UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol.2, No 7/2, 1997 pp. 449 - 458
Editor of series: Katica (Stevanovi}) Hedrih, e-mail: katica@masfak.masfak.ni.ac.yu
Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: (018) 547-095, Fax: (018)-547-950
http:// ni.ac.yu/Facta

NORMAL VIBRATIONS IN NEAR CONSERVATIVE SELF-EXITED AND NON-AUTONOMOUS NONLINEAR SYSTEMS

UDC: 531.3; 534.01

Yu. V. Mikhlin

Dept. of Appl. Mathematics, Kharkov Polytechnical
University Kharkov, Ukraine

Abstract. *In paper analysis of normal vibrations in broad classes near-conservative finite-dimensional systems is done by utilized perturbation methodology.*

1. INTRODUCTION

Nonlinear normal vibrations modes are a generalization of normal (principal) vibrations of linear systems. In the normal mode a finite-dimensional system behaves like a conservative one having a single degree of freedom. In this case all position coordinates can be well defined from any one of them by

$$x_i = p_i(x) \quad (x \equiv x_1,\ i = 1,2,3,...,n)\,, \tag{1}$$

$p_i(x)$ being analytical functions [1-3]. Rosenberg [4,5] was the first to introduce broad classes of essentially nonlinear conservative systems allowing normal vibrations with rectilinear trajectories in a configuration space:

$$x_i = k_i x_1 \quad (i = 2,3,...,n) \tag{2}$$

For instance, homogeneous systems whose potential is an even homogeneous function of the variables belong to such a class. It is interesting to note that the number of modes of normal vibrations in the nonlinear case can exceed the number of degrees of freedom. In systems of a more general type, trajectories of normal vibrations are curvilinear. In the [1-3] the power series method was proposed for the construction of above mentioned trajectories.

Here, a perturbation methodology is utilized to the analysis of normal vibrations in

Received September 20, 1997

broad classes near-conservative finite-dimensional nonlinear systems.

2. NORMAL VIBRTATIONS IN NEAR-CONSERVATIVE NON-AUTONOMOUS NONLINEAR SYSTEMS

It is well-known that forced vibrations in nonlinear systems with one degree of freedom are similar to those in an undisturbed conservative system (for small periodic disturbances in the resonance region). In this subsection, we shall construct resonance periodic solutions for finite-dimensional non-autonomous systems that will be close to normal vibrations in the corresponding conservative systems. It turns out that in these modes, a non-autonomous system behaves similarly to a conservative system with one degree of freedom. Therefore, it is worthwhile to consider normal vibrations in non-autonomous systems. Rauscher's ideas are used in the construction of the resonance solutions, the method leading to equations for trajectories in a configurational space.

The ideas underlying the Rauscher method were used for analyses of normal vibrations in non-autonomous systems previously [6]. The external periodic action is written in terms of a function of a positional coordinate. To solve the problem, one assumes that the force is proportional to the coordinate raised to such a power that the resultant autonomous system is homogeneous.

It should be noted that periodic modes in non-autonomous systems close to Lyapunov systems were thoroughly investigated by Malkin [7].

1. Let us consider a non-autonomous system defined by

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2, ..., x_n) + \varepsilon f_i(x_1, \dot{x}_1, x_2, \dot{x}_2, ... x_n, \dot{x}_n, t) = 0 \quad (i = 1,2,...,n) \tag{3}$$

The functions f_i are assumed to be analytical in x_i, $\dot{x}_i$ and periodical in t with a period T; a potential energy Π is subject to the limitations shown above, and ε is a small parameter.

Let us assume that a non-disturbed conservative system ($\varepsilon = 0$) allows normal modes of vibrations $x_{i0} = k_i x_{10}$ ($i = 2,...,n$; k_i are constants). Any summand may be regarded as a disturbance if, due to its presence, the system ceases to belong to a class that allows normal vibrations with rectilinear trajectories.

Once again, for the ease of analysis a coordinate system is introduced in which all $k_i = 0$, so that $x_{i0} \equiv 0$ ($i = 2,3,...,n$).

Let us find a resonance periodic mode in which the non-autonomous system (3) shows behavior similar to that of a conservative system with one degree of freedom [8]. Let us describe the motion in this "arbitrary" conservative system by a variable $x \equiv x_i$.

