UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol.2, No 8, 1998 pp. 715 - 728
Editor of series: Katica (Stevanovi}) Hedrih, e-mail: katica@masfak.masfak.ni.ac.yu
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http:// ni.ac.yu/Facta

LYAPUNOV STABILITY ROBUSTNESS CONSIDERATION FOR LINEAR SINGULAR SYSTEMS: NEW RESULTS

UDC: 621.3.016.532:62-52.

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Abstract. *In this paper, the stability robustness of particular class of linear systems in the time domain, is addressed using the Lyapunov approach. The bounds of unstructured perturbation vector function, that maintain the stability of the nominal system with attractivity property of subclass of solutions are obtained both for regular and irregular linear singular systems.*

Key words. *Generalized state-space systems, attraction property, Lyapunov stability, robustness*

1. INTRODUCTION

Linear singular systems are those systems whose dynamics is governed by a mixture of differential and algebraic equations. These systems are known as generalized, descriptor as well as semi-state systems. They naturally arise in many practical engineering disciplines and applications, such as electrical networks, aircraft dynamics, robotics, optimization problems, feedback control systems, large-scale systems, as a limiting case of singularly perturbed systems, etc. and in biology, economy and demography.

The survey of updated results concerning different aspect of treatment of linear singular systems and the broad bibliography on this subject can be found in the books of Aplevich (1991), Bajić (1992), Campbell (1980, 1982), Dai (1989) and Debeljković *et al.* (1996) and in the two special issues of the journal *Circuit, Systems and Signal Processing*

Received March 16, 1998

(1986, 1989).

Physical systems are very often modeled by idealized and simplified models, so that information obtained on the basis of such models is not always sufficiently accurate. This makes motivation for investigation of robustness of examined system properties with respect to the model inaccuracies.

Patel and Toda (1980) first reported the robustness bounds on unstructured perturbations of linear continuous-time systems.

Yedavalli and Liang (1986) improved Patel's result for linear perturbations with known structure and proposed similarity transformation method to reduce robustness bounds conservatism.

In the paper of Yedavalli (1986), the aspect of "stability robustness" is analyzed in the time domain. A bound on the structured perturbation of an asymptotically stable linear system is obtained to maintain stability using Lyapunov matrix equation solution. For the special case of nominal system matrix, some other results have been also obtained.

Zhou and Khargonekar (1987) considered the robust stability analysis problem by linear state-space methods. They derived some lower bounds on allowable perturbations that maintain the stability of nominally stable system with structured uncertainty. It has been shown that those bounds are less conservative than the existing ones.

Recently, Chen and Han (1994) using iterativity approach, derived new results in the same area of interest for the linear system with unstructured time-varying perturbations. In comparison with some existing methods, less conservative results have been obtained.

This was the short overview of the problems related to continuous linear systems.

A general overview of results concerning the stability robustness problems in the area of nonlinear time-varying singular systems can be found in Bajić (1992), while some other similar considerations for linear singular systems are presented by Dai (1989).

In this paper, the existence of solution of both regular and irregular singular systems, that are attracted by the origin of the state space, is examined. A weak domain of attraction of the origin consisting the points of the state-space which generate at least one solution convergent to the origin, is estimated using Lyapunov's second method.

It has been shown that the same results can be efficiently used for determining quantitative measures of robustness for such class of system. In that sense, these results represent natural extension of results presented in Debeljković *et al*. (1994.a, 1994.b), as well as the application of results derived in Toda and Patel (1980), Yedavalli (1986) and Zhou and Khargonekar (1987) to the linear singular systems.

2. PRELIMINARIES

Consider the linear singular system represented by:

$$E\dot{\mathbf{y}} = A\mathbf{y}(t), E, A \in \mathcal{R}^{m\times n}, \mathbf{y}(t_0) = \mathbf{y}_0, \tag{1}$$

where $\mathbf{y} \in \mathcal{R}^n$ is the phase vector (i.e. generalized state–space vector). The matrix E, when $m = n$, is possibly singular. When this is the case, then rank $E = p < n$, *nullity* $E = n - p = q$. If the matrix E is invertible, then (1) can be written in the normal form as

$$\dot{\mathbf{y}}(t) = E^{-1}A\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \tag{2}$$

Behavior of solutions of (2) is very well documented in modern literature on this subject. However, this is not the situation for the system (1), where $m \neq n$ or when $m = n$ with $\det E = 0$.

