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THE CAPACITY LIMIT OF THE CYLINDRICAL SHELLS *

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Abstract. *In this paper, the theorem of the lower limit carrying capacity for a special class of the thin cylindrical shells is given, applied on the uniformly distributed radial load. In this method, the lower stress limit is searched for by determining the statically and plastically permissible stress field which fulfills the equilibrium conditions, boundary conditions of each force and the given yield condition. The Mises yield condition applied to a homogeneous isotropic material is used as an illustration. The problem given in this form comes to a typical non-linear programming NLP problem and for its solution the procedure as well as the computer program package SLLIM is used. The obtained limiting stress values are compared with the experimental ones given in the References available to the authors.*

1. THE HYPOTHESES OF THE THIN SHELL THEORY

The elastic thin shell theory is based on the hypothesis of the linear theory of elasticity; namely, that:

1. the material is homogenous and isotropic,
2. stress and strain relations are linear (Hooke's law),
3. displacements are small so that squares and higher degree displacements and their derivatives can be ignored,
4. displacements of points of application of internal and external forces in equilibrium conditions can be neglected.

Additionally, some new hypotheses are introduced:

5. the elements which in the strainless state normal to the middle surface remain normal to the deformed middle surface and their length is not changed
6. the stresses in the cross-section which are parallel to the middle surface could be ignored.

The fifth hypothesis results from the thin plate theory as Kirchhoff-Lawe's

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hypothesis. The essential consequence of this hypothesis is that the displacements of the middle surface shell point are known, such that the shell strain is completely defined by the strain in it's middle surface.

2. THE CIRCULAR CYLINDRICAL SHELLS

On the basis of the thin shell general theory [1,6], and due to the adequate characteristics of the other basis surface form by which the circular cylindrical shell geometry is defined, all general theory expressions are considerably simplified. The position of the middle surface point is determined by x and s coordinates, that is , by the distance along generatrix and the length of directress arch. The length square of the linear element of middle surface is $ds^2 = dx^2 + ds^2$, while Lamé's parameters and curve radii are: $A = 1$, $B = R$, $R_1 = R_x \to \infty$, $R_2 = R_\varphi = R = const.$ The distance from the middle surface is measured by ordinate z, which is positive in the direction of the external normal **n**, (fig. 1).

2.1 The equilibrium conditions

By section x and x+dx, that is φ and $\varphi+d\varphi$, vertical to the middle surface volume element $dxRd\varphi 2h$ is formed. The normal stress $\sigma_z = 0$, according to the hypothesis 6. The relations by which section forces are defined, are in function of component stresses and the element width $ds_\varphi^{\#} = (R + z)ds_\varphi /R$ to the distance z from the middle surface. The volume element of the shell must be in equilibrium under the action of the cross section forces and external tractions $p = [p_x, p_\varphi, p_r]$. Ten forces in the cross section should satisfy six equilibrium conditions, which have been transformed in the following three conditions :

$$1)\ R\frac{\partial N_x}{\partial x} + \frac{\partial N_{\varphi x}}{\partial \varphi} + Rp_x = 0 \qquad 2)\ \frac{\partial N_\varphi}{\partial \varphi} + R\frac{\partial N_{x\varphi}}{\partial x} - \frac{1}{R}\frac{\partial M_\varphi}{\partial \varphi} - \frac{\partial M_{x\varphi}}{\partial x} + Rp_\varphi = 0$$

$$3)\ N_\varphi + \frac{1}{R}\frac{\partial^2 M_\varphi}{\partial \varphi^2} + \frac{\partial^2 M_{x\varphi}}{\partial x \partial \varphi} + R\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{\varphi x}}{\partial x \partial \varphi} + Rp_r = 0 \tag{1}$$

2.2 The strain

During the shell deformation the points of its middle surface are displaced by **u**, **v**, and **w**, respectively, in the direction of a derivative, a tangent on a circle and a normal. These displacements are positive in a sense of x and s increasing , that is, in the direction of the normal to the middle surface. Based on hypothesis 4, displacements of all shell points are determined when displacements of the middle surface shell points are known, that is, the deformation is defined by the three tangential strain components :

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_\varphi = \frac{1}{R}\left(\frac{\partial v}{\partial \varphi} + w\right) \qquad \gamma_{x\varphi} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \varphi} \tag{2}$$

as well as by the three components of the middle surface bending strain, which also include surface bending and a curve change :

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} \qquad \kappa_\varphi = \frac{1}{R^2}\left(-\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial v}{\partial \varphi}\right) \qquad \kappa_{x\varphi} = \frac{1}{R}\left(-\frac{\partial^2 w}{\partial x \partial \varphi} + \frac{\partial v}{\partial x}\right) \tag{3}$$

The strain components must satisfy the three strain compatibility conditions. For the equidistant surface at distance z from the middle surface of the shell, the strain components of the following form

$$\varepsilon_x^* = \varepsilon_x + z\kappa_x \qquad \varepsilon_\varphi^* = \varepsilon_\varphi + z\kappa_\varphi \qquad \gamma_{x\varphi}^* = \gamma_{x\varphi} + 2z\kappa_{x\varphi} \tag{4}$$

relate components of bending strain with the principal strain values of the general theory of elasticity (dilation and shear).

2.3 The section forces and strain values connections

Taking into the consideration the expressions for the strains for an equidistant surface, neglecting h/R members and their higher powers, we reach the following relations for the cross section forces and the strain values of the circular cylindrical shell:

$$\begin{aligned} &N_x = D(\varepsilon_x + \nu\varepsilon_\varphi),\ N_{x\varphi} = 0{,}5(1-\nu)D\gamma_{x\varphi},\ M_x = K(\kappa_x + \nu\kappa_\varphi),\ M_{x\varphi} = -(1-\nu)K\kappa_{x\varphi}, \\ &N_\varphi = D(\varepsilon_\varphi + \nu\varepsilon_x),\ N_{\varphi x} = 0{,}5(1-\nu)D\gamma_{x\varphi},\ M_\varphi = K(\kappa_\varphi + \nu\kappa_x),\ M_{\varphi x} = -(1-\nu)K\kappa_{x\varphi}. \end{aligned} \tag{5}$$

2.4 The limiting conditions

The most frequent shell boundary conditions are:

- a partly or completely free boundary: $N_x = \overline{N}_{x\varphi} = \overline{T}_x = 0,\ M_x = 0$
- a strictly encastered boundary: $u = v = w = 0, \quad \gamma_1 = 0$
- a boundary is connected by pin joint with the support: $u = v = w = 0, \quad M_x = 0$
- a boundary is free supported with diaphragm: $u = 0,\ \overline{N}_{x\varphi} = \overline{T}_x = 0,\ M_x = 0$

Under assumption that boundary is traction-free and supports are rigid, homogeneous boundary conditions have been obtained. That is, some force components and displacements on shell boundary are equal to zero. When boundary is not traction-free, or when supports are subjected to certain displacements, the boundary conditions are not homogenous.

