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ONE APROACH TO THE PROBLEM OF TWO BODIES *

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Abstract. *The motion of two bodies, viewed as material points, is known in the celestial mechanics as "the two bodies' problem". Kepler's Laws, Newton's Law of Gravity as well as Galilee's theorems about the free fall of the body are exactly related to this problem. In this paper, however, it can be taken as an example of two material points, but its relation to the above-mentioned laws makes it a problem of great importance. The formula for the gravitational force is obtained in the form:* $F = \chi \frac{m_1 m_2}{\rho}$ *which obviously differs from the Newton's one. For assumptions which are not in agreement with the first of Kepler's laws, the formula for F simply reduces to the Newton's law of gravitation. Under these assumptions an analytic expression for gravitational "constant" is obtained. It is shown that the numerical value of* κ*, for all planets of the Sun-system, is very close to the widely accepted value of gravitational constant.*
Two bodies as material points with masses m_1 *and* m_2 *are moving towards each other in the plane* $z = 0$ *so that the distance between their centers of inertia is a known function of time* $\rho(t)$.

Key words: *Two body problem, gravitational force, gravitational constant.*

The condition of the problem requiring that " the distance between the bodies is a function of time" is observed as the analytically ideal constraint $\rho(t)$:

$$f = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - \rho(t) = 0 \tag{1}$$

The differential equation of motion of these two material points affected by this constraint can be reduced to the form (see, for example [3], p. 313)

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$$\left.\begin{aligned} m_1\ddot{x}_1 &= \frac{\lambda}{\rho}(x_1 - x_2) \\ m_1\ddot{y}_1 &= \frac{\lambda}{\rho}(y_1 - y_2) \end{aligned}\right\}, \tag{2}$$

$$\left.\begin{aligned} m_2\ddot{x}_2 &= -\frac{\lambda}{\rho}(x_1 - x_2) \\ m_2\ddot{y}_2 &= -\frac{\lambda}{\rho}(y_1 - y_2) \end{aligned}\right\}. \tag{3}$$

The condition of velocities is

$$(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + (y_1 - y_2)(\dot{y}_1 - \dot{y}_2) = \rho\dot{\rho}\ . \tag{4}$$

The acceleration condition is

$$(x_1 - x_2)(\ddot{x}_1 - \ddot{x}_2) + (y_1 - y_2)(\ddot{y}_1 - \ddot{y}_2)(\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2 = \rho\ddot{\rho} + \dot{\rho}^2\ . \tag{5}$$

By substituting acceleration from the equation (2) and (3) into the previous equation we obtain

$$\begin{aligned} &\upsilon_r^2 + (x_1 - x_2)\left[\frac{\lambda}{m_1\rho}(x_1 - x_2) + \frac{\lambda}{m_2\rho}(x_1 - x_2)\right] + \\ &+ (y_1 - y_2)\left[\frac{\lambda}{m_1\rho}(y_1 - y_2) + \frac{\lambda}{m_2\rho}(y_1 - y_2)\right] = \dot{\rho}^2 + \rho\ddot{\rho}, \end{aligned} \tag{6}$$

where

$$\upsilon_r^2 = (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2\ . \tag{7}$$

Hence it follows that the multiplier

$$\lambda = \frac{m_1 m_2}{m_1 + m_2}\frac{\dot{\rho}^2 + \rho\ddot{\rho} - \upsilon_r^2}{\rho}\ . \tag{8}$$

has the dimension of force, $\dim\lambda = \mathrm{MLT}^{-2}$.

Let the letter χ denote the expression

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - \upsilon_r^2}{m_1 + m_2}\ . \tag{9}$$

and substitute it in the equations (2) and (3); the following form of the differential equations of motion

$$\left.\begin{aligned} m_1\ddot{x}_1 &= \chi\frac{m_1 m_2}{\rho^2}(x_1 - x_2) \\ m_1\ddot{y}_1 &= \chi\frac{m_1 m_2}{\rho^2}(y_1 - y_2) \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} m_2\ddot{x}_2 &= -\chi\frac{m_1 m_2}{\rho^2}(x_1 - x_2) \\ m_2\ddot{y}_2 &= -\chi\frac{m_1 m_2}{\rho^2}(y_1 - y_2) \end{aligned}\right\} \tag{11}$$

will be obtained.

The right sides of the equations (10) are the coordinates of the vector $\mathbf{F_1}$ affecting the body of the mass m_1, so that the magnitude of the force F_1 is equal to

$$F_1 = \chi\frac{m_1 m_2}{\rho^2}. \tag{12}$$

The force of the same magnitude F_2, but of the opposite sense, affects the body of the mass m_2 as described by the opposite sign. However, a problem emerges when these equations are integrated if the structure of the function $\chi(t, \dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)$ is taken into consideration, including its comparison with the Newton's force of gravity

$$F = \chi\frac{m_1 m_2}{\rho^2}. \tag{13}$$

In this case let's deal only with the questions which might be of interest for the laws of dynamics.

