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CONSERVATION LAW OF J INTEGRAL TYPE FOR MULTYLAYERED SHELLS

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Abstract. *The balance laws for multilayered shells is analysed. Using the Euclidean group of transformation, the equivalence between the balance laws and the Euclidean invariance is demonstrated. An example is considered and extension of one of these balance laws is carried out for simple problems of plates theory.*

1. INTRODUCTION

Conservation laws (or balance laws) have been the subject of considerable research in recent years. One of these laws, the J-integral, has been applied extensively to the fracture mechanics problems with much success. In this paper, we analyse similar type of integrals for multilayered shells in the context of multidirector surfaces theory, based on the assumption of piece-wise linearity od displacements field across the thickness. This approach identifies each layer with a two-dimensional (2-D) array of material vectors so that the shell is regarded as a surface endowed with *n*-director fields, *n* being the number of layers. This strongly suggests the concept of a multidirector Cosserat surface.

Conservation laws for classical shells have been considered by Bergez and Radenkovic [1], and Bergez [2] Lo [3] introduced path-independent integrals for cylindrical shells and shells of revolution. Studies made by Kienzler and Golebievska-Herrmann [4] show that conservation laws are derived from variational principle in the context of higher-order shells theories. Based on the Naghdi's theory of thin shells, Sedmak, Berković and Jarić [5] have derived path independent integral for generally shaped shells.

The aim of this paper is to derive conservation laws (or balance laws) using invariant characteristic of variational principle in relation to the Euclidean group of transformation.

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Using Euclidean group of transformation, the equivalence between the conservation law and the Euclidean invariance is demonstrated. As a consequence a novel result for the conservation law (or the balance law) for multilayered shells has ben obtained. Finally, one of the laws is used as an example to illustrate its application.

2. EQUATIONS OF VARIATIONAL INVARIANCE

Let $\xi = (\xi_i) \in R_i$, $i = 0$, α, be the independent and $\phi = (\phi_\alpha) \in R_\alpha$, $\alpha = 1$, m, dependent vector variables, describing the behaviour of material system under consideration.

We define now the following action integral

$$A(\phi) = \int_T \int_B L dS dt = \int_R L(Y) d\xi \tag{1}$$

where L represents real scalar function of ξ, ϕ, ϕ_α, defined and differentiable for all values of its arguments and $Y = Y(\xi, \phi, \phi_\alpha)$.

For the action integral (2,1), the small transformations of dependent and independent variables are introduced as follows:

$$\begin{aligned} \xi^{*i} &= \xi^i + \delta\xi^i = \xi^i + \alpha^i \eta + 0(\eta^2) \\ \phi^* &= \phi^i + \delta\phi = \phi + b\eta + 0(\eta^2) \end{aligned} \tag{2}$$

where the quantities $\delta\xi^i = \alpha^i$, $\delta\phi = b$ etc. are taken to be of infinitesimal order and η is a small parameter.

Now a special form of Noether's theorem can be defined, which is used here to derive the conservation laws (the proof of this theorem can be found on [6]):

Noether's theorem:

If the fields ϕ satisfy the corresponding Euler-Lagrange equations $E(L)_\phi = Q$, then the functional (1) remains infinitesimally inveriant at ϕ under the small transformations (2), if and only if ϕ, satisfies

$$\frac{\partial}{\partial \xi} i[\{L,_{\phi,i}, m\} + L\alpha^i] - \{m, Q\} = 0 \tag{3}$$

where the vector m is defined as

$$m = b - \phi,_i \alpha^i \tag{4}$$

It was conveient to use abbreviated notation in eqs (3) and (4), suggested by Ericksen [7]:

$$\{\phi_1, \phi_2\} = \{(a_1, b_1)(a_2, b_2)\} = a_1 a_2 + b_1 b_2$$

3. CONSERVATION LAWS FOR MULTILAYERED SHELLS

The starting point for conservation laws introduction is elastic multilayered shell theory by Epstein and Glockner [8], and Ericksen and Truesdell [9]. Only the basic elements of the theory are given here and details can be found in [8,9].

Let $R = R(X^\alpha)$ be the position vector of a generic point of the reference surface S of a shell in the reference configuration, with curvillinear Gaussian coordinates X^α (α=1,2). Associated with it is a complete description of the shell and the supplementary director fields $D_I = D_I(X^\alpha)$, $_{I = 1,2,\dots n.}$

A motion of he shell is defined by specifying the possition vector, r, of the deformed surface and the deformed directors, d_I, as function of the curvillinear coordinates, X^α and time t:

$$r = r(X^\alpha, t) = r(\xi^i)$$
$$d_I = d_I(X^\alpha, t) = d_I(\xi^i) \tag{5}$$

Let us assume that m constraints are imposed on the deformation in

$$\psi_i(r_{,\alpha}; d_I, d_{I,\alpha}) = 0 \quad I = 1,\dots m \tag{6}$$

which must satisfy frame indifference.