Let us first write the energy integral for the "arbitrary" conservative system with one degree of freedom:

$$\frac{\dot{x}^2}{2} + \Pi(x,0,...0) + \varepsilon F(x,\varepsilon) = h(\varepsilon) \tag{4}$$

where $h(\varepsilon)$is the energy constant.

All variables pertaining to the periodic solution are introduced as single-valued functions of x, at least within a half of the period:

$$x_i = x_i(x,\varepsilon)\ ,\ \dot{x} = \dot{x}(x,\varepsilon)\ ,\ x_i = x_i(x,\varepsilon)\ ,\ t = t(x,\varepsilon)\ ,$$

Here, $\dot{x}$ is defined as a function of x by (4) while t, as a function of x, is defined in accordance with the principal idea of the Rauscher method by (4) through the following quadrature:

$$t(x,\varepsilon) + \phi = \frac{1}{\sqrt{2}} \int_{x(0)}^{x} \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi,0,...0) - \varepsilon F(\xi,\varepsilon)}} \tag{5}$$

In what follows, the phase ϕ is so selected that the condition $x(0) = 0$, when $x(0) = x$, is satisfied. In this case $h(\varepsilon)$ and x are related by the following equation which is implied by (4):

$$\Pi(x,0,...0) + \varepsilon F(x,\varepsilon) = h(\varepsilon) \tag{6}$$

Since the series expansion of the potential begins with even-pover terms, for each given value of $h(\varepsilon)$ there are two corresponding amplitude values $x = x_j$ (j = 1,2).

The equations governing the trajectory $x_i(x)$ (i = 1,23,...,n) may be obtained from (3) by the use of the energy integral (4) of the arbitrary conservative system. On introducing a new independent variable x instead of t, one obtains

$$\begin{aligned} &x_i''(x(x))^2 + x_i'[-\Pi_x(x,x_2(x),...x_n(x)) - \varepsilon f_1(x,\dot{x}(x),x_2(x),...,t(x))] + \\ &+ \Pi_{x_i}(x,x_2(x),...x_n(x)) - \varepsilon f_i(x,\dot{x}(x),\dot{x}_2(x),...,t(x)) = 0 \qquad (i = 2,3,...,n) \end{aligned} \tag{7}$$

Here and henceforth, a prime denotes differentiation with respect to x.

When $x = x_j$ (j = 1,2) and $\dot{x} = 0$, we obtain additional conditions from (7):

$$\begin{aligned} &\{x_i'[-\Pi_x(x,x_2(x),...x_n(x)) - \varepsilon f_1(x,\dot{x}(x),x_2(x),\dot{x}_2(x)...,t(x))] + \\ &+ \Pi_{x_i}(x,x_2(x),...x_n(x)) - \varepsilon f_i(x,\dot{x}(x),x_2(x),\dot{x}_2(x)...,t(x))\}\,|_{x=x_j} = 0 \\ &(i = 2,3,...n;\ \ j = 1,2) \end{aligned} \tag{8}$$

Eqs. (7) in combination with conditions (8) allow a univalent determination of $x_i(x)$.

It is evident from $x_i(x) = x_i'(x)\ \dot{x}(x)$ that the functions $x_i(x)$ are unambiguously defined by the functions $x_i(x)$ and $\dot{x}(x)$.

Finally, two other conditions should be added. Firstly, the periodicity condition is needed that follows from (5):

$$T + \phi = \frac{1}{\sqrt{2}} \oint \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi,0,...0) - \varepsilon F(\xi,\varepsilon)}} \tag{9}$$

An equation of motion for a resonance mode is obained from (3):

$$\ddot{x} + \Pi_{x_i}(x,x_2(x),...,x_n(x)) + \varepsilon f(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...x_n(x),\dot{x}_n(x),t(x)) = 0 \tag{10}$$

Comparing (2) with (8), we find that

$\Pi_x(x,x_2(x),...,x_n(x)) + \varepsilon f(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...t(x)) = \dfrac{d}{dx}[\Pi(x,0,...0) + \varepsilon F(x,\varepsilon)]$, whence

$$\varepsilon \frac{dF}{dx}(x,\varepsilon) = \Pi_x(x, x_2(x),..., x_n(x)) + \varepsilon f(x, \dot{x}(x), x_2(x), \ldots t(x)) - \frac{d}{dx}[\Pi(x,0,...0)] \quad (11)$$

$F(x,\varepsilon)$ may be found from (11).

