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# THE PRINCIPLE OF ACTION

*UDC 531*

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**Abstract**. *The most common expression for action in physics and mechanics now is* $J = \int_{t_0}^{t_1} L(x, x, t)dt$ *, where L is Lagrange's function, and for integral variational principle the relation* $\delta J = 0$. *We define action as the functional* $\int_{t_0}^{t_1} A dt$ *, where A is the work of the forces along real path and emphasize clear difference between elementary work* $dA = X\,dx$ *of forces X,* $X \in R^{n+1}$*, along real displacement and elementary work* $\delta A = X\,\delta x$ *of the same forces on possible variations of coordinates x,* $x \in M^{n+1}$*. We define the principle of acton by the relation*

$$\delta \int_{t_0}^{t_1} [A(x) - A(I)]\, dt = 0 \,, \qquad (W)$$

*where A(X) is the work of all forces X and A(I) is the work of the force of intertia I and show that known variational principles of mechanics are corollaries of the relation (W).*

## 1. INTRODUCTION

In the expressions such *as under the action of the force* or *inter-action of the bodies or action equal reaction*, the term action implies the presence of the forces and their inducement rather than some particular concept of *action*. On the other hand, in analytical mechanics, theoretical physics or even mathematics, the term *action* implies a more or less accurately determined functional whose definition makes no reference of a force. For this reason it is necessary here to determine the concept of action.

The origin of the principle of least action was a paper L.S. Polak (in russian) published in the proceeding book *Variational Principles of Mechanics*. The concept of *action* can be found in the work given by Leibnitz (1669) as *action formalis*, whose

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dimension is product of mass, velocity and way (Mathematische Schriften, Harausg. von Gerhard, t. II, 1; t. III, 1860). Christian Wolff (1726) wrote: action consist of mass, velocity and space. In "Accord de differentes lois de la Nature qui avaient jusqu'ici paru incompatibles". Pierre Maupertius (1744-1746) was the first who write about the *principle of least action*. Two years later (1748) in "Reflexions sur quelques lois generales de la Nature qui s'observent dans les effects des forces quelconques", Euler found the functional form of the action as $\int Tdt$, where is $T$ kinetic energy and $t$ is time, and gave a definition of principle of least action.

Analytical form of this principle

$$\delta(M\int udt) = \delta(M\int u^2 dt) = 0\ ;\quad \delta\Sigma M_i \int u_i ds_i = 0 \tag{1}$$

where $u$ is velocity, $m$ mass of body, was given by Lagrange (1760).

Other contributions to the principle of least action were given by: Rodrigues Olinde (1816), $\int Tdt$; W.R. Hamilton (1834), $\int 2Tdt$; C. Jacobi, (1842/43), $\int Pdx_I$; J.H. Poincare (1889), et all. Jacobi wrote: "Lagrange's principle of least action is mama of the analytical mechanics". In further development of Analytical mechanics the concept of action usually accepted the functionales

$$J_1 = \int_{t_0}^{t_1} E_k dt \quad \text{and} \quad J_2 = \int_{t_0}^{t_1} Ldt \tag{2}$$

where $E_k$ is kinetic energy of mechanical system, and $L(q,q,t)$ is kinetic potential, often called lagrangian or Lagrange's function. Physics dimensions of action are $\mathrm{ML^2T^{-1}}$.

The relation $\delta J_1 = 0$ is named The principle of least action, and the relation $\delta J_2 = 0$ the principle of stationary action or Hamilton's principle.

## THE PRINCIPLE OF ACTION

For this reason it is necessary here to determine the concept of action.

**Definition 1.** The action of mechanical system is

$$J = \int_{t_1}^{t_2} A(X)dt \tag{3}$$

where

$$A = \int_s Xdx \tag{4}$$

is the active force' work ($X \in R^{n+1}$, $x \in M^{n+1}$) along real poth $s$.

**Definition 2.** Elementary work of the same forces $X$ on possible variations of coordinate $x$ is.

$$\delta A(X) = X\delta x. \tag{5}$$

The principle of the action can be determine by the relation

$$\delta\int_{t_0}^{t_1} [A(X) - A(I)]dt = 0 \tag{6}$$

where $A(I)$ is the work of inertial force.

Work of the inertial force

$$I_i = -a_{ij}(x)(D\dot{x}^j/dt) = -a_{ij}(d\dot{x}^j/dt + \Gamma_{i,jk}\dot{x}^k\dot{x}^i) \tag{7}$$

is equal to negative kinetic energy $E_k$.

