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A QUALITATIVE METHOD FOR STABILITY ANALYSIS OF NONLINEAR DISCRETE DYNAMICAL SYSTEMS

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Abstract. *Main results are presented by two theorems determining sufficient conditions for stability, and necessary conditions for instability of equilibrium point of a nonlinear discrete dynamical system. By introducing the concept of discrete divergence as a difference operator, a discrete analogue of the Liuvill theorem on expansion of phase volume was derived. This theorem was a base for the derivation of the presented results.*
Key Words: *Stability, nonlinear system, discrete system, field theory, diveregence*

1. INTRODUCTION

Methods for quantitative examination of stability of continuous dynamical systems by using field theory are considered in [1-3].Basic idea is to consider the set of dynamical systems trajectories in phase space for a given set of initial conditions, involving equilibrium point, as a flux of the vector field. Otherwise, stability conditions of the equilibrium point are derived with regard to the sign of divergence of the field. Theorems on stability are proved by using Liuvill theorem [4]. By using this method, some of results are generalized. For example, the well known Bendixson criterion [4] on existence of closed trajectories in phase plane is easily generalized. Advantage of these method is in a simple application.

In the other side, the idea of discrete-time has attracted the attention of physics [5]. In particular, the assumption of space-time coordinates with integer values requires the translation of relativistic mechanics and electrodynamics into the form of finite difference equations. By defining the operators of discrete divergence and rotation, a special study of the covariance of these equations under the inhomogeneous Lorente group has been carried out.

In this paper, by introducing a new concept of discrete divergence as difference operator, a discrete analogue of the Liuvill theorem on expansion of phase volume is

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obtained. This theorem is used for derivation of sufficient conditions for stability, and necessary conditions for instability of equilibrium points of a nonlinear discrete dynamical system.

2. DISCRETE LIUVILL THEOREM

Let N denote the set of non-negative integers $N_0=\{0\}\cup N$, and let $\Re^n$ denote the n-dimensional real euclidean space. We consider the system

$$\Delta_k \mathbf{x}(k) = \mathbf{f}(\mathbf{x}(k)) \tag{1}$$

where X and **f** are n-vectors, and the continuous function **f** is defined on a subset of $\Re^n$ and

$$\Delta_k x_i(k) = x_i(k+1) - x_i(k)$$

Denote by X the n-dimensional phase space of the system (1), by G, $G\subset X$ the area to which belong the trajectories of the system (1), and $G_0\subset G$ the set of equilibrium states of (1). Assume that $\mathbf{mes}(G/G_0)\neq 0$. We consider the set of trajectories of the system (1) for a given set of initial conditions G_p, $G_0\subset G_p\subset G$ as a vector field.

A volume which forms the phase trajectories for $k\in N_0$,starting from the initial conditions G_p, is called the phase volume. We call the elementary cell the n-dimensional cube, containing the set G_p.

Theorem 1 *Denote by $V(k)$ a volume in the phase space formed by the flux of* **f**, *and by $S(k)$ a closed surface involving $V(k)$. Then,*

$$\Delta_k V(k) = \sum_{S(k)} \mathrm{DIV}\mathbf{f}\Delta V \tag{2}$$

where

$$\mathrm{DIV} f = \begin{vmatrix} D_{x_1} f_1 + 1 & D_{x_2} f_2 & \dots & D_{x_1} f_n \\ D_{x_2} f_1 & D_{x_2} f_2 + 1 & \dots & D_{x_2} f_n \\ \vdots & \vdots & \dots & \vdots \\ D_{x_n} f_1 & D_{x_n} f_2 & \dots & D_{x_n} f_n + 1 \end{vmatrix} - 1 \tag{3}$$

$$D_{x_i} f_j = (f_j(x_1,\dots,x_i,x_i+\Delta x_i,\dots,x_n) - f_j(x_1,\dots,x_i,\dots,x_n))/\Delta x_i$$

i.e.

$$D_{x_i} f_j = \frac{\Delta_{x_i} f_j}{\Delta x_i}$$

This theorem is a discrete analogue of the Liuvill theorem on the phase space expansion. In two dimensional case (3) becomes:

$$\mathrm{DIV}\,\mathbf{f} = D_{x_1} f_1 + D_{x_2} f_2 + D_{x_1} f_1 D_{x_2} f_2 - D_{x_1} f_2 D_{x_2} f_1$$

Notice that in [5] divergence was defined by

$$\mathrm{DIV}\mathbf{f} = \sum_{i=1}^{n} D_{x_i} f_i$$

This is formal discrete analogue of the continuous case. This definition cannot be use for obtaining of the discrete Liuvill theorem.