2.5 The limiting state of the circular cylindrical shell

The limiting analysis of a rigid-plastic solid is based on the determination of yield conditions, the allowable static stress field and kinematically admissible field of the strain velocity. This analysis can be simplified by using the fact that the first condition is a scalar mode while two conditions per field can be described without the stress tensor analysis and the velocity tensor in each point. The simplification of this type has already been adopted in elasticity, and will be carried out here. Namely, in the domain of Hooke’s law, internal forces have been defined for the two orthogonal cross-sections. Thus, in order to completely describe the shell stress state it is enough to know six expressions (5), which must satisfy the equilibrium equations (1). To describe the strain state, the stresses can always be transformed into strains via the generalized Hooke’s law. Also, Bernoulli’s law is valid as well as its generalization, that is, the law of keeping of a normal (hypothesis 5), both in elasticity and plasticity domain, which have been shown in theoretical [Massonnet] and experimental papers [Rianitsyn]. On the basis of these hypotheses the adequate simplifications of the shell limiting analysis have been introduced.

Replacing the volume element of **dxdydz** value by the shell element of **dxRdφH**, that is the stress tensor $\boldsymbol{\sigma}_{ij}$ by the generalized stress $\mathbf{Q}_i$ and strain velocity tensor by $\mathbf{q}_i^*$ generalized strain speed, the fundamental plasticity theorem does not change. For further analysis it is important to mention the following theorems: maximal power dissipation theorem: upper and lower carrying limit theorem as well as fundamental characteristics, yield surface convexity and plastic potential law.

Thus, the stress tensor $\boldsymbol{\sigma}_{ij}$ is replaced by the generalized stress symmetrical tensor $\mathbf{Q}_i$, with the following components $\mathbf{Q}_i = (N_x , N_\varphi , N_{x\varphi} , M_x , M_\varphi , M_{x\varphi})$, where $N_{x\varphi} = N_{\varphi x}$ and $M_{x\varphi} = M_{\varphi x}$. Transversal forces T_x and T_φ are not generalized stresses. According to the hypothesis 5, that is, thin shell theory, the normal on the middle surface of the shell is principal axis of the velocity strain tensor at all levels within shell thickness. Ziegler has shown, [2], that this direction is principal for the stress tensor in the shell element wholly plasticified, both for Treska and Huber-Hencky-von Mises's yield condition. This means that the tangential stresses vertical to the middle surface should be equal to zero and, consequently, T_x and T_φ transversal forces are zero, when they are treated as generalized stresses in the generalized stresses tensor. Also, the conclusion is that transversal forces do not participate in dissipation power, and they do not work in elasticity domain so they do not participate in yield condition. This is, certainly, opposite to equilibrium conditions, in which they are different from zero, but they really exist.

Understanding this problem with shell and plates, theoretically and experimentally, is still very limited. However, it is well known that the relative thickness of surface girders, measured as the ratio "the span per thickness", is in general much greater than the ratio "the span per height" of beam girders. With thin shells this ratio can reach the value of 500 while with beams rarely it exceeds 30. For this reason, it is logical to suppose that transversal forces will not have significant influence on shell plastification so that they can be ignored.

Generally the yield condition of shell element is the following function:

$$\varphi = (N_x, N_\varphi, N_{x\varphi}, M_x, M_\varphi, M_{x\varphi}, C_1, ..., C_2) = 0 \,, \tag{6}$$

where $C_1,...,C_2$ are the characteristic constants of material. This condition is represented by convex surface in space with 6 dimensions that refer to $ON_x ,...,OM_{x\varphi}$ axes.

In general case the ratio between stress and plastic strains is given in differential mode, that is, by yield law. In case of circular cylindrical shells, plastic strains are determined through known equations, in which strain components have been connected to displacements, in accordance with hypothesis 3 on small strains. During a time interval dt from yield appearance, strain components vary. Plastic yield is characterized by the strain velocity field with the following components:

$$\begin{aligned}
&\dot{\varepsilon}_x = \frac{\partial \dot{u}}{\partial x}, \quad \dot{\varepsilon}_\varphi = \frac{1}{R}\left(\frac{\partial \dot{v}}{\partial \varphi} + w\right), \quad \dot{\gamma}_{x\varphi} = \frac{\partial \dot{v}}{\partial x} + \frac{1}{R}\frac{\partial \dot{u}}{\partial \varphi} \\
&\dot{\kappa}_x = \frac{\partial^2 \dot{w}}{\partial x^2}, \dot{\kappa}_\varphi = \frac{1}{R^2}\left(-\frac{\partial^2 \dot{w}}{\partial \varphi^2} + \frac{\partial \dot{v}}{\partial \varphi}\right), \quad \dot{\kappa}_{x\varphi} = \frac{1}{R}\left(-\frac{\partial^2 \dot{w}}{\partial x \partial \varphi} + \frac{\partial \dot{v}}{\partial x}\right)
\end{aligned} \tag{7}$$

In order to completely define the strain process, the fact that plastic strain occurs at a constant volume should be taken into consideration. As the consequence of plastic potential law, application to the yield condition this condition is stated as linear tensor invariant of strain velocity through equation: $\dot{e}_x + \dot{e}_\varphi + \dot{e}_z = 0$.

Three equilibrium equations (1) and equation (6) form a system of four equations with six unknown generalized stresses $\mathbf{Q}_i$. Since the equation system (1) is a system of differential equations, the adequate boundary conditions per generalized stresses should be stated. The problem of determining the internal forces defined in this way is, certainly, statically undeterminate. By applying the yield law [12] to the yield condition, (6), six relations will be obtained,

$$q_i^* = \lambda \frac{\partial \Phi}{\partial Q_i} \tag{8}$$

where $q_i^* = (\varepsilon_x^*, \varepsilon_\varphi^*, \gamma_{x\varphi}^*, \kappa_x^*, \kappa_\varphi^*, \kappa_{x\varphi}^*)$ is a function of six generalized strain velocity components, and λ is a positive scalar. Strain velocity components q_i^* are partial derivatives of yield conditions (6), that is they are the functions of generalized stresses $N_x,...,M_{x\varphi}$ The generalized stresses are via relations (5) connected to strain components, and they are through equations (7) connected to components of displacements velocity vectors u^*, v^* and w^*.

At last , a recapitulation of available equations can be done and also the comparison with the number of unknown values made.