Assumption 1: H1: For the distance $\rho = R =$ const and the velocity $\upsilon_r =$ const it follows that

$$\chi = -\frac{\upsilon_r^2}{m_1 + m_2} \quad \text{and} \quad F = -\frac{m_1 m_2}{m_1 + m_2}\frac{\upsilon_r^2}{R}.$$

Assumption 2: H2: m_1 is the Earth's mass, $m_1 = m$, m_2 is the Sun's mass, $m_1 = M$ while R is the distance between the centers of the Earth and the Sun; the velocity υ_r is equal to the average velocity of the Earth's around the Sun. In its general form the formula for the force can be written, as:

$$F = -\frac{mM}{m+M}\frac{\upsilon_r^2}{R} = -\frac{mMR^3}{m+M}\frac{4\pi^2}{T^2R^2} = -\frac{mM}{m+M}\frac{R\upsilon_r{}^2}{R^2} = \kappa^*\frac{mM}{R^2} \tag{14}$$

where

$$\kappa^* = \frac{4\pi^2 R^3}{(m+M)T^2}, \tag{15}$$

and T is the period of the Earth's circle around the Sun. With the introduced hypotheses κ^* is a constant. For the case when R is a large half-axis of the elliptic path we find it in the classical Celestial Mechanics ([2], p. 56). "This equation expresses one important relation of the parameters a and T which is not completely identical to the Kepler's Law. This Law, namely, assumes that the quotient a^3/T^2 for all the planets is one and the same, which would not be the case, regarding the forward equation, since the presence of the mass m in it changes the value of the mentioned quotient from one planet to another. Still, since the masses of the planets are very tiny comparing to that of the Sun, thus in the above given equation m can be neglected beside M and thus the identity of the third

Kepler's Law with the laws of the Celeistal mechanics can be obtained."

While neglecting the mass m the constant (15) is evaluated as:

$$\kappa = \frac{4\pi^2 a^3}{MT^2} . \tag{16}$$

and it is called the universal gravitational constant which has the numerical value:

$$\kappa = 6.67 \times 19^{-8} \text{ cm}^3 \text{ gr}^{-1} \text{ sec}^{-2} .$$

The difference between κ^* and κ can be determined with great accuracy

$$\frac{1}{m+M} = \frac{1}{M(1+m/M)} = \frac{1}{M}\left(1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2 + ...\right)$$

Accordingly,

$$\kappa^* = \kappa - \kappa\varepsilon + \kappa\varepsilon^2 - \tag{17}$$

where $\varepsilon = m/M$. Since the relation between the masses of the earth and the Sun is

$$\frac{m}{M} = \frac{1}{333432} = 299.112263\times10^{-8}$$

then in the first approximation it is $\kappa^* = 0.999997\kappa = 6.66997999\times10^{-8}$. For the Jupiter-Sun ratio it is m_j/M = 318.36/330000 = 95479.7379×10^{-8} and $\kappa^* = 0.999045202\kappa = 6.663565264\times10^{-8}$. For the above stated assumptions, we get from (14):

$$\kappa^* = \frac{R\upsilon_r^2}{m+M} \tag{18}$$

where R is an average distance of the planet from the Sun, while υ_r is an average orbital velocity. The mass of the Sun is most often taken in the referential literature in three ways, namely: $M = 330000m_\oplus$, $M = 2\times10^{33}$ gr, $M = 333432m_\oplus$ [2]. If we take m as the mass of the planets and its satelite, κ^* is readily evaluated by means of (18) and using data from "Evolution of the Solar System" by Honnes A. et al:

	$330000m_\oplus$	2×10^{33}	$333432m_\oplus$
Mercury	6.7386	6.6423	6.6737
Venus	6.7493	6.6528	6.6843
Earth	6.7568	6.6603	6.6917
Mars	6.7730	6.6762	6.7078
Jupiter	6.7658	6.6993	6.7008
Saturn	6.7388	6.6426	6.6739
Uranus	6.7511	6.6547	6.6861
Neptune	6.7547	6.6582	6.6897
Pluto	6.7524	6.559	6.6874
Earth - Moon, Jupiter - Europe		6.63	
Average	6.7534	6.6569	6.6864
values κ^*		6.69	

Therefore, for the above given assumptions concerning the motion of two bodies, we obtain from (18) numerical values which can be averaged to the accepted gravitational constant.

For the average value $\kappa^* = 6.6864$, the radius of the earth $R = 6.38$ ([1], p. 17) and the Earth's mass $m_\oplus = 5.974$ ([2]. p. 197) we find that the velocity square of the body's circle round the earth in its immediate vicinity would be

$$\upsilon_r^2 = \kappa^* \frac{m + m_\oplus}{R}$$

so that for $m \ll m_\oplus$ the acceleration due to gravity is:

$$g = \frac{\upsilon_r^2}{R} = 6.6864 \frac{5.974 \times 10^{27}}{(6.38 \times 10^9)^2} 10^{-8} = 9.8133\,.$$

The given numerical data, along with the given hypotheses show the conditions in which the classical formula for the force of gravity is obtained.