Secondly, the periodicity condition should now be supplemented by a condition stating that the work of all forces performed during the period equals zero:

$$\oint \frac{dF}{d\xi}(\xi,\varepsilon)d\xi \equiv \oint [\Pi_x(\xi, x_2(\xi),...,x_n(\xi))] - \frac{d}{d\xi}[\Pi(\xi,0,...,0) + \varepsilon f(\xi, \dot{x}(\xi), x_2(\xi),...,t(\xi))]d\xi = 0 \quad (12)$$

Here, the integral refers to the period of the vibrations. Conditions (9) and (12) are needed in order to find the energy $h(\varepsilon)$and the phase ϕ for the solution.

Using relationships (4) through (12), one may find all characteristics of the resonance mode (of normal vibrations) in the system (3) by the iteration process.

2. As stated above, $x_{i0} = 0$ in the zeroth approximation. Eq. (4) implies that $\dot{x}_0^2 = 2[h - \Pi(x,0,...,0)]$. From (5) one finds

$$t_0(x,0) + \phi_0 = \frac{1}{\sqrt{2}} \int_{x(0)}^{x} \frac{d\xi}{\sqrt{h_0 - \Pi(\xi,0,...,0)}}$$

Finally, the periodicity condition has the following from:

$$T + \phi_0 = \frac{1}{\sqrt{2}} \oint_0 \frac{d\xi}{\sqrt{h_0 - \Pi(\xi,0,...,0)}} \quad (13)$$

Let us define the constant h_0 from this condition. Since for a conservative system (at $\varepsilon = 0$) condition (12) hold identically, one clearly may assume that $\phi_0 = 0$. Condition (6) in the yeroth approximation takes the form

$$\Pi(x,0,...0) = h_0 \quad (14)$$

that permits to obtain the amplitudes $x_j^{(0)}$ $(j = 1,2)$.

Let us now write all the necessary relationships for the first approximation in ε, first of all (7) that define the trajectory $x_{i1}(x)$, together with the "boundary" conditions (8):

$$\begin{aligned} &x_{i1}''(x_0(x))^2 + x_{i1}'[-\Pi_x(x,0,...0) - f_1(x, \dot{x}_0(x), 0,..., t_0(x))] + \\ &+ \sum_{k=2}^{n} x_{k1} \Pi_{x_i x_k}(x,0,...0) + f_i(x, \dot{x}_0(x), 0,...,0, t_0(x)) = 0 \end{aligned} \quad (15)$$

$$\begin{aligned} &\{x_i'[-\Pi_x(x,0,...,0) - f_1(x, \dot{x}_0(x),0,...,0,t_0(x))] + \\ &+ \sum_{k=2}^{n} x_{k_1} \Pi_{x_i x_k}(x,0,...,0) + f_i(x, \dot{x}_0(x),0,...,0,t_0(x))\}|_{x=x_j^{(0)}} = 0 \quad (i = 2,3,...n;\ j = 1,2) \end{aligned} \quad (16)$$

These equations and boundary conditions are formally identical to the respective relationships for a conservative system [1-3].

This implies, however, that a single-valued analytical solution may be found that will satisfy Eqs. (15) and additional conditions (16) under the same limitations imposed on the

generating system as in [1-3].

The next step in our search for solution is to find the "additional potential energy" $F(x,\varepsilon)$ from (11) in the first approximation with respect to ε :

$$\frac{dF_1}{dx} = \sum_{j=2}^{n} x_{k_1}(x)\Pi_{x x_j}(x,0,...0) + f_1(x,\dot{x}_0(x),0,...,0,t_0(x))$$

The function $F_1(x,\varepsilon)$ is obtained by integrating; the integration constant here has a meaning of a correction to the energy h_0 of the generating solution (recall that we are dealing with the energy integral of an "arbitrary" conservative system with one degree of freedom). In the first approximation with respect to ε, one obtains

$$t_1(x,\varepsilon) + \phi_1 = \frac{1}{\sqrt{2}} \int_{x(0)}^{x} \frac{d\xi}{\sqrt{h_0 + \varepsilon h_1 - \Pi(\xi,0,...,0) - \varepsilon F_1(\xi)}}$$