The Lagrangian density H associated with the multilayered shell is given by

$$H = L(Y) - \lambda^i \psi_i \,,\;\; Y = Y(r_{,i}; d_I, d_{I,i}) \tag{7}$$

ant $\lambda^i = \lambda^i\ (X^\alpha, t)$ is the Lagrange multiplier associated with the i-th constraint, eqn (6). The laws of motion, given by eqs (15a,b) in [8], are equivalent to the Euler-Lagrange equations

$$\frac{\partial}{\partial \xi}\alpha \frac{\partial H}{\partial \phi_{,\alpha}} - \frac{\partial H}{\partial \phi} - Q = 0 \tag{8}$$

Then the Noether's theorem can be applied to our case. To confirm this statement we choose

$$L = H\,,\;\; \phi = (r, d_I)\,,\;\; Q = (F, F^I)\,,\;\; m = (p, q_I)\,,\;\; b = (\beta, \gamma_I)$$
$$P = \frac{\partial H}{\partial r}\,,\;\; P^I = \frac{\partial H}{\partial d_I}\,,\;\; T^\alpha = \frac{\partial H}{\partial r_{,\alpha}}\,,\;\; T^{I\alpha} = \frac{\partial H}{\partial d_{I,\alpha}} \tag{9}$$

Before proceeding further, the integral form of the conservation laws is given, applying the Gauss theorem to (3):

$$\frac{d}{dt}\int_s (P_p + P^I q_I + L\alpha_0) ds + \int_c (T^\alpha p + T^{I\alpha} q_I + L\alpha^\alpha) n_\alpha dl + \int_s (Fp + F^I q_I) ds = 0 \tag{10}$$

where c is the smooth closed curve, bounding s and n is the unit normal (in s) to c.

Following Toupin [10], one can postulate that the action density L is invariant under the group of Euclidean displacements. Since the group of Euclidean displacements is a connected Lie group, it is sufficient to require that the action density is invariant under infinitesimal transformations of the group of Euclidean displacements in order to be invariant under arbitrary, finite transformations of the group. An infinitesimal transformation of the group has the form:

$$X^* = X + C\eta ;\ \ \phi^* = \phi + (\Omega\phi + D)\eta ;\ \ \phi = (r, d_I) ;\ \ t^* = t + C_0\eta \tag{11}$$

where Ω is an antisymmetric tensor, $\Omega = \Omega^T$, and Ω, C, C_0 and D are arbitrary constants. By taking all of the arbitrary constants in (11) to be equal to zero, except the one in turn, we obtain the corresponding conservation laws:

(I) $D \neq 0$, $\alpha_0 = 0$, $\alpha^\alpha = 0$, $\beta = D$, $\gamma_I = 0$, $p = D$

The corresponding conservation law (10) now reads

$$\frac{d}{dt}\int Pds + \int T^\alpha n_\alpha dl - \int Fds = 0 \tag{12}$$

(II) $\Omega \neq 0$, $\alpha_0 = 0$, $\alpha^\alpha = 0$, $\beta = \Omega r$, $\gamma_I = \Omega d_I$, $p = \Omega r$, $q_I = \Omega d_I$.

This transformation represents rigid body rotation, and the corresponding conservation law reads

$$\frac{d}{dt}\int Sds - \int_c (r \times T^\alpha + d_I \times T^{I\alpha}) n_\alpha dl - \int_s (r \times F + d_I \times F^I) ds = 0 \tag{13}$$

where

$$S = r \times P + d_I \times P^I .$$

(III) $\alpha_0 = C_0 \neq 0$, $\alpha = 0$, $\beta = 0$, $\gamma_I = 0$, $p = -C_0 \dot{r}$, $q_I = -C_0 \dot{d}_I$.

This transformation represent a shift of time, and the corresponding law reads

$$\frac{d}{dt}\int_S Eds - \int_c (T^\alpha \dot{r} + T^{I\alpha} \dot{d}_I) n_\alpha dl - \int_s (F\dot{r} + F^I \dot{d}_I) ds = 0 \tag{14}$$

where $E = P\dot{r} + P^I \dot{d}_I - L$.

The conservation laws (12-14) represent the conservation of linear momentum, moment of momentum and energy, respectively. Thus, we have established the basic theorem of equivalence between conservation and invariance [10].

As a special case we consider

(IV) $\alpha^\alpha = C^\alpha \neq 0$, $\alpha_0 = 0$, $\beta = 0$, $\gamma_I = 0$, $p = -r,_\alpha C^\alpha$, $q_I = -d_{I,\alpha} C^\alpha$.