There are ten equations in total:

- yield condition (6) .... one equation
- equilibrium conditions (1) .... three equations
- generalized strain velocity relations (8)..six equations

The unknown values are:

- six generalized stresses $\mathbf{Q}_i = (N_x , N_\varphi , N_{x\varphi} , M_x , M_\varphi , M_{x\varphi})$
- three components of displacement velocity vector......................................u^*, v^* w^*
- scalar parameter... λ.

Since the number of the unknown values is equal to the number of equations, the problem of determining the boundary carrying capacity can be considered statically and kinematically determined with boundary conditions per forces and strain velocities satisfied (theoretically the solution is possible in a closed form). However, relations for connection of velocity vector component with strain velocity components, and also differential connections(5), are such that it is possible to obtain analytical solution by use of a very complicated mathematical apparatus, that is, practically, the problem is possible to be solved in principle. For that reason many scientists used to resort to approximate solutions in such a way that they introduced corresponding simplifications [3,5]. Very often these solutions were directed to determination of boundary load approximate values by applying the upper and lower limit theorem (kinematical and static method). In the following paragraph for the defined simultaneous problem [12], that is the equations (1), (6) and (8), the procedure for determination of the lower limit carrying capacity for the circular cylindrical shell is proposed.

3. THE PROCEDURE FOR DETERMINATION OF THE APPROXIMATE VALUE OF THE BOUNDARY CARRYING CAPACITY FOR CYLINDRICAL SHELLS

3.1 The Generalized stresses

The state of stress is determined by the field of generalized stresses

$$Q_i = (N_x, N_\varphi, N_{x\varphi}, N_{\varphi x}, M_x, M_\varphi, M_{x\varphi}, M_{\varphi x})\,. \tag{9}$$

Numerical approximation of physical values is realized through development of these values into complete sixth degree polynomials per variables x and s. Due to numerical treatment of a problem, instead of real variables, dimensionless values have been used. The normed point coordinates have the following form: ζ = x/R, φ = s/R, and the complete sixth degree polynomial per ζ and φ is of the following form:

$$\begin{aligned} P = \big(&1, \xi, \varphi, \xi^2, \xi\varphi, \varphi^2, \xi^3, \xi^2\varphi, \xi\varphi^2, \varphi^3, \xi^4, \xi^3\varphi, \xi^2\varphi^2, \xi\varphi^3, \varphi^4, \\ &\xi^5, \xi^4\varphi, \xi^3\varphi^2, \xi^2\varphi^3, \xi\varphi^4, \varphi^5,, \xi^6, \xi^5\varphi, \xi^4\varphi^2, \xi^3\varphi^3, \xi^2\varphi^4, \xi\varphi^5, \varphi^6\big) \end{aligned} \tag{10}$$

The generalized stresses become dimensionless values when they are reduced by the corresponding plasticity modulus $N_p = 2h\Re_e$ and $N_p = 2h\Re_e$ as follows: $\mathbf{n}_p = N_i / N_p$ and $\mathbf{m}_p = M_i / M_p$; while the field of dimensionless generalized stresses is defined by the expression

$$Q_i = (n_x, n_\varphi, n_{x\varphi}, n_{\varphi x}, m_x, m_\varphi, m_{x\varphi}, m_{\varphi x})\,. \tag{11}$$

Adding to the above polynomial members (11) corresponding variables $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{S}_k$ and $\mathbf{J}_k$, respectively from $\mathbf{n}_x$ to $\mathbf{m}_{x\varphi}$, expressions for stress generalization are finally formed, having the following general form

$$n_x = \frac{N_x}{N_p} = \sum_{i=0}^{6}\sum_{j=0}^{i} A_k \xi^{i-j}\varphi^j, m_x = \frac{M_x}{M_p} = \sum_{i=0}^{6}\sum_{j=0}^{i} C_k \xi^{i-j}\varphi^j, \left[k = \frac{1}{2}(i+1)i + j + 1\right]. \tag{12}$$

By introducing dimensionless values for coordinates and generalized stresses as well as the assumption that load components are $p_x = p_\varphi = 0$, while $p_r \neq 0$, the equations (1) are transformed:

$$\begin{aligned} &1)\ \frac{\partial n_x}{\partial \xi} + \frac{\partial n_{\varphi x}}{\partial \varphi} = 0 \qquad 2)\ \frac{\partial n_\varphi}{\partial \varphi} + \frac{\partial n_{x\varphi}}{\partial \xi} - \kappa\frac{\partial m_\varphi}{\partial \varphi} - \kappa\frac{\partial m_{x\varphi}}{\partial \xi} = 0 \\ &3)\ n_\varphi + \kappa\left(\frac{\partial^2 m_\varphi}{\partial \varphi^2} + \frac{\partial^2 m_{x\varphi}}{\partial \xi \partial \varphi} + \frac{\partial^2 m_x}{\partial \xi^2} + \frac{\partial^2 m_{\varphi x}}{\partial \xi \partial \varphi}\right) = -p_r. \end{aligned} \tag{13}$$

Equilibrium condition $\Sigma M_n = 0$ should be added to equations (13) in the form

$$4)\ n_{x\varphi} - n_{\varphi x} + \kappa m_{\varphi x} = 0 \tag{14}$$

For the method applied in this paper, as well as according to the principal assumptions of the lower limit carrying capacity theorem, the equation (14) is not an identity. Different from stress field $\boldsymbol{\sigma}_{ij}$ when according to the elasticity theory assumption the stress tensor is symmetrical, that is $\tau_{x\varphi} = \tau_{\varphi x}$, the generalized stress field $\mathbf{Q}_i$, by definition, does not satisfy this assumption, since $N_{x\varphi} \neq N_{\varphi x}$ and $M_{x\varphi} \neq M_{\varphi x}$, so that the use of supplemental equilibrium equation (14) is completely justified.

According to the lower limit theorem there is no need to define the permissible strain velocity field, therefore relations between generalized stresses and strain velocity are unnecessary. Also, according to strain work and plastic potential formulation, forces that make work (generalized stresses $\mathbf{Q}_i$, here) participate in the yield condition, and the yield condition itself, defined by expression (19), must be satisfied.

In general case radially distributed external load $p_r(x,s)$ can be subjected in the form of a full polynomial with a degree lower for two from the polynomial by which generalized stresses approximation has been carried out (the third equilibrium equation, (13), is the second degree differential equation). External load, thus, can, be subjected in the form of the fourth degree full polynomial along ζ and φ coordinates, that is , $p_r(x,s) = p_1(1+p_2\xi+p_3\varphi+...+p_{15}\varphi^4)$, where every $p_2, p_3,...,p_{15}$ is divided by p_1.