Introducing a suitable nonsingular transformation *TEQ*, Dai (1989), or sometimes just:

$$T\mathbf{x}(t) = \mathbf{y}(t), T \in \mathcal{C}^{n\times n}, \tag{3}$$

a broad class of singular systems (1) can be transformed to the following form:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t), \tag{4a}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t), \tag{4b}$$

$$AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad ET = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{4c}$$

where $\mathbf{x}(t) = [\mathbf{x}_1^T(t) \quad \mathbf{x}_2^T(t)]^T \in \mathcal{R}^n$ is a decomposed vector, with $\mathbf{x}_1(t) \in \mathcal{R}^{n_1}$, $\mathbf{x}_2(t) \in \mathrm{R}^{n_2}$, and $n = n_1 + n_2$. The matrices A_i, $i = 1 \ldots 4$, are of appropriate dimensions. Comparing (4) with (1), it is obvious that if $m = n$ we consider the case when $\det E = 0$. This conclusion stems from the fact that $\det (ET) = \det E \det T = 0$, and that $\det T \neq 0$. When the matrix pencil $(cE - A)$ is regular, i.e., when:

$$\det (cE - A) \neq 0, \quad c \in \mathcal{C}, \tag{5}$$

then solutions of (1) exist, and they are unique for so-called consistent initial values $\mathbf{x}_0$ of $\mathbf{x}(t)$, and moreover, the closed form of these solutions is known. If A_4 is regular, the condition (5) is reduced to:

$$\begin{aligned} &\det\ (cI_{n_1} - A_1)\det(-A_4 - A_3(cI_{n_1} - A_1)^{-1} A_2) = \\ &= (-1)^{n_2} \det\ A_4\ \det((cI - A_1) - A_2A_4^{-1}A_3) \neq 0. \end{aligned} \tag{6}$$

Let us denote the set of the consistent initial values of (4) by $\mathcal{W}_I$. Also, consider the manifold $\mathcal{M} \subseteq \mathcal{R}^n$ determined by (4b) as $\mathcal{M} = \aleph([A_3 \ A_4])$, where $\aleph(\cdot)$denotes the kernel (null space) of the operator $(\cdot)$. For the system governed by (4), the set $\mathcal{W}_I$ of the consistent initial values is equal to the manifold $\mathcal{M}$, that is $\mathcal{W}_I = \mathcal{M}$. In other words, a consistent initial value $\mathbf{x}_0$ has to satisfy $\mathbf{0} = A_3\mathbf{x}_{10} + A_4\mathbf{x}_{20}$, or in equivalent notation:

$$\mathbf{x}_0 \in \mathcal{W}_I \equiv \mathcal{M} = \aleph([A_3\ A_4]). \tag{7}$$

However, if :

$$\operatorname{rank}\ [A_3\ A_4] = \operatorname{rank} A_4, \tag{8}$$

then $\mathcal{W}_I = \mathcal{M} = \aleph[A_3\ A_4]$ and the determination of the $\mathcal{W}_I$ obviously requires no additional computation, except those necessary to convert (1) into the form (4). Assuming that $\operatorname{rank} A_4 = r \leq n_2$, it is clear on the basis of (7), that $(n_1 + n_2 - r)$ components of the vector $\mathbf{x}_0$ can be chosen arbitrarily to achieve no impulsive solutions of the system, governed by (4). Note, also, that then $\operatorname{rank} A_4 = r < n_2$, the uniqueness of solutions is not guaranteed, Bajić *et al.* (1997), Debeljković *et al.* (1997).

3. PROBLEM FORMULATION

Since the transformation (3) is nonsingular, the convergence of solutions $\mathbf{y}(t)$ of (1) and $\mathbf{x}(t)$ of (4) toward the origin of (1) and (4), respectively, is an equivalent problem. Thus, for the null solution of (4), we are going to investigate the weak domain of attraction. The *weak domain of attraction* of the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of (4) is defined by

$$\mathcal{S} = \left\{ \mathbf{x}_0 \in \mathcal{R} : \mathbf{x}_0 \in \mathcal{M},\ \exists \mathbf{x}(t,\mathbf{x}_0),\ \lim_{t\to\infty} \| \mathbf{x}(t,\mathbf{x}_0) \| \to 0 \right\}. \tag{9}$$

We use the term *weak* because solutions of (4) need not be unique, and thus for every $\mathbf{x}_0 \in \mathcal{S}$ there also may exist solutions which do not converge toward the origin. In our case $\mathcal{S} = \mathcal{M} = \mathcal{I}_{\mathrm{I}}$, and we may think of weak domain of attraction as of weak global domain of attraction. Note that this concept of global domain of attraction used in the paper, differs considerably with respect to the global attraction concept known for state-space systems, in normal form (1).