However, the formulas (5) and (8) show that the force of attraction depends, among other things, upon the velocity and acceleration of the change of the distance between the bodies. In the case of a free fall of the body with the mass m, for $\upsilon_r = \dot{\rho} = (R + \xi) = \dot{\xi}$, regarding (5) and (8) it follows that

$$F_1 = \chi \frac{mM_\oplus}{R + \xi} = \frac{(R+\xi)\ddot{\xi} m M_\oplus}{(m + M_\oplus)(R + \xi)} = \frac{m\ddot{\xi}}{1 + \dfrac{m}{M_\oplus}} \approx m\ddot{\xi}\,, \quad \frac{m}{M_\oplus} \approx 0\,.$$

As Galilee found out a long time ago that $\xi = (1/2)gt^2$ for the force of the Earth's gravity the expected value $F = mg$ is obtained.

Another characteristic case is that of the two moving bodies of the masses m_1 and m_2 whose distance ρ is changed according to the formula $\rho = A\cos(\Omega t + \gamma)$ where A, Ω and γ are constants. In this case $\upsilon_r = \dot{\rho}$ and by means of the formula (5) it is found that

$$\chi = -\frac{\lambda^2 \Omega^2}{m_1 + m_2}\,.$$

By substitution into the differential equations of motion (10) and (11) we obtain

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2 (x_1 - x_2) \\ \ddot{y}_1 &= -\omega_1^2 (y_1 - y_2) \\ \ddot{x}_2 &= -\omega_2^2 (x_1 - x_2) \\ \ddot{y}_2 &= -\omega_2^2 (y_1 - y_2) \end{aligned}$$

where, for the sake of brevity, the notations

$$\omega_1^2 = \frac{m_2 \Omega^2}{m_1 + m_2} \quad \text{and} \quad \omega_2^2 = \frac{m_1 \Omega^2}{m_1 + m_2}$$

are introduced.

Moreover, if we also denote $x = x_1 - x_2$ and $y = y_1 - y_2$ the above system of equations can be reduced to two homogenous linear differential equations

$$\ddot{x} = -\Omega^2 x\,,\quad \ddot{y} = -\Omega^2 y\,.$$

Their solutions

$$x = C_1 \cos\Omega t + C_2 \sin\Omega t\,,\quad y = C_3 \cos\Omega t + C_4 \sin\Omega t\,,$$

as it is known, for various initial conditions determine various trajectories. For instance:

a) For $t_0 = 0$ and $x(t_0) = x_0$, $y(t_0) = y_0$; $\dot{x}_0 = 0$, $\dot{y}_0 = 0$, we get oscillations $x = x_0\cos\Omega t$, $y = y_0\cos\Omega t$ are obtained along the straight line $y_0 x - x_0 y = 0$.

b) For $y_0 = 0$ and $\dot{x}_0 = 0$ motion is determined by the equations $x = x_0\cos\Omega t$ and $y = \frac{\dot{y}_0}{\Omega}\sin\Omega t$ along the ellipse

$$\frac{x^2}{x_0^2} + \frac{\Omega^2 y^2}{\dot{y}_0^2} = 1\,,$$

that is:

$$\frac{(x_1 - x_2)^2}{(x_1 - x_2)_0^2} + \frac{\Omega^2 (y_1 - y_2)^2}{(y_1 - y_2)_0^2} = 1\,.$$

Contemporary knowledge about the motion of artificial satellites can confirm the above-mentioned new equations or refute the basic differential equations of motion in mechanics (2) and (3).

In the same way the problem of three or more bodies can be resolved. This problem will be elaborated and realised for publications into more details in 1998.

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JEDAN PRISTUP PROBLEMU DVA TELA

Veljko A. Vujičić

Razmatra se zadatak određivanja sile uzajamnog dejstva dva tela - materijalne tačke, čije je međusobno rastojanje funkcija vremena. Pomoću Lagranžovih diferencijalnih jednačina kretanja prve vrste lako se određuje da je tražena sila obrnuto proporcionalna rastojanju materijalnih tačaka. U slucaju da posmatrane materijalne tacke pretstavljaju nebeska tela Sunčevog sistema ovaj zadatak dobija značaj problema dva tela. Nađena sila $F = \chi\frac{m_1 m_2}{\rho}$ *se razlikuje od Njutnove sile gravitacije. Tek za pretpostavku da je pomenuto rastojanje konstatno, što nije u saglasnosti sa Keplerovim zapažanjima dobijaju se formule za Njutnovu silu gravitacije i odgovarajuću gravitacionu konstantu.*

Ključne reči: *problem dva tela, gravitaciona sila, gravitaciona konstanta*