One can now refer to the periodicity condition (9) and the condition (12) in the first approximation with respect to ε:

$$T + \phi_1 = \frac{1}{\sqrt{2}} \oint \frac{d\xi}{\sqrt{h_0 + \varepsilon h_1 - \Pi(\xi,0,...,0) - \varepsilon F_1(\xi)}}$$

$$P(\phi) \equiv \oint \left[\sum_{k=2} x_k \binom{n}{\xi} \Pi_{\xi\, x_k}(\xi,0,...0) + f_1(\xi,\dot{x}_0(\xi),0,...,0,t_0(\xi)) \right] d\xi = 0 .$$

Here, the last integral may also be calculated with respect to t by integration from 0 to T on substituting $d\xi = \dot{x}_0(\xi)dt$.

The first equality allows to find a "correction" to the energy h_1, and the second one will yield the phase ϕ_1 of the solution in the first approximation with respect to ε. Moreover, the amplitudes $x_j^{(1)}$ $(j = 1,2)$ in this approximation are defined by

$$h_0 + \varepsilon h_1 = [\Pi(x,\varepsilon x_{21}(x),...\varepsilon x_{n1}(x)) + \varepsilon f_1(x,\dot{x}_0(x),0,...0,t_0(x))]\big|_{x=x_j^{(1)}}$$

The periodicity condition (9) is not solvable in the first and subsequent approximations unless h_0 is a nonmultiple root of (14) and the amplitudes $x_{1,2}^{(1)}$ related to h_0 are nonmultiple roots of (13).

This implies, however, that the generating systems connot be linear, for the latter have only multiple roots of the periodicity equation (13) due to the isochronism. This conclusion is another proof of the importance of studying normal vibrations in nonlinear conservative systems.

It follows from the roots of (13) being nonmultiple that the case in which equilibrium positions occur at the boundray should be excluded from consideration.

It goes without saying that the condition $P'(\phi) \neq 0$ should be satisfied as well.

In the subsequent approximations with respect to ε, the solution is calculated by the iteration process or through the use of small parameter series in a similar way.

If conditions of the form (11) are satisfied and the roots of (6), (9) and (12) are

nonmultiple, resonance normal vibration mode in a non-autonomous system (3) thus correspond to any normal mode of vibrations in a nondisturbed conservative system in the limit of $\varepsilon = 0$. The number of resonance modes is defined by the number of roots of the equation that governs the phase of the generating solution. For each value of the phase, there is a special corresponding value of the amplitudes $x_{1,2}$ of the energy h and, accordingly, a special resonance normal mode of vibrations.

It should be noted that the application of the Rauscher method results in the initial system being reduced in the sought-for mode to a conservative one at each step of the construction process.

3. NORMAL VIBRATIONS IN SELF-EXITED SYSTEMS

It will be demonstrated that the algorithms for analysis of normal vibrations are applicable to near-conservative nonlinear autonomous systems with small self-excited disturbances. Consider the following quasiconservative system:

$$x_i^{'} + \Pi_{x_i}(x_1, x_2, ..., x_n) + \varepsilon g_i(x_1, \dot{x}_1, ..., x_n, \dot{x}_n) = 0 \tag{17}$$

where ε is a small parameter, and the functions g_i may be nonlinear with respect to $\dot{x}_i$. The system may involve friction of any physical nature, such as viscous, dry or turbulent. The conditions that assure occurrence of self-excited vibrations (limiting cycles) are discussed later.

Assume that a conservative system ($\varepsilon = 0$) permits normal modes of vibrations of the form (2).

Consider one of these solutions as a generating one. Rotating the coordinate axes, so that the new x-axis is directed along the rectilinear trajectory, one obtain a generating solution in the new coordinates as, $x_{j0} = 0$ ($i = 2,3,...,n$), $x_1 \equiv x = x(t)$.