For the radial uniformly distributed load all polynomial members, except for the first one, are equal to zero. The free member on the right side of equation (13) is dimensionless value, that is $p_r = -Rp_1/N_p$, which depends on $\boldsymbol{\kappa}$ coefficient. The corresponding equilibrium equation members are multiplied by $\boldsymbol{\kappa}$ coefficient. In equations (13) and (14) $\boldsymbol{\kappa}$ coefficient is calculated by the expression $\boldsymbol{\kappa} = M_p/(RN_p)$ and the function has physical and geometrical characteristics of a shell.

3.2 The expressions for the transverse forces

The transverse forces at the cross-sections with ζ and φ normals are the bending moment and torsion moment functions. On shell contours the substituting transverse forces, normed by N_p, become dimensionless values in the following form:

$$\bar{t}_x = \kappa\left(\frac{\partial m_{x\varphi}}{\partial\varphi} + \frac{\partial m_x}{\partial\xi} + \frac{\partial m_{\varphi x}}{\partial\varphi}\right) i \bar{t}_\varphi = \kappa\left(\frac{\partial m_{\varphi x}}{\partial\xi} + \frac{\partial m_\varphi}{\partial\varphi} + \frac{\partial m_{x\varphi}}{\partial\xi}\right) \tag{15}$$

3.3 The boundary conditions

In order to determine uniformly the generalized stresses from differential equations (13) and (14) it is necessary to satisfy boundary conditions. Since the middle shell surface is limited by lines which coincide with principal coordinate directions, for each contour it is possible to set maximum four nonhomogenous boundary conditions per form forces:

for contour $\zeta = \pm 1/R$ $\quad n_x = n_x^s;\ \ m_x = m_x^s;\ \ \bar{n}_{x\varphi} = \bar{n}_{x\varphi}^s,\ \ \bar{t}_x = \bar{t}_x^s$ (16)

for contour $\varphi = \pm\ \varphi_o$ $\quad n_\varphi = n_\varphi^s;\ \ m_\varphi = m_\varphi^s;\ \ \bar{n}_{\varphi x} = \bar{n}_{\varphi x}^s,\ \ \bar{t}_\varphi = \bar{t}_\varphi^s$ (17)

The normed substituting shear force is given by the expression:

$$\bar{n}_{x\varphi} = \frac{\overline{N}_{x\varphi}}{N_p} = n_{x\varphi} - \kappa m_{x\varphi} \tag{18}$$

Forces with ^{s} indication in index represent subjected forces that load the boundary. Along the boundary they can be concentrated forces of a constant intensity per boundary length, or the sixth degree at the most per corresponding variable ζ or φ, which is caused by the accepted polynomial degree for the generalized stresses. For example normal force $n_x^s = {}_x n_1 + {}_x n_2 \varphi + {}_x n_3 \varphi^{2+} + {}_x n_4 \varphi^3 + {}_x n_5 \varphi^4 + {}_x n_6 \varphi^5 + {}_x n_7 \varphi^6$ etc.

3.4 The yield condition

For the defined field of generalized stresses, (9), the yield condition is eight component function,

$$\Phi(n_x, n_\varphi, n_{x\varphi}, n_{\varphi x}, m_x, m_\varphi, m_{x\varphi}, m_{\varphi x}) = 0\,. \tag{19}$$

When choosing yield conditions for a given construction physical characteristics of material are dominant. On the basis of hypothesis 6 and stress tensor for circular cylindrical shell subjected to cylindrical coordinate system x, φ and R, the yield condition is simplified and has the following form $\sigma_x^2 - \sigma_x\sigma_\varphi + \sigma_\varphi^2 + 3\tau_{x\varphi}^2 = \Re_e^2$, where $\Re_e$ is a referent value which depends on physical characteristics of material. Similar to generalized stresses the yield condition should be reduced to dimensionless values normed by $\Re_e^2$. Besides the yield condition should be the function of the same variables like equilibrium and boundary conditions and they are generalized stresses. At last, after a series of transformation the yield condition obtains the following form:

$$\begin{aligned}&\left[\frac{m_x^2}{2} + m_x\left(\frac{m_x^2}{4} + n_x^2\right)^{\frac{1}{2}} + n_x^2\right] + \left[\frac{m_\varphi^2}{2} + m_\varphi\left(\frac{m_\varphi^2}{4} + n_\varphi^2\right)^{\frac{1}{2}} + n_\varphi^2\right] - \\ &- \left[\frac{m_x}{2} + \left(\frac{m_x^2}{4} + n_x^2\right)^{\frac{1}{2}}\right]\cdot\left[\frac{m_\varphi}{2} + \left(\frac{m_\varphi^2}{4} + n_\varphi^2\right)^{\frac{1}{2}}\right] + \\ &+ 3\left[\frac{m_{x\varphi}}{2} + \left(\frac{m_{x\varphi}^2}{4} + n_{x\varphi}^2\right)^{\frac{1}{2}}\right]\cdot\left[\frac{m_{\varphi x}}{2} + \left(\frac{m_{\varphi x}^2}{4} + n_{\varphi x}^2\right)^{\frac{1}{2}}\right] = 1\end{aligned} \tag{20}$$

3.5 The possible stress state

The stress field, defined by $\mathbf{Q}_i$ generalized stress, is statically permissible if stresses satisfy equilibrium equations with in the whole shell area and if boundary conditions per forces are fulfilled. Beside the stated for plastically possible field it is necessary conditions, that the yield condition in that area and at the boundary be satisfied. Depending on the boundary relation with the support the following boundary conditions can be set:

the partly or fully free contour

$$\text{for } \zeta = \pm l/R \quad 1)\ n_x = n_x^s, \quad 2)\ m_x = m_x^s, \quad 3)\ \bar{n}_{x\varphi} = \bar{n}_{x\varphi}^s, \quad 4)\ \bar{t}_x = \bar{t}_x^s \tag{21}$$

the contour is connected by joint with the supports

for $\zeta = \pm l/R$, it is always $\mathbf{n}_x = \mathbf{n}_x^s$, and from the remaining three conditions: $\mathbf{m}_x = \mathbf{m}_x^s$,

$\bar{n}_{x\varphi} = \bar{n}^{s}_{x\varphi}$, $\bar{t}_x = \bar{t}^{s}_x$ none of them is needed to be set, or plus two at most, which depends on which force direction coincides the support direction:

the fixed contour

on this contour, conditions per forces can not be set. However, kinematical solution for failure line distribution based on upper limit carrying capacity theorem offers a possibility to set a condition, for example, for the contour whose normal is **k**, $m_k = m^{s}_k$, that is $m_k = 1$ where $M_k/M_p = 1$, since M_k is moment value of full cross-section plastification on the contour.