Really,

$$\begin{aligned} A(I) &= \int_s I_i dx^i = -\int_s a_{ij}(x)(D\dot{x}^j/dt)dx^i = \\ &= -\int a_{ij}(D\dot{x}^j/dt)\dot{x}^i dt = -\hat{\int} a_{ij}\dot{x}^i D\dot{x}^j = -\hat{\int} D\left(\frac{1}{2}a_{ij}\dot{x}^i\dot{x}^j\right) = -E_k . \end{aligned} \tag{8}$$

where $\hat{\int}$ is tensorial integral [1] and $D\dot{x}^j = d\dot{x}^j + \Gamma_{kl}\dot{x}^k dx^l$ is natural differential of vector $\dot{x}$.

**Equivalentcy of principle of action and other variational principles of Mechanics**

1. The principle of action (6) is equivalent to Dalamber-Lagrange principle

$$[Q_i - a_{ij}(D\dot{q}^j/dt)]\delta q^j = 0, \quad i,j = 0,1,\dots,n \tag{9}$$

where $Q_i$ are generalized forces. It is easy to prove if we know that

$$\delta A(Q) := Q_i \delta q^i , \tag{10}$$

$$\delta A(I) = -\delta E_k = -\delta\left(\frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j\right) = \frac{\partial a_{ij}}{2\partial q^k}\dot{q}^i\dot{q}^j\delta q^k + a_{ij}\dot{q}^i\delta\dot{q}^j \tag{11}$$

and

$$\delta(dq/dt) = d(\delta q)/dt \quad \text{and} \quad \delta q(t_1) = 0 \;\&\; \delta q(t_2) = 0 \tag{12}$$

So, the relation (7) reduced to

$$\int [Q_i - a_{ij}(D\dot{q}^j/dt)]\delta q^i dt = 0 \tag{13}$$

and we conclude that (6) and (9) are equivalent

2. The principle of action (6) is equivalent to Hamilton-Ostrogradsky's principle

$$\int(\delta E_k + Q_i\delta q^i)dt = 0 . \tag{14}$$

It immediately follows from (5) and (9), since $\delta A(I) = -\delta E_k$.

3. In the case of the potential forces, when the work is equal to non positive potential energy, $E_q$ (6) reduces to Hamilton's principle.

$$\delta\int(E_k - E_p)dt = \delta\int L dt = 0 . \tag{15}$$

4. A difference between classical definition (2) and (3) appears in the case of inertial motion of a system. It is clear in the example of motion of a material point of a mass $m$ in a fixed orthonormal system $y$. In such a case the inertial force is $m\ddot{y}_i = 0 \rightarrow \dot{y}_i = \; const = c_i$ .

The action (1) is

$$J = \int_{t_0}^{t_1} A(I)dt = -\int_{t_0}^{t_1} m \int_s (\ddot{y}_i dy_i) dt = 0 \tag{16}$$

and action $J_1$ is

$$J_1 = \int E_k dt = \frac{mc^2(t_2 - t_1)}{2} \tag{17}$$

since we suppose that mass and velocity are constant.

5. So, for inertial motion or for a static case the principle of action reduced to:

$$\int Q_i dq^i dt = 0 \tag{18}$$

This equivalent to generalized conditions of equilibrium

$$Q_i = 0, \quad i = 0, 1, \dots , n \, . \tag{19}$$

6. For systems with scleronomic constraints all relations above have the same forme except that the indices *i*, *j* go over 1, 2, ... , *n* (instead 0, 1, 2, ... , *n*), when n is number of degrees of freedom of the scleronomic system.

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# PRINCIP DEJSTVA

**Veljko Vujičić**

*Za pojam dejstva u fizici i mehanici najčešće se uzima integral* $J = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$, *gde je L Lagranžova funkcija, a za integralni varijacioni princip uzima se relacija* $\delta J = 0$.

*Ovde se definiše dejstvo sila kao funkcional* $\int_{t_0}^{t_1} A dt$, *gde je A rad sila duž stvarnog puta; ističe se jasna razlika između elementarnog rada* $dA = Xdx$ *sila X duž stvarnog pomeranja dx i elementarnog rada tih sila na mogućim varijacijama* $\delta x$.

*Princip dejstva sila, ili prosto princip dejstva iskazuje se relacijom*

$$\delta \int_{t_0}^{t_1} [A(x) - A(I)]dt = 0,$$

*gde je A(X) rad sila X i A(I) rad sila inercije I; pokazuje se da se poznati integralni varijacioni principi klasične mehanike javljaju kao posledice ovako ustanovljenog principa.*