This transformation represents the family of coordinate translations, and leads us to the conservation laws which are of a special interest for us.

$$\frac{d}{dt}\int_S (Pr,_\alpha + P^I d_{I,\alpha}) ds - \int_c (L\delta_\alpha^\beta - T^\beta r,_\alpha - T^{I\beta} d_{I,\alpha}) n_\beta dl - \int_s (Fr,_\alpha + F^I d_{I,\alpha}) ds = 0 \tag{15}$$

The expressions (12-15) represent novel conservation laws for multilayered shells. Of special interest in fracture mechanics is the expression (15) which represent the conservation law of J integral type.

4. APPLICATION

A. Special case

Bearing in mind the application to elastic multilayered shells, it is convenient to assume a Lagrangian density decompossable as

$$L = W - K \tag{16}$$

with

$$2K = R^{00}\dot{r}\dot{r} + 2R^{0I}\dot{r}\dot{d}_I + R^{IJ}\dot{d}_I\dot{d}_J \tag{17}$$

and

$$W = W(r_{,\alpha}, d_I, d_{I,\alpha}, X^{\alpha}) \tag{18}$$

where the inertia coefficients R^{00}, R^{01}, R^{IJ} are time independent. Under these circumstances K satisfies Euclidean invariance, what can be verified easily, and since L has been assumed to be Euclidean invariant, it follws from (16) that the same is true for W.

For the case $n = 1$, eqn (8) reduces to the theory of Cosserat surface with one variable director [7,8], which has been used for the theory of sandwich shell (9). Indeed, for $n = 0$ and no constraints, eqn (8) and (16-18) reduce to:

$$\left(\frac{\partial W}{\partial \phi_{,\alpha}}\right)_{,\alpha} + \frac{\partial W}{\partial \phi} + G = \kappa\dot{\phi} \tag{19}$$

where $\phi = (r,d)$, $G = (f,g)$ and $\kappa(a,b) = (R^{00}a+R^{01}b, R^{01}a+R^{11}b)$.

B. Elastic plates

We are interested in homogeneous flat plates, for which the reference configurations is of the following form

$$r = r_R(X^1, X^2, 0)\,; \quad d = d_R = const\,. \tag{20}$$

X^1 and X^2 being rectangular Cartesian coordinates. Then, from (16) W and T will represent energies/unit undeformed or reference area. To describe its homogeneity, we restrict W by the condition that

$$W = W(r_{,\alpha}, d, d_{,\beta}\,; X^{\gamma}) = W(r_{,\alpha}, d, d_{,\beta}\,; 0) \tag{21}$$

where the coordinates are chosen so that the original lines are within the plate.

Recalling that W does not depend explicitely on r and has to be Euclidean invariant [7], one obtains:

$$W(\phi, \phi_{,\alpha}) = W^*(U, U_{,\alpha}) \tag{22}$$

The equation (19) then becomes

$$\left(\frac{\partial W^*}{\partial U_{,\alpha}}\right)_{,\alpha} + \frac{\partial W^*}{\partial U} + G^* = \kappa U \tag{23}$$

and the corresponding conservation law (10) in becomes

$$\frac{d}{dt}\int_s \{\kappa U, U,_\alpha\}ds + \int_c (L\delta_\alpha^\beta - \{T^\beta, U,_\alpha\})n_\beta dl - \int_s \{G, U,_\alpha\}ds = 0 \tag{24}$$

where the curly brackets denote the inner product.

Another simple case can be obtained if (23) reduces to a system of ordinary differential equations. An obvious possibility is to put:

$$U = U(x)\,,\quad G^* = 0\,,\quad x = n_\alpha X^\alpha - Vt$$

where n_α and V are constant. Then (24) transforms to a integral:

$$\int_c (W^* + V^2\{\kappa U, U''\})dX^2 - \{T^\alpha n_\alpha, U'\}dl - \int_s V^2\{\kappa U, U'''\}ds = 0 \tag{25}$$

which is path-independent for any path $c(t)$ around the crack tip and $t > t_0 > 0$.

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ZAKONI ODRŽANJA TIPA J-INTEGRALA ZA VIŠESLOJNE LJUSKE

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Koristeći osobinu invarijantnosti varijacionog principa u odnosu na grupu infinitezimalnih transformacija za vektorska polja, u ovom radu je uveden odgovarajući oblik teoreme Emi Neter. Potom se na primeru višeslojne ljuske sa zadatim ograničenjima koristi Euklidova grupa transformacija i njoj pridružuju odgovarajući zakoni konzeravcije. Na kraju se analiziraju dobijeni rezultati.