3.6 The unknown values and conditions

The unknown values are generalized stresses $\mathbf{Q}_i$ and load parameter $\mathbf{p}_r$. Each generalized stress comprises 28 unknown values, and they are the sixth degree polynomial coefficient, (12), marked as $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{S}_k$ and $\mathbf{J}_k$ and there are 8×28 = 224 total. The unknown load parameter p_r should be joined to their total number, that is, the total number of the unknown values is 225. The following conditions should be met:

the equation of I-form: equilibrium conditions (13) and (14) and boundary conditions (21) as well

the inequality of II-form: yield condition (20).

In equation form conditions point coordinates ζ and φ do not figure. Their overall number in scalar form is 98 equations from equilibrium condition (13) and (14), (the first and second per 21, the third and fourth per 28); 7 equations for each of the first three boundary conditions (21); and 6 equations for each of the fourth boundary conditions (21).

$$U_j = 98 + 7(L_n + L_m + L_{nt}) + 6L_t,$$

where U_j denotes the overall number of equations, and L_n, L_m and L_{nt} the first three form conditions (21), and L_t denotes the number of equations of the last form condition (21) that can be set for a concrete shell case. For the shell from figure 1 it is obvious that the number of independent equations cannot exceed the number of the unknown values from $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{S}_k$ and $\mathbf{J}_k$, that is, $U_j \le 224$. For complicated shell forms the case $U_j \ge 224$ is possible, especially if polynomial degree is considerably reduced.

3.7 The mathematical problem formulation

Load parameter $\mathbf{p}_r$ is via equilibrium condition in correlation with the unknown values $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{S}_k$ and $\mathbf{J}_k$, and indirectly via yield conditions with point coordinates of shell area ζ and φ,

$$\mathbf{p}_r = f(\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k, \mathbf{D}_k, \mathbf{E}_k, \mathbf{F}_k, \mathbf{S}_k, \mathbf{J}_k, \zeta, \varphi), \quad (k=1,2,...,28) \tag{22}$$

If all the unknown values $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{S}_k$ and $\mathbf{J}_k$ are denoted with uniform marks $\mathbf{X}_i$, where i = 1,2,...,*n*, *n* = 224, the expression (22) becomes,

$$\mathbf{p}_r = f(\mathbf{X}_i, \zeta, \varphi) \tag{23}$$

The equation form conditions (13), (14) and (21) are scalar conditions and are denoted with

$$h_j(\mathbf{X}_i) = 0, \quad (j = 1,2,...,U_j) \tag{24}$$

The yield condition which is the function of the unknown values and coordinates, according to the marks introduced, can be written in the form of

$$G = G(\mathbf{X}_i, \zeta, \varphi) \geq 0 \tag{25}$$

and applies for each ζ and φ. A mark has-been introduced for the yield condition G as well as in computer program SLLIM, that is $G \equiv \Phi$.

However, thus defined problem can be solved by applying some of the optimization methods by which the peak value and also minimum or maximum of a function is determined, that is, the target function, at a set of allowable solutions, which have been subjected by in advance known functions, which represents a typical problem of a non-linear programming (NLP problem). Therefore, the function (23) shall be maximized, and conditions (24) and (25) satisfied in the whole shell area.

The usual optimization method, based on the variation calculation, cannot be used in this case since the yield condition is given in the form of an inequality due to its character. For this reason as well as for the non-linear relation in the same yield condition, the method of the non-linear programming is very indicative. However, the main difficulty is not about a high degree of inequalities nonlinearity (fourteen: X_i^2 , ζ^{12} and φ^{12}) (25), but in the fact that the condition (25) should be fulfilled in all the area points of the shell which means that there is an infinite number of these conditions. Thus, some simplifications should be introduced so that the problem set in this way could be practically solved.

3.8 The method for the lower limit carrying capacity solution

If the problem from (22) to (25) is differently formulated so that the yield condition (25) is satisfied only in a chosen point series instead of the whole shell area, then the point coordinates do not appear as variables. The yield point becomes much more simplified (second degree only), while the inequality number reduces to the number of points chosen n. The problem can be formulated in the following way:

$$\begin{aligned} &\mathbf{p}_r = f(\mathbf{X}_i) \qquad (\mathbf{X}_i = X_1,...,X_{224}) \\ &h_j(\mathbf{X}_i) = 0, \qquad (j = 1,2,...,U_j) \\ &G_n = G(\mathbf{X}_i) \geq 0 \quad (n = 1,2,...,\tilde{n}) \end{aligned} \tag{26}$$

By solving the problem (26) the values for all variables $\mathbf{X}_i$, as well as the maximal possible load parameter value $\mathbf{p}_r$ is obtained. It means that statically possible load field is obtained. However, the condition is that the field is plastically feasible, which is also fulfilled, but only in the chosen shell points series. In a strictly mathematical view this condition is not fulfilled in the whole shell area, since it may be disturbed in one of the "middle points" area.

The obtained load parameter value $\mathbf{p}_r$ is the highest statically and plastically possible load parameter value, based on the lower limit carrying capacity theorem, that is, it is for sure the best static solution within the limitations of the selected procedure. Certainly, it refers to the approximate solution, which usually results from the boundary carrying capacity theorems. Thus, the solution accuracy can, in general case, be evaluated by the difference between the solutions obtained by the lower and upper carrying capacity limit.

When there are exact solutions, a possibility remains to compare thus obtained values with the exact solutions.

4. AN EXAMPLE OF THE CIRCULAR CYLINDRICAL SHELL ANALYSIS

In figure 1 a shell form and geometry are defined. The load is uniformly rotationally symmetrical, that is, $\mathbf{p}_r = \mathbf{p}_r(\xi,\varphi) = p_1 = const$ along the whole shell surface. For a boundary $\varphi = \varphi_o$, which is free, the boundary conditions per forces are: 1) $\mathbf{n}_\varphi = 0$, $\mathbf{m}_\varphi = 0$, $\mathbf{n}_{\varphi x} = 0$ and 2) $\mathbf{n}_{\varphi x} = 0$. On boundaries $\xi = \pm l/R$ the shell is freely supported, with the conditions per forces, $\mathbf{n}_x = 0$, $\mathbf{m}_x = 0$.

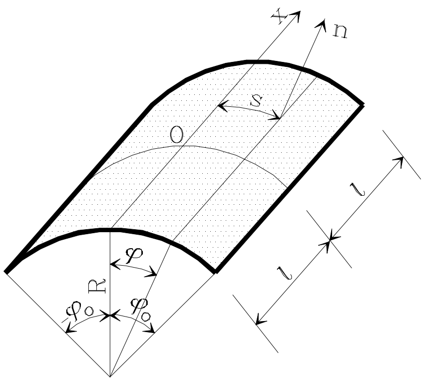

Fig. 1.