On introducing a new independent variable x instead of t, one obtains the equations describing the trajectories of normal vibrations

$$\begin{aligned} &x_i^{''}(x(x)) + x_i^{'}[-\Pi_x(x, x_2(x), ...x_n(x)) - \varepsilon f_1(x, \dot{x}(x), \dot{x}_2(x), ...)] + \\ &+ \Pi_{x_i}(x, x_2(x), ...x_n(x)) - \varepsilon f_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ...) = 0 \qquad (i = 2, 3, ..., n) \end{aligned} \tag{18}$$

Here and henceforth, a prime denotes differentiation with respect to x; $x_2(x) = x_2^{'}(x)\,\dot{x}$.

When $x = x_j$ ($j = 1,2$) and $\dot{x} = 0$, we obtain additional conditions from (18):

$$\begin{aligned} &\{x_i^{'}[-\Pi_x(x, x_2(x), ..., x_n(x)) - \varepsilon f_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x)...)] + \Pi_{x_i}(x, x_2(x), ...x_n(x)) + \\ &+ \varepsilon f_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ...,)\}|_{x=x_j} = 0 \qquad (i = 2, 3, ...n;\ \ j = 1, 2) \end{aligned} \tag{19}$$

Here (x, $x_2(x)$,...,$x_n(x)$) being the trajectory return points. Eqs. (18) in combination with conditions (19) allow a univalent determination of $x_i(x)$.

All variables pertaining to the periodic solution are introduced as single-valued functions of $x \equiv x_1$:

$$x_i = x_i(x, \varepsilon),\ \dot{x} = \dot{x}(x, \varepsilon) \tag{20}$$

In the zeroth approximation $\dot{x}_0^2 = 2[h - \Pi(x,0,...,0)]$, where h is the energy of the nondisturbed system. We shall restrict our consideration of (20) to the first approximation in ε:

$$2x_{i1}''[h - \Pi_x(x,0,...,0)] + x_i'[-\Pi(x,0,...0) - g_1(x,\dot{x}_0(x),0,...,0)] + \\ + \sum_{k=2}^{n} x_{k1}\Pi_{x_i x_k}(x,0,...,0) + g_i(x,\dot{x}_0(x),0,...,0) = 0 \qquad (i = 2,3,...,n) \tag{21}$$

In this approximation, the boundray conditions are obtained from (19) at $\dot{x}_0 = 0$, $x = x_j$ (j = 1,2):

$$\{x_{i1}'[-\Pi_x(x,0,...,0) - g_1(x,\dot{x}_0(x),0,...,0,)] + \sum_{k=2}^{n} x_{k_1}\Pi_{x_i x_k}(x,0,...,0) + \\ + g_i(x,\dot{x}_0(x),0,...,0,)\}|_{x=x_{1,2}} = 0 \qquad (i = 2,3,...n) \tag{22}$$

This type of equations and boundary conditions was discussed in [1-3]. The power series method may be applied to finding $x_{i1}(x)$. Let the trajectory $x_{i1}(x)$ be defined in the first approximation with respect to ε. We shall now turn to an analysis of the motion $x(t)$ in time:

$$\ddot{x} + \Pi_x(x, x_2(x),..., x_n(x)) + \varepsilon g_1(x,\dot{x}(x), x_2(x), \dot{x}_2(x),...) = 0$$

Here, the equation of phase trajectories is of the form ($\dot{x} = v$)

$$\frac{dv}{dx} = -\frac{\Pi_x(x, x_2(x),..., x_n(x)) + \varepsilon g_1(x, v, x_2(x), x_2'(x)v...)}{v}$$

In the general case, one has to employ approximate methods to find all phase trajectories, and due allowance is to be made for the small values of ε. However, this equation allows ecact integration if g_1 is a linear function with respect to $w = v^2$.

At a closed phase trajectory of the limiting cycle, the system behavior is similar to that of a conservative system with one degree of freedom. Therefore, the condition that the work of all forces over the period is equal to zero should hold:

$$\oint g_1(x,\dot{x}(x), x_2(x), \dot{x}_2(x),...)dx = 0 \tag{23}$$

The integral is taken over the period of vibrations. This condition is used to find the value of the energy h (and accordingly the amplitude values of x_j) for the generating solution of the conservative system, that is the limiting solution (at ε tending to zero) for the sought-for self-excited vibrations of the system (17).