Our task is to estimate the set $\mathcal{S}$. We will use Lyapunov direct method to obtain the underestimate $\mathcal{S}_u$ of the set $\mathcal{S}$ (i.e. $\mathcal{S}_u \subseteq \mathcal{S}$). Our development will not require the regularity condition (5) of the matrix pencil $(sE - A)$.

4. MAIN RESULTS

This section introduces a stability result which will be employed for the robustness analysis. For the systems in the form (4), the Lyapunov-like function can be selected as

$$V(\mathbf{x}(t)) = \mathbf{x}_1^T(t) P \mathbf{x}_1(t), P = P^T > 0, \tag{10}$$

where P will be assumed to be positive definite and real matrix. The total time derivative of V along the solutions of (4) is then

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)(A_1^T P + P A_1)\mathbf{x}_1(t) + \mathbf{x}_1^T(t) P A_2 \mathbf{x}_2(t) + \mathbf{x}_2^T(t) A_2^T P \mathbf{x}_1(t) . \tag{11}$$

Brief consideration of the attraction problem shows that if (11) is negative definite, then for every $\mathbf{x}_0 \in \mathcal{I}_{\mathrm{I}}$ we have $\|\mathbf{x}_1(t)\| \to 0$ as $t \to \infty$. Then, $\|\mathbf{x}_2(t)\| \to 0$ as $t \to \infty$, for all those solutions for which the following connection between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ holds

$$\mathbf{x}_2(t) = L\mathbf{x}_1(t), \qquad \forall t \in \mathcal{R}. \tag{12}$$

If the rank condition (8) holds, which implies $\mathcal{I}_{\mathrm{I}} = \aleph[A_3 \;\; A_4]$, then there exist a matrix L being any solution of matrix equation

$$0 = A_3 + A_4 L , \tag{13}$$

where *0* is null matrix of dimension the same as A_3.

It is obvious that the solutions of (4) have to belong to the set $\aleph([L \; -I_{n_2}])$ as well as potential domain of attraction is given by:

$$\mathcal{S}_u = \{\mathbf{x} \in \mathcal{R} : \ \mathbf{x}(t) \in \aleph([L \;\; -I_{n_2}])\} \subseteq \mathcal{S}. \tag{14}$$

We are now in position to state the following result.

Theorem 1. Let (8) hold. Then, the underestimate $\mathcal{S}_u$ of the potential domain $\mathcal{S}$ of attraction of the null solution of singular system (4) is determined by (14) provided L is any solution of (13) and $(A_1 + A_2L)$ is Hurwitz matrix. Moreover, $\mathcal{S}_u$ contains more than one element.

Proof. If the rank condition (8) is satisfied, it follows that $\mathcal{W}_I = \aleph([A_3\ \ A_4])$. Let L be any solution of (13). Note that such L always exist when (8) holds. Select now $\mathbf{x}_0 \in \aleph([L\ \ -I_{n_2}])$. This is consistent initial condition at $t = t_0$ since

$$\aleph([L\ \ -I_{n_2}]) \subseteq \aleph([A_3\ \ A_4]) = \mathcal{W}_I . \tag{15}$$

Then solutions $\mathbf{x}(t,\ \mathbf{x}_0)$ of (4) that emanate from point $\mathbf{x}_0$ exist. To examine the behavior of these solutions, the agregation function, defined by (10), is used. Now, we employ (11) and (12) to obtain

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)((A_1 + A_2L)^T P + P(A_1 + A_2L))\mathbf{x}_1(t)\,, \tag{16}$$

which is negative definite with respect to $\mathbf{x}_1$ if and only if

$$\Omega^T P + P\Omega = -\,Q\,, \quad \Omega = A_1 + A_2\mathrm{L}\,, \tag{17}$$

where Q is real symmetric positive definite matrix. Hence $V(\mathbf{x}(t))$, defined by (10), is positive definite function and its total time derivative, along the solutions of (4) that satisfy (12), is negative definite. So

$$\lim_{t\to\infty} ||\mathbf{x}_1(t,t_0)|| \to 0\,, \tag{18}$$

as long as $\mathbf{x}_0 \in \aleph([L\ \ -I_{n_2}])$. But (12) implies also

$$\lim_{t\to\infty} ||\mathbf{x}_2(t,t_0)|| = \lim_{t\to\infty} ||L\mathbf{x}_1(t,t_0)|| \le \lim_{t\to\infty} ||L||\,||\mathbf{x}_1(t,t_0)|| \to 0\,. \tag{19}$$

As $\aleph([L\ \ -I_{n_2}])$ is not singleton, then there are solutions of (4) with the initial value $\mathbf{x}_0 \neq \mathbf{0} \in \mathcal{R}^n$ that converge toward the origin of phase space as $t \to \infty$. Thus, $\mathcal{S}_u$ has more than one element.