For the purpose of reducing the unknown values number it is helpful in this example to use the possibility offered by symmetry either by the shell geometry or by the load form. The number of the unknown values, uniformly denoted by $\mathbf{X}_i$, is reduced and it is n = 64. However, the load parameter p_r is also one of the n unknowns.

The overall number of scalar equations is: 28 equations from equilibrium conditions (13) and (14), (the first, second and fourth per 6, the third 10); 4 equations for each form boundary condition (21/1,2,3) and 3 equations for each form boundary condition (21/4).

$U_j = 28 + 4\times5 + 3\times1 = 51$

However, since $U_j \leq 64$, the problem is statically undetermined.

Based on the problem stated by (26), and according to NLP method, an optimization equation should be chosen. Since the ultimate aim is to determine the shell carrying capacity, the target function will be equilibrium equation comprising the unknown load parameter p_r, that is, the first scalar equation from the third equilibrium condition (13), where $\mathbf{p}_r = -Rp_1/N_p$. This equation should be maximized, or, since its value is negative, it should be minimized, which is essentially the same, that is,

$$\min p_r = \min f(X_i) = \min\left(-\frac{Rp_1}{N_p}\right) = p_r^* \tag{27}$$

From the remaining 50 scalar equations, which do not depend on ξ and φ coordinates, it is necessary to extract those unknown values $\mathbf{X}_i$ which are linearly dependent. After linearly dependent value elimination n = 17 independently variables remain, which points out that there were 33 linearly dependent equations among the scalar equations U_j. The generalized stresses then have the following form:

$$n_x = 0{,}5\kappa{,}5\kappa(\xi^2 - l^2)\{(\varphi_o^2 - 3\varphi^2)[8X_2 + 4X_3 + X_8 + X_{14} + 0{,}5(1{,}\dot{3}X_4 + X_{10} + X_{16})] + \\ + [20(\varphi^2 - 3\varphi_o^2) + (\varphi^4 - 5\varphi_o^4)](6X_5 + X_{11} + X_{17})\}$$

$$n_{\varphi} = -\kappa(\varphi^2 - \varphi_o^2)^2[2X_2 + 30X_5 + 5(X_{11} + X_{17})]$$

$$m_x = (\xi^2 - 1^2)\{X_1 + [(\varphi^2 - 2\varphi_o^2)X_2 - 1{,}5(4X_3 + X_8 + X_{14}) - 30\varphi_o^2 X_5 - 5\varphi_o^2(X_{11} + X_{17})] + $$
$$+ (\xi^2 + 1^2)(\varphi_o^2 - 3\varphi^2)[0{,}3X_4 + 0{,}25(X_{10} + X_{16})]\}$$

$$m_{\varphi} = 0{,}5(\varphi^2 - \varphi_o^2)\{2(\varphi^2 - \varphi_o^2)(X_3 + \xi^2 X_4) - X_6 - X_{12} - [3(X_7 + X_{13}) + 5\xi^2(X_9 + X_{15})] -$$
$$- \varphi_o^2[X_8 + X_{14} + 3\xi^2(X_{10} + X_{16}) + \varphi_o^2(X_{11} + X_{17})]\} + (\varphi^6 - 2\varphi_o^6 - 3\varphi^2\varphi_o^4)X_5$$

$$n_{x\varphi} = \kappa\xi\varphi\{[20\varphi^2 - \varphi_o^2) + (\varphi^4 - \varphi_o^4)](6X_5 + X_{11}) + [20(\varphi^2 - \varphi_o^2) - \varphi_o^4]X_{17} - (\varphi^2 - \varphi_o^2)$$
$$[\xi^2(1{,}3X_4 + X_{10}) + 8X_2 + 4X_3 + X_8] - X_{12} - \xi^2 X_{13} - \xi^4 X_{15} - \xi^2\varphi_o^2 X_{16} - \varphi_o^2 X_{14}\}$$

$$n_{\varphi x} = \kappa\xi\varphi\{(\varphi^2 - \varphi_o^2)[\xi^2(1{,}3X_4 + X_{10} + X_{16}) + 8X_2 + 4X_3 + X_8 + X_{14}] +$$
$$+ [20(\varphi^2 - \varphi_o^2) + (\varphi^4 - \varphi_o^4)](6X_5 + X_{11} + X_{17})\}$$

$$m_{x\varphi} = \xi\varphi(X_6 + \xi^2 X_7 + \varphi^2 X_8 + \xi^4 X_9 + \xi^2\varphi^2 X_{10} + \varphi^4 X_{11})$$

$$m_{\varphi x} = \xi\varphi(X_{12} + \xi^2 X_{13} + \varphi^2 X_{14} + \xi^4 X_{15} + \xi^2\varphi^2 X_{16} + \varphi^4 X_{17})\,. \quad (28)$$

The target function (27) has the following form:

$$_{\min} p_r = _{\min}[\kappa\{2X_1 - \varphi_o^2(4X_3 + X_8 + X_{14}) - 2\varphi_o^4[X + 3(6X_5 + X_{11} + X_{17})]\}] = p_r^* \quad (29)$$

The yield condition of the form (20), $G(\mathbf{X}_i) \geq 0$, has to be fulfilled in a chosen shell "points" series, (ñ =81). NLP problem form (26) is now formulated:

$$\max \mathbf{p}_r = \max f(\mathbf{X}_i) \qquad (\mathbf{X}_i = X_1,...,X_{17})$$
$$G_n(\mathbf{X}_i) \geq 0 \qquad (n = 1,2,...,81) \quad (30)$$

5. THE INTERPRETATION AND ANALYSIS OF THE RESULTS OBTAINED

In paragraphs 3 an 4 the problem is mathematically stated and illustrated by the example, and by SLLIM computer program a numerical analysis is carried out by use of NLP method. However, the qualitative and quantitative estimate of the acceptability of solutions obtained by the suggested procedure is possible only by comparing them with the upper limit or with the known experimental values for the same problem. Since the determination of the upper limit and the experiment conduct were not the subject of this paper, it was necessary to find a qualitative solution by which comparison should be made. Unfortunately there was no exact solution for the problem, as well as the solution for the upper carrying capacity limit in the references used. For this reason, the results from experiment [3] have been used as verified values for comparing solutions obtained by the suggested procedure. According to the description of the author himself, shell models have forms as shown in fig. 1, and are made of cement mortar. In order to avoid the possibility that shell stability becomes a dominant influence upon their carrying capacity, adequate model forms and dimensions are chosen to be analyzed.