4. EXAMPLES

1. As a simple example, let us consider two linearly bound Van-der-Paul oscillators with nonlinear elastic characteristics:

$$\ddot{x}_1 + x_1 + \alpha x_1^3 + \beta(x_1 - x_2) + \alpha(1 - x_1^2)x_1 = 0$$
$$\ddot{x}_2 + x_2 + \alpha x_2^3 + \beta(x_2 - x_1) + \alpha\gamma(1 - x_2^2)x_2 = 0$$

Two vibration modes may be singled out in the zeroth approximation: $x = \pm x_2$. Let us introduce new coordinates x, y as follows:

$$x_1 + x_2 = x\,, \qquad x_1 - x_2 = y\,.$$

In the new coordinates, one of the vibration modes is governed by $y = 0$, $x_0 = x_0(t)$ in the zeroth approximation, and

$$\ddot{x}_0 = -x_0 - \frac{\alpha}{4}x^3\,, \quad \dot{x}_0^2 = 2\left(h - \frac{x_0^2}{2} - \frac{\alpha x_0^4}{16}\right)$$

An equation describing the trajectory $y(x)$ in the first approximation with respect to ε has the form

$$y_1''\dot{x}_0^2 + y_1'\ddot{x}_0 + y_1\left(1 + 2\beta + \frac{3\alpha}{4}x_0^2\right) + \left(1 - \frac{x_0^2}{4}\right)\frac{\dot{x}_0}{2} - \gamma\left(1 - \frac{x_0^2}{4}\right)\frac{\dot{x}_0}{2} = 0 \tag{24}$$

The equation should be solved together with boundary conditions of the form (22) (at $\dot{x}_0 = 0$, $x_0 = Q$):

$$y_1'(Q)\,\ddot{x}_0(Q) + y_1(Q)\left(1 + 2\beta + \frac{3\alpha}{4}Q^2\right) = 0 \tag{25}$$

Substituting the solution $y(x)$ as a power series into (21), we obtain a set of recurrent equations in the expansion coefficients. One must also use the condition (22). Let $\alpha = 1.5$; $\beta = 1.5$; $\gamma = 0.5$; $\varepsilon = 0.2$. In the first approximation, the equations yield a trajectory $\gamma \cong 0.1317 - 0{,}0072x^2 - 0{,}0001x^4$; $x(0) = Q = 4$ (quasiharmonic approximation) for one vibration mode and $x \cong 0.0577 - 0{,}0023y^2 - 0{,}0001y^4$; $y(0) = Q = 4$; for the other via similar calculations. There is a good agreement between the approximation solution and checking numerical calculations by the computer. The numerical calculations show that only “in-phase mode” (which is close to the trajectory $x_2 = x_1$) is stable.

2. Another example refers to the problem of vibrations of a plate in a flow of gas. The system is assumed to have two degrees of freedom. Here, q_1 is the vertical displacement of the plate and q_2 is the angle of rotation. Both the restoring force $P(q_1)$ and the restoring moment $M(q_2)$ is nonlinear:

$$P(q_1) = (d_1 + d_2 q_1^2)q_1\,, \quad M(q_2) = (d_3 + d_4 q_2^2)q_2$$

where d_1, d_2, d_3, d_4 are coefficients depending on the elastic characteristics of the plate.

The aerodynamic lift is

$$A(q_2) = \sigma v^2 \sin 2q_2 \cong \sigma v^2\left(2q_2 - \frac{2q_2^3}{3}\right)$$

where v is the flow velocity, and σ is dependent of the flow density and the foil aerodynamic chord.

Let ρ be the radius of inertia of the section mass with respect to the bend center axis, l the distance from the center of rigidity to center of mass, and m is the mass of the plate

per unit length. The equations of motion for the system are

$$\begin{aligned} & m\ddot{q}_1 - ml\ddot{q}_2 + (d_1 + d_2 q_1^2) q_1 = \sigma v^2 \left(2q_2 - \frac{2q_2^3}{3} \right) \\ & - ml\ddot{q}_1 + m(\rho^2 + l^2)\,\ddot{q}_2 + (d_3 + d_4 q_2^2) q_2 = 0 \end{aligned} \tag{26}$$

The analysis of vibration modes for the system (26) is based on the assumption that the values of d_3 and d_4 are close to those of d_1 and d_2 respectively, the sum $\rho^2 + l^2$ is close to unity and l^2 is much smaller than unity.