This proof is based on the results firstly reported in Debeljković *et al*. (1997).

5. ROBUSTNESS OF ATTRACTION PROPERTY

To analyze robustness of attraction property of the phase space origin, let us consider the perturbed system (1) which for this purpose can be represented in the following form:

$$E\dot{\mathbf{y}}(t) = A\mathbf{y}(t) + \mathbf{f}_p\big(\mathbf{y}(t)\big) = A\mathbf{y}(t) + G_p\mathbf{y}(t), \tag{20}$$

where the vector $\mathbf{f}_p(t)$ represents model perturbation and matrix G_p is of appropriate dimension.

To simplify formulation of the stability robustness results we first transform (20) to

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + G_1(t)\mathbf{x}(t)\,, \tag{21.a}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) + G_2\mathbf{x}(t), \tag{21.b}$$

as it has been done with (1) to (4). G_1 and G_2 are matrices of dimension $n_1 \times (n_1 + n_2)$ and $n_2 \times (n_1 + n_2)$ respectively, determined by the following expression

$$G_1 = \begin{bmatrix} G_{11} & G_{12} \end{bmatrix},\ G_2 = \begin{bmatrix} G_{21} & G_{22} \end{bmatrix},\ G_p T = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{22}$$

Then we introduce the following assumption.

Assumption 1. Let L be matrix which satisfies (13) and let $G_2 \equiv 0$, so that

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\mathbf{x}(t) = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} = \begin{bmatrix} (G_{11} + G_{12}L)\mathbf{x}_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} G_L\mathbf{x}_1(t) \\ 0 \end{bmatrix}. \tag{23}$$

Now we state results on robustness stability as follows.

Theorem 2. Let the rank condition (8) and Assumption 1 hold. Then the underestimate S_u of the potential domain of attraction of system (21) is given by (14), if one of the following conditions is fulfilled

$$\text{a)}\ \|G_L\|_S < \mu, \quad \text{b)}\ \|G_L\| < \mu, \quad \text{c)}\ |g_{Lij}| < \mu/n_1\ , \tag{24}$$

where g_{Lij} is the (i, j) element of matrix G, and

$$\mu = \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}, \tag{25}$$

and where $P = P^T > 0$, is symmetric, positive definite, real matrix, being unique solution of Lyapunov matrix equation

$$(A_1 + A_2L)^T P + P(A_1 + A_2L) = -2Q, \tag{26}$$

for any real, symmetric, positive definite matrix Q. The set S_u contains more than one element. $\|(\cdot)\|$ and $\|(\cdot)\|_S$ denotes Euclidean and spectral norm of matrix $(\cdot)$ respectively and $\sigma_{(\cdot)}(\cdot)$ corresponding singular value.

Proof. Let Lyapunov-like function candidate be chosen as in (10). Then, using Assumption 1, equations (11) and (26), one can easily get

$$\dot{V}(\mathbf{x}_1(t)) = -2\mathbf{x}_1^T(t)Q\mathbf{x}_1(t) + 2\mathbf{x}_1^T(t)PG_L\mathbf{x}_1(t). \tag{27}$$

From (24.a) it is obvious that

$$\|G_L\|_S\ \sigma_{\max}(P) < \sigma_{\min}(Q)\ , \tag{28}$$

as well as

$$\|PG_L\|_S \le \|G_L\|_S\ \sigma_{\max}(P)\ . \tag{29}$$

Moreover, Patel and Toda (1980):

$$\sigma_{\min}(Q)\mathbf{x}_1^T(t)\mathbf{x}_1(t) \le \mathbf{x}_1^T(t)Q\mathbf{x}_1(t), \tag{30}$$

$$\mathbf{x}_1^T(t)PG_L\mathbf{x}_1(t) \le \|PG_L\|_S \ \ \|\mathbf{x}_1(t)\|^2, \tag{31}$$