Models are without edge beams, which has enabled the study of a failure of the shell itself. All shell types are circular cylindrical shells, free supported on boundaries x = ±l and free on boundaries $\varphi = \pm\varphi_o$. Supporting has been performed through diaphragms, which for the purpose of experiment can be considered absolutely rigid. Different types of shells are realized through shell length variations, L, and their opening, which is practically done with the change of the boundary angle $\pm\varphi_o$. In this way shells of long and short type have been formed, which define L:a ratio, that is, the proportion of arch of shell and shell width (f:a). All models belong to a shallow shell class. It should be emphasized that experiments have shown that failure mechanisms consist of rigid parts connected by pin joints along "lines of pin joints", that is, lines which underwent a failure. According to this fact it could be said that failure mechanisms are similar to mechanisms met with flat plates.

Table 1. Shell model geometric characteristics

shell type	**φ** [°]	**R** [cm]	**f** [cm]	**a** [cm]	**L=2l** [cm]	**H=2h** [cm]	**f:a**	**L:a**
A	30	93	12,5	93	188	2,2	0,134	2,00
B	30	93	12,5	93	90	2,2	0,134	1,00
C	17	93	4,1	54	188	2,2	0,076	3,48
D	17	93	4,1	54	90	2,2	0,076	1,67

f-arch of shell a-shell hole width L-shell length

Beside experiment results, the author in the same paper [3], suggests an analytic solution for the lower limit carrying capacity. This analytic solution is based on the usage of discontinuity in the generalized stress field for the cross section forces $\mathbf{n}_x$ and $\mathbf{m}_x$ which belong to the cross section along "lines of pin joints" of failure mechanism. When these analytic solutions are concerned, without analyzing their theoretical assumptions, it is necessary to mention that principal assumptions of the thin shell theory and also the lower limit theorem do not allow the discontinuity of stress and strain fields, since then the compatibility conditions are not satisfied. However, there are also such approaches to the boundary carrying capacity problem solution [5]. This problem was especially studied by Prager and Hill. These authors consider the mentioned A. Sawczuk's analytic solution acceptable, so that it has been adopted as a comparative parameter for the estimate of solution quality obtained according to NLP procedure, suggested in this paper. By choosing the comparative solutions, the choice of the same shell types for the analysis by suggested procedure, that is, by SLLIM program, is caused. The Huber-Hencky-von-Mises yield condition is used in the examples, which from theoretical aspect is considered the most strict, regarding the fact that in this paper the yield condition which certainly suits the reinforced cement mortar, was unknown. Numerical values of the obtained and comparative results are given in table 2 . By $\mathbf{p}^e$ and $\mathbf{p}^s$ signs the boundary load intensities are denoted, obtained by experimental, that is, analytical procedure respectively, expressed in the real measures. Since according to the experiment description, the support along boundary couldn't be clearly defined x = ±l, that is, it is not clear whether the displacement is $u \neq 0$ or $u = 0$, the solution for both cases has been found by SLLIM computer program. These values have $\mathbf{f}^*$ and $\mathbf{f}^{**}$ signs respectively in table 2. The real values for boundary load parameters $\mathbf{f}^*$ an $\mathbf{f}^{**}$ are obtained according to the expression (27). However, the determination of $\Re_e$ value, that is, boundary values of N_p and M_p forces per length unit of shell cross section for real material, such as the reinforced cement mortar, somehow represents a difficulty. The reason for this is lack of information on plastic behavior of the reinforced cement mortar, the material of which the analyzed shell models are made.

Missing the adequate yield condition for this material, an attempt has been made to determine, at least approximately, boundary values for N_p and M_p forces for the observed shell models, by applying a method from BAB'87 regulations, which otherwise apply for the reinforced concrete. Thus, it is possible to express dimensionless values of boundary load, $\mathbf{p}_r$, as real values of boundary load $\mathbf{f}^*$ an $\mathbf{f}^{**}$ and represent them in table 2.

Mechanical characteristics of the model material are:

- R'_{br} = 1,40 kN/cm^2 calculating strength of cement mortar at pressure
- $\Re_e$ = 100,00 kN/cm^2 yield limit reinforcement
- The reinforcement is divided into two orthogonal networks parallel to principal coordinates directions x and φ with holes of square form, 13 × 13mm (n = 2 × 78 number of wires/m).

N_p normal force and M_p moment have been calculated, N_p = 460,86 kN/m, M_p = 1,596 kNm/m, and coefficient $\boldsymbol{\kappa} = M_p/(RN_p) = 0,0037237$.

The boundary load intensity $\mathbf{f}^*$, determined by a static method with application of non-linear programming (NLP method), for all shell types A, B, C and D is lower than the boundary load intensity p^e from the experiment (by 60, 43, 49 and 23%). Thus the theorem of the lower limit of the intensity of the boundary load is confirmed. For all shell types $\mathbf{f}^{**} > \mathbf{f}^*$, which is a consequence of some relations of shells with diaphragms, that is, displacements **u** at supports in the direction of x have been prevented. This is especially prominent for shell type B, where ratio is L:a = 1 and $\mathbf{f}^{**} \approx 2\mathbf{f}^*$; and the smallest difference is for C type (2,6%) where ratio is L:a = 3,48 which could be expected. With respect to the experiment, the boundary load intensity is lower, that is, it is always $\mathbf{f}^{**} < \mathbf{p}^e$, which according to the table sequence is 39, 16, 48 and 14,5%.

Table 2.

shell type	$\mathbf{p^e}$ [kN/m^2]	$\mathbf{p^s}$ [kN/m^2]	$\mathbf{p^s}$ [%]	$\mathbf{f^*}$ [kN/m^2]	$\mathbf{f^*}$ [%]	$\mathbf{f^{**}}$ [kN/m^2]	$\mathbf{f^{**}}$ [%]
A	10,5	15,7	+49,0	**4,213**	*-60,0*	**6,428**	*-39,0*
B	35,0	68,7	+96,0	**19,878**	*-43,0*	**40,754**	*+16,0*
C	7,0	5,0	-28,0	**3,526**	*-49,0*	**3,620**	*-48,0*
D	19,5	21,5	+10,0	**14,854**	*-23,0*	**16,664**	*-14,5*

$\mathbf{p}^e$ - EXPERIMENT [3]
$\mathbf{p}^s$ - ANALYTIC SOLUTION ACCORDING TO SAWCZUK'S SUGGESTION [3]
$\mathbf{f}^*$ - VALUES OBTAINED BY NLP METHOD (ON BOUNDARYS x = ±l: BOUNDARY CONDITIONS $\mathbf{n}_x = \mathbf{m}_x = 0$)
$\mathbf{f}^{**}$ - VALUES OBTAINED BY NLP METHOD (ON BOUNDARYS x = ±l: BOUNDARY CONDITION IS $\mathbf{m}_x = 0$)

As already said, the boundary load p^s represents the lower limit carrying capacity. For A, B and D shell types, boundary load p^s values are higher than the experimental ones p^e and expressed in the same sequence in percentages are 49, 96 and 10%. This deviation is lower than the experimental value for C type shell for 28% and this solution can be considered the only acceptable value, since this refers to the procedure based on the lower limit carrying capacity theorem.

