poluobratna metoda, probni funkcional.*