so

$$\mathbf{x}_1^T(t)PG_L\mathbf{x}_1(t) < \mathbf{x}_1^T(t)Q\mathbf{x}_1(t), \tag{32}$$

and finally

$$-2\mathbf{x}_1^T(t)Q\mathbf{x}_1(t) + 2\mathbf{x}_1^T(t)PG_L\mathbf{x}_1(t) < 0, \tag{33}$$

that is, $\dot{V}(\mathbf{x}_1(t)) < 0$, so $\|\mathbf{x}_1(t,\mathbf{x}_0)\| \to 0$ when $t \to \infty$, as well as $\|\mathbf{x}_2(t)\|$ for any $\mathbf{x}_0 \in \aleph([L\ -I_{n_2}])$, since S_u is not singleton. This ends the proof.

To prove (24.b) and (24.c) one has just to use

$$\|G_L\|_S \le \|G_L\| = \left[\sum_{i,j=1}^{n_1} |g_{Lij}|^2\right]^{1/2}, \tag{34}$$

what ends the proof.

Theorem 3. Let the rank condition (8) and Assumption 1 hold. Then the underestimate S_u of potential domain of attraction of system (21) is given by (14), if the following condition is fulfilled

$$|g_{Lij}|_{\max} = \varepsilon < \frac{1}{\sigma_{\max}\left[|P|U\right]_S} \equiv \eta, \tag{35}$$

where P satisfies the Lyapunov matrix equation given by:

$$(A_1 + A_2L)^T P + P(A_1 + A_2L) = -2I, \tag{36}$$

I being $n_1 \times n_1$ identity matrix with U being $n_1 \times n_1$ matrix whose entries are unity. $[(\cdot)]_S$ means symmetric part of matrix $(\cdot)$.

Proof. For the system of the form (4) the Lyapunov function candidate can be selected as

$$V(\mathbf{x}_1(t)) = \mathbf{x}_1^T(t)P\mathbf{x}_1(t), \ P = P^T > 0. \tag{37}$$

The total time derivative along the solutions of (4) is then

$$\dot{V}(\mathbf{x}_1(t)) = \mathbf{x}_1^T(t)(-2I + G_L^T P + PG_L)\mathbf{x}_1(t). \tag{38}$$

Let matrix Ψ be defined in the following manner

$$\Psi = \varepsilon U, \tag{39}$$

and suppose that the first condition of Theorem is fulfilled, i.e.,

$$|g_{Lij}| = \varepsilon < \frac{1}{\sigma_{\max}(|P|U)_S}. \tag{40}$$

Then, it is obvious that

$$\sigma_{\max}(|P|\psi)_S < 1, \quad \sigma_{\max}(PG_L)_S < 1, \tag{41}$$

$$\sigma_{\max}[-(PG_L)_S] < 1, \quad \sigma_{\max}[(PG_L)_S(-I)^{-1}]_S < 1, \tag{42}$$

so, according to Lemma 1 (see Appendix), $[-I + (PG_L)_S]$ is negative definite.

Moreover, since

$$(PG_L)_S = \frac{G_L^T P + PG_L}{2}, \tag{43}$$

it is clear that the matrix $(-2I + G_L^T P + G_L)$ is negative definite, as well as it is then $\dot{V}(\mathbf{x}_1(t)) < 0$, what had to be proved.

The analysis of this result is identical to that presented in the proof of Theorem 1 and leads to the same conclusion.

Theorem 4. Let the rank condition (8) and Assumption 1 hold. Moreover, let the matrix G_L be defined in the following manner

$$G_L = \sum_{i=1}^{m} k_i G_{Li}, \tag{44}$$

where G_{Li} are constant matrices and k_i are uncertain parameters varying in some intervals around zero, i. e., $k_i \in [-\varepsilon_i, +\varepsilon_i]$. Then, the underestimate S_u of potential domain of attraction of system (21) is given by (14) when P satisfies the Lyapunov matrix equation

$$(A_1 + A_2 L)^T P + P(A_1 + A_2 L) = -2I, \tag{45}$$

and if one of the following conditions is fulfilled

a) $$\sum_{i=1}^{m} k_i^2 < \frac{1}{\sigma_{\max}^2(P_e)}, \tag{46}$$

or

b) $$\sum_{i=1}^{m} |k_i| \sigma_{\max}(P_i) < 1, \tag{47}$$

or

c) $$|k_j| < \frac{1}{\sigma_{\max}\left(\sum_{i=1}^{m} |P_i|\right)}, \quad j = 1, 2, \dots m. \tag{48}$$

where P_i and P_e are given by

$$P_i = \frac{1}{2}(G_{Li}^T P + PG_{Li}) = [PG_{Li}]_S \tag{49}$$

and

$$P_e = [P_1 \quad P_2 \quad \cdots \quad P_m]. \tag{50}$$

Moreover S_u contains more than one element.