Beside boundary load parameters, the output values of SLLIM computer program are the values of yield conditions and the generalized stress Q_i in the discrete shell point system. Their change is graphically represented in Fig. 2 and 3. In all figures the middle shell surface is represented, without thickness, which is considered to be zero surface, and vertically to it in a selected discrete point sequence, (ñ = 81), ordinates are directed which correspond to physical values.

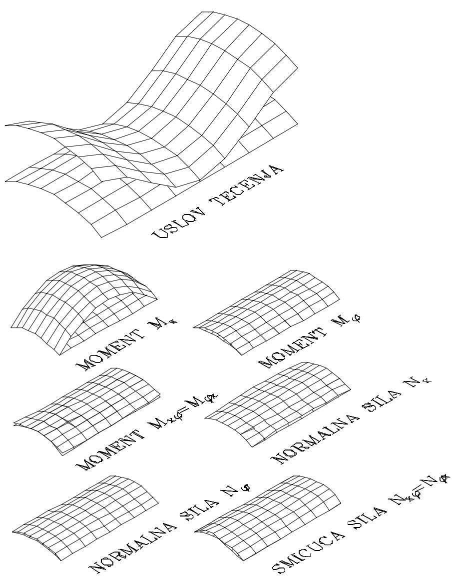

Fig. 2. Tip A (displacements **u** at supports in the direction of x have been not prevented)

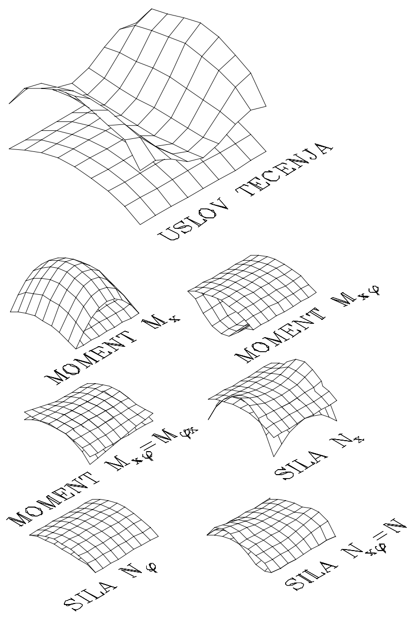

Fig. 3. Tip B (displacements **u** at supports in the direction of x have been prevented)

In figures 2 and 3 the yield conditions for A and B type shells are represented. Since the yield condition function in the whole shell area fulfills the condition $G(\mathbf{X}_i, \xi, \varphi) \geq 0$, the ordinates are always positive, which is clearly seen in the figures. It should also be mentioned that each middle surface point, whose ordinate is equal to zero, corresponds to a cross-section of the exhausted carrying capacity.

Also, the distribution of the generalized stresses Q_i is represented in the figures. The ordinates numerical values are calculated in a selected points sequence (ñ = 81). From the generalized stress fields Q_i which have been normed by N_p and M_p modules, the real cross section forces values can be easily calculated, that is N_x,..., $M_{\varphi x}$, by which the shell stress is defined, via ordinates multiplication with N_p and M_p .

The changes of the yield condition function on shell surface, represented in Fig. 2 and 3, could be used for the estimate of a possible failure mechanism form, since the failure will occur in the points where this condition reached its maximum, that is, where it is equal to zero. On the basis of the figures, photos and descriptions [5] of real failure mechanisms on shell models from comparative experiments [3] , a great similarity has been found between the real construction behavior and the results obtained by the suggested procedure. This fact, certainly, confirms the acceptability of the results obtained.

The suggested analysis procedure has been separately carried out by the generalized stress development into the fourth degree polynomial [13] . Then some differences have been observed in regard to the procedure derived by static values approximations by the sixth degree polynomial.

*All figures enclosed are made by **grafl** and **acad** computer programmes.*

6. CONCLUSION

Based on these observations a conclusion is that the increase of degree of the developing polynomial will probably contribute to obtaining a more qualitative solution. Perhaps, some other development functions forms should be considered instead of polynomials, which will describe the stress field in more details, and certainly, the yield condition.

A completely correct check of a suggested procedure value for determination of the lower limit carrying capacity for the circular cylindrical shell, however, can be realized only upon carrying out a series of experiments on prototype construction or models of material which largely satisfies hypotheses and the yield condition used in the analysis. In a concrete case, this should be the reinforced cement mortar or, if model check is done, the reinforced gypsum as was done by the experiment in the paper [10].

The analysis of the results obtained has shown that the non-linear programming method suggested in this paper gives, as a rule, lower values of boundary carrying capacity than the experimental ones. This should have been expected since the method is based on the theorem of the lower limit carrying capacity. Different from the comparative analytical solution $\mathbf{p}^s$, also for the lower limit carrying capacity, by NLP method only in one case $\mathbf{f}^{**}$ for the C type shell, a result is obtained which is higher than the experimental value. This value is otherwise determined for C type shell, but with a hypothesis that the shell displacement in x direction is prevented, which probably do not correspond to the real support conditions. Certainly, the procedure for solving the problem by means of static method determines the obtained solution as an approximate solution of boundary carrying capacity of circular cylindrical shells, but these solutions are within acceptability limits.

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GRANIČNA NOSIVOST KRUŽNO CILINDRIČNIH LJUSKI

Mileva Grubić, Petar Dančević, Milić Milićević

U radu se primenjuje teorema o donjoj granici nosivosti za posebnu klasu tankih ljuski cilindričnog oblika, pod dejstvom uniformnog radijalnog opterećenja. Ovom metodom donja granica opterećenja traži se određivanjem statički i plastički dopuštenog polja naprezanja, koje zadovoljava uslove ravnoteže, konturne uslove po silama i usvojeni uslov tečenja. Za ilustraciju je odabran Mises-ov uslov tečenja, primenjen na homogen izotropan materijal. Ovako postavljen problem svodi se na tipičan problem nelinearnog programiranja, NLP, za čije rešenje je korišćen postupak i kompjuterski programski paket SLLIM. Dobijene vrednosti parmetra graničnog opterećenja upoređene su sa vrednostima iz eksperimenata, koji su objavljeni u literaturi[3] dostupnoj autorima.