In order to find vibration modes, let us use equations (21) and boundary conditions (22). The results of the calculations haven`t written out but it should be noted that the vibration modes are close to straight lines. Therefore it seems worthwhile to employ the equations for finding approximate forms of vibrations of the type $q_2 = kq_1$, $k = const$. In order to find k, one substitute the solution $q_2 = kq_1$ into the equations (4) and ensure that the orthogonality condition is satisfied at $q_1 = q_1(0) = Q$, $q_1(0) = 0$.

One thus obtains an algebraic equation of the form

$$\begin{aligned} & k^4 \left[ld_4 + \frac{4}{3}(\rho^2 + l^2)\sigma v^2 \right] Q^2 - k^3 \left[d_4 + \frac{4}{3} l\sigma v^2 \right] Q^2 + k^3 [ld_3 - 2\sigma v^2 (\rho^2 + l^2)] + \\ & + k[-d_3 + 2\sigma l v^2 + (\rho^2 + l^2)(d_1 + d_2 Q^2)] - l(d_1 + d_2 Q^2) = 0 \end{aligned} \tag{27}$$

Note that when the value of v is not too high, we obtain four real values of k with four vibration modes corresponding to them. If the values of parameters are same as given above and $Q = 1$ ($q_2 \cong kq_1$), one arrives at:

$v^2 = 0$: $k_1 = -1.025$, $k_2 = 0{,}465$, $k_3 = 1.151$, $k_4 = 3.409$,
$v^2 = 0.2$: $k_1 = -1.043$, $k_2 = 0{,}492$, $k_3 = 1.109$, $k_4 = 2.185$,
$v^2 = 0.42$: $k_1 = -1.059$, $k_2 = 0{,}536$, $k_3 = 1.008$, $k_4 = 1.630$,
$v^2 = 0.6$: $k_1 = -1.073$, $k_2 = 0{,}687$, $k_3 = 0.721$, $k_4 = 1.403$.

As v^2 increases, the modes $q_2 \cong k_2 q_1$ and $q_2 \cong k_3 q_1$ vanish in a certain limiting point.

It is easy to check that the potentiality condition (23) for the system at hand is identically satisfied for any vibration mode defined by an analytical function $y(x)$ or $x(y)$, and therefore normal vibrations are dependent on two free parameters, namely the vibration amplitude Q and the phase ϕ.

REFERENCES

1. A.F. Vakakis, L.I. Manevich, Yu.V. Mikhlin, V.N. Pilipchuk, A.A. Yevin. (1996) *Normal Modes and Localiyation in Nonlinear Systems*. New York: Wiley.
2. Manevich L.I., Mikhlin Yu.V. and Pilipchuk V.N. (1989) *The method of normal oscillation for essentially nonlinear systems*. Moscow: Nauka (in Russian).
3. Mikhlin Yu.V. (1996) *Normal vibrations of a general class of conservative oscillators*. Nonl. Dynamics 11(1), 1-16.
4. Rosenberg R.M. (1962) *The normal modes of nonlinear n-degree-of freedom systems*. Journal of Appl. Mech., 30, 7-14.
5. Rosenberg R.M. (1966) *Nonlinear vibrations of systems with many degrees of freedom* - In: Advances of Appl. Mech., 9, New York: Academic Press, 156-243.

6. Kinney W. and Rosenberg R.M. (1966) *On the steady state vibrations of nonlinear systems with many degrees of freedom*. J.Appl. Mech. 33(2), 406-412.
7. Malkin I.G. (1956) *Certain Problems of the Theory of Nonlinear Vibrations*. GTI: Moscow (in Russian).
8. Mikhlin Yu.V. (1974) *Resonance modes of near-concervative nonlinear systems*. J. Appl. Math. and Mech. (PMM) 38(3), 425-429.

NORMALNE OSCILACIJE U SKORO KONZERVATIVNOM SAMOPOBUĐENOM I NEAUTONOMNOM NELINEARNOM SISTEMU

Yu. V. Mikhlin

U radu su pomoću perturbacione metodologije analizirane normalne oscilacije u širokoj klasi konačno dimenzionih skoro konzervativnih sistema.