Proof. If one use (37) and (44), it is clear that

$$\dot{V}(\mathbf{x}_1(t)) = 2\mathbf{x}_1^T(t)\left(\sum_{i=1}^{m} k_i P_i - I\right)\mathbf{x}_1(t), \tag{51}$$

is negative definite when

$$\sigma_{\max}\left(\sum_{i=1}^{m} k_i P_i\right) < 1 . \tag{52}$$

$\sum_{i=1}^{m} k_i P_i$ can be transformed to

$$\sum_{i=1}^{m} k_i P_i = \begin{bmatrix} P_1 & P_2 & \dots & P_k \end{bmatrix}\begin{bmatrix} k_1 I & k_2 I & \dots & k_m I \end{bmatrix}^T = P_e \begin{bmatrix} k_1 I & k_2 I & \dots & k_m I \end{bmatrix}^T \tag{53}$$

so that

$$\sigma_{\max}(P_e)\left(\sum_{i=1}^{m} k_i^2\right)^{1/2} \geq \sigma_{\max}\left(\sum_{i=1}^{m} k_i P_i\right), \tag{54}$$

what means that when (46) is fulfilled, then (52) is also. Moreover

$$\sum_{i=1}^{m} |k_i| \sigma_{\max}(P_i) \geq \sigma_{\max}\left(\sum_{i=1}^{m} k_i P_i\right), \tag{55}$$

so when (47) is satisfied, then (40) is satisfied too. Finally, since

$$\max_j |k_j| \, \sigma_{\max}\left(\sum_{i=1}^{m} |P_i|\right) \geq \sigma_{\max}\left(\sum_{i=1}^{m} |k_i P_i|\right) \geq \sigma_{\max}\left(\sum_{i=1}^{m} k_i P_i\right) \tag{56}$$

is obvious, inequality (48) guarantees (52), what ends the proof.

5. NUMERICAL EXAMPLES

In order to illustrate the presented results, some suitable examples have been worked out.

Example 1. Consider a singular system given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{y}}(t) = \begin{bmatrix} -1 & -2 & 0 & -1 \\ 1 & -2 & 1 & -4 \\ 1 & -1 & 0 & 1 \\ 3 & -5 & 2 & -3 \end{bmatrix} \mathbf{y}(t) + \begin{bmatrix} 2k & -6k & 3k & -6k \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{y}(t) . \tag{57}$$

Since det $(cE - A) \neq 0$ this is regular singular system.

Let us examine the behavior of this system according to the results obtained. Using the transformation matrix

$$T = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{58}$$

which is nonsingular since det $T = 1$, the system (39) can be transformed to

$$\dot{\mathbf{x}}_1(t) = \begin{bmatrix} -4 & -2 \\ 0 & -3 \end{bmatrix} \mathbf{x}_1(t) + \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2(t) + \begin{bmatrix} -2k & -4k & 3k & 2k \\ 1 & 2 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \tag{59.a}$$

$$\mathbf{0} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1(t) + \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \mathbf{x}_2(t) \tag{59.b}$$

Since the rank condition (8) is satisfied, one can find

$$L = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \tag{60}$$

from (13), and then

$$\mathrm{S}_u = \left\{ \mathbf{x} \in \mathrm{R}^4 : \mathbf{x}(t) \in \aleph\left(\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 2 & 0 & -1 \end{bmatrix} \right) \right\} \subseteq \mathrm{S}. \tag{61}$$

if conditions of Theorems 2, 3 or 4 are satisfied.

Let's show that. Since

$$G_L = G_{11} + G_{12} L \Big|_{G_2 \equiv 0} = k \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \tag{62}$$

Assumption 1 is satisfied.

For $Q = I$, from (21) one can have

$$P = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} = P^T > 0, \tag{63}$$

so that

$$|g_{Lij}| \le |k| \le \frac{\sigma_{\min}(Q)}{n_1 \sigma_{\max}(P)} = \frac{1}{2 \cdot \frac{1}{2}} = 1. \tag{64}$$

The Theorem 3 gives better result. Namely,

$$|g_{Lij}| < 1.19\,, \tag{65}$$

since

$$\begin{gathered} |P| = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \sigma_{\max} \left[|P| U \right]_S = 0.8416. \\ |P| U = \begin{bmatrix} 1/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}, \ \left[|P| U \right]_S = \begin{bmatrix} 1/3 & 5/12 \\ 5/12 & 1/2 \end{bmatrix}. \end{gathered} \tag{66}$$

To apply Theorem 4, one needs to find the following data

$$G_L = k \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = k \cdot G_{L_1} \tag{67}$$

$$P_1 = \left[PG_{L_1}\right]_S = \begin{bmatrix} -1/3 & 1/6 \\ 1/6 & 0 \end{bmatrix} = P_e , \qquad \sigma_{\max}(P_e) = 0.1618 \tag{68}$$

$$k^2 < \frac{1}{\sigma^2_{\max}(P_e)} \;\Rightarrow\; |k| < 2.48. \tag{69}$$

Fig. 1 and Fig. 2 represent system trajectories for possible values of uncertain parameter k.

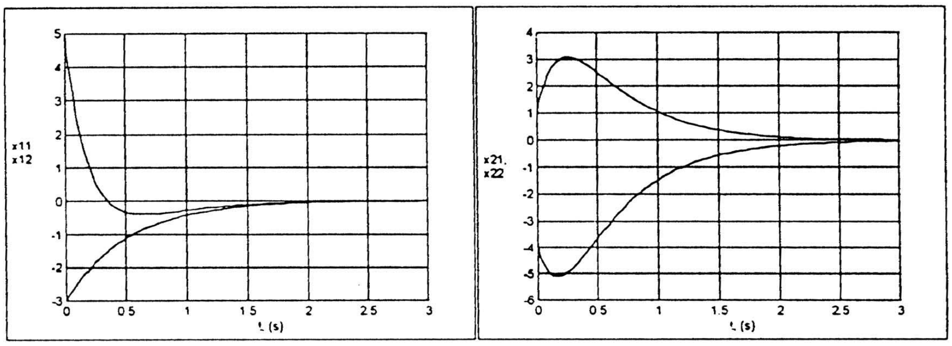


Fig. 1. $k = 2.45; \mathbf{x}_{10} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \mathbf{x}_{20} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{x}_0 \in \aleph([L - I_{n_1}])$

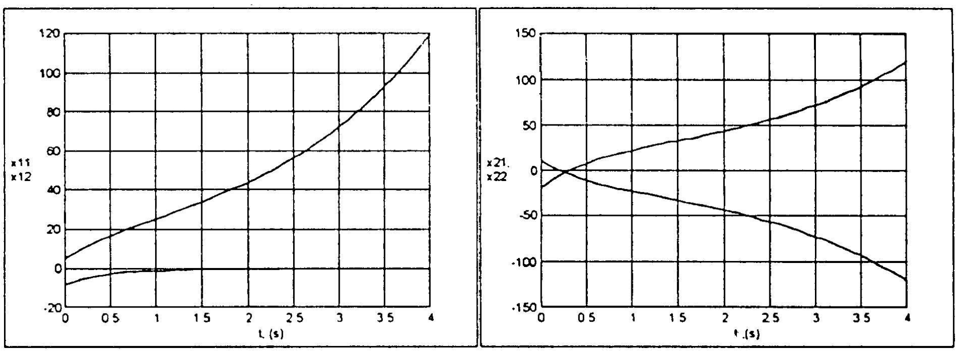


Fig. 2. $k = -3.5; \mathbf{x}_{10} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}, \mathbf{x}_{20} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}, \mathbf{x}_0 \in \aleph([L - I_{n_2}])$

In the first case (Fig. 1), parameter k is chosen in such a way that condition of Theorem 3 is satisfied, so the stability robustness of attraction property of origin is proved. It can be shown that quantitative measures obtained by Theorem 3 are less conservative than the others two, Zhou and Khargonekar (1987).

Second case (Fig. 2), shows that required property is not achieved, since the choice of parameter k was not adequate.

Example 2. Consider a singular system given by

$$\dot{\mathbf{x}}_1(t) = \begin{bmatrix}-1\end{bmatrix}\mathbf{x}_1(t) + \begin{bmatrix}-1 & -3\end{bmatrix}\mathbf{x}_2(t) + G_1\mathbf{x}(t) \tag{70.a}$$

$$\mathbf{0} = \begin{bmatrix}-1\\ 1\end{bmatrix}\mathbf{x}_1(t) + \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\mathbf{x}_2(t) \tag{70.b}$$

Since det $(cE - A) = 0$ for any c, this is a irregular singular system and solutions are not unique.

The following results can be easily obtained

$$\text{rank}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & -1\end{bmatrix} = \text{rank}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix} = 1 \leq 2. \tag{71}$$

$$L = \begin{bmatrix}a\\ 1-a\end{bmatrix}, a \in \mathrm{R}, G_2 \equiv 0. \tag{72}$$

From (20) one can get

$$P = -\frac{1}{a-4}, \quad a < 4 \tag{73}$$

in order to have $P = P^T > 0$.

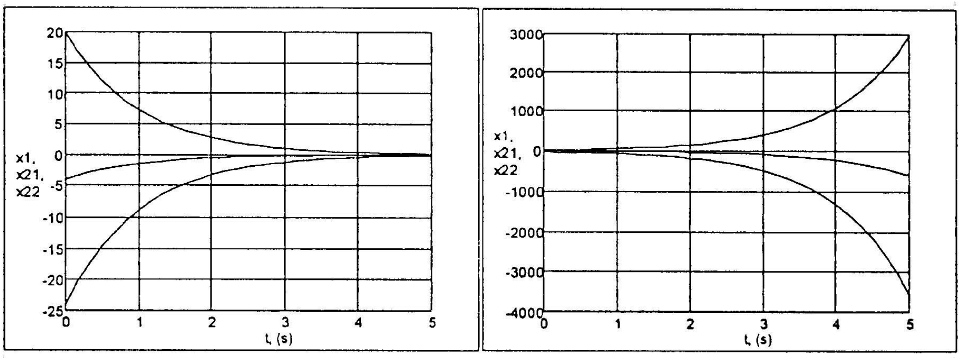
So

$$\|G_L\| \leq \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} = -(a-4), \tag{74}$$

and

$$\mathrm{S}_u = \left\{\mathbf{x} \in \mathrm{R}^3 : \mathbf{x}(t) \in \aleph\left(\begin{bmatrix}a & -1 & 0\\ 1-a & 0 & -1\end{bmatrix}\right) a < 4\right\} \subseteq \mathrm{S}. \tag{75}$$

Two different values of parameter a have been chosen and corresponding system responses has been depicted in Fig. 3.



a) $a = -5, G_L = 8\,; \mathbf{x}_0 \in \aleph([L - I_{n_2}])$ b) $a = -5, G_L = 10\,; \mathbf{x}_0 \in \aleph([L - I_{n_2}])$

Fig. 3.

In the first case (Fig. 3.a), condition given by (56) is satisfied and system has required property. In the second case (Fig. 3.b), G_L is chosen to contradict (56) and system response diverge.

6. CONCLUSION

Simple sufficient algebraic conditions are presented for testing the existence of solutions of linear singular systems which converge toward the origin. The estimate of weak domain of attraction is given.

It has been shown that, under some particular conditions, these results can be efficiently used in checking stability robustness of the linear singular systems. In that sense, they represent natural extension of the results derived earlier, for ordinary linear systems.

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APPENDIX

Theorem A.1. System given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t, \mathbf{x}(t)), \quad t \in [t_0, +\infty[\tag{A.1}$$

is stable if

$$\frac{\| \mathbf{f}(t,\mathbf{z}) \|}{\| \mathbf{z} \|} \leq \mu_1 \equiv \frac{\min \lambda(Q)}{\max \lambda(P)}, \forall (t,\mathbf{z}) \in \mathrm{R}^{n+1}, \tag{A.2}$$

where R is unique positive definite solution of Lyapunov equation

$$A^T P + PA^T = -2Q, \tag{A.3}$$

and where Q is some positive definite matrix.

Lemma A.1. The bound in (A.2) is maximum when the matrix $Q = I$ in (A.3), where I is $n \times n$ identity matrix.

For proofs, see Patel and Toda (1980).

LJAPUNOVSKA ROBUSTNOST STABILNOSTI LINEARNIH SINGULARNIH SISTEMA: NOVI REZULTATI

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U ovom radu razmatrana je osobina privlačenja nultog ravnotežnog stanja linearnog singularnog sistema i pripadajuća osobina robusnosti u odnosu na linearne nestrukturne perturbacije. Izvedene i dokazane teoreme predstavljaju znatno proširenje rezultata do kojih su autori došli ranije, a istovremeno jasno ukazuju na široke mogućnosti primene postojećih rezultata na analizu robusnosti stabilnosti razmatrane klase sistema.