3. SUFFICIENT CONDITIONS FOR STABILITY AND NECESSARY CONDITIONS FOR INSTABILITY

Consider the region G_p containing the equilibrium point of the system (1). The set of phase trajectories for the initial conditions belonging to G_p can be considered as the flux of the vector field which forms the phase volume. In this case, the operator of divergence of vector field, can be considered as the intensity of source of the vector field set into the equilibrium point.

If DIV**f**>0, the phase volume of the starting cell will increase for $k\in N_0$. Since the basic cell containing the equilibrium point increases for $k\in N_0$ we conclude that the equilibrium point of the system (1) is unstable in the Liapunov sense.

Theorem 2 *Let the system* (1) *has the equilibrium point in the area G of the phase space X. If*, DIV**f**>0 *where* **f** *is the right side of equation* (1), *then the equilibrium point is unstable in the sense of Liapunov*.

The proof of these theorem is derived by a direct application of the relation (2), i.e., by using the property that the increment of the phase volume $\Delta_k V(k)>0$ for DIV**f**>0.

From (2), it follows that the one of the necessary conditions for stability of equilibrium point of (1) is given by DIV**f**<0.

If DIV**f**=0, the phase volume is constant. This case corresponds to the Hamiltonian discrete systems, that means to the conservative discrete systems. That is the case where there is no the increase non description of the energy of the isolated system.

If DIV**f**=*const*, from (2) follows

$$\Delta_k V(k) = V(k)\mathrm{DIV}\mathbf{f}$$

i.e.

$$V(k) = V(0)(1+\mathrm{DIV}\mathbf{f})^k$$

For the contraction of the phase volume for, which is the necessary condition for the stability of the system (1), it is necessary that

$$|1+\mathrm{DIV}\mathbf{f}|<1 \tag{4}$$

Since the nonlinear systems where DIV**f**<0 is variable, involve systems with DIV**f**=*const*, is a particular case, the theorem on the necessary conditions for stability of equilibrium points of (1) follows directly:

Theorem 3 *Necessary condition for stability of equilibrium point of system* (1) *is*:

$$-2<\mathrm{DIV}f<0 \tag{5}$$

In the phase space may be exists regions where DIV**f**<0 and those where DIV**f**>0. The boundaries between these regions can be determined trough the solution of the equation DIV**f**=0, (see example 2). In that way, the stable and unstable equilibrium points can be determined (there move than one).

Example 1 Let be given the system (1) where

$$\mathbf{x}(k)=\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix},\quad \mathbf{f}(\mathbf{x}(k))=\begin{bmatrix}x_2(k)\\ -x_1(k)+x_2(k)-\frac{1}{3}(x_1(k))^3\end{bmatrix}$$

Since $\mathrm{DIV}\mathbf{f}=(x_1(k))^2+x_1(k)+\frac{4}{3}$ for each $x_1(k)$, we conclude from Theorem 1, that the equilibrium point $x_1(k)=0$, $x_2(k)=0$ is unstable.

Example 2 Let be given the system

$$\Delta_k x_1(k)=0.5x_2(k)$$
$$\Delta_k x_2(k)=0.5x_1(k)-0.5x_2(k)-0.5(x_1(k))^2$$

i.e.

$$\Delta_k\mathbf{x}(k)=\mathbf{f}(x(k))$$

where

$$\mathbf{x}(k)=\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix},\quad \mathbf{f}(\mathbf{x}(k))=\begin{bmatrix}0.5x_2(k)\\ 0.5x_1(k)-0.5x_2(k)-0.5(x_1(k))^2\end{bmatrix}$$

Since $\mathrm{DIV}\mathbf{f}=x_1(k)$, we conclude from Theorem 3, that necessary conditions for stability of equilibrium point is $x_1(k)=0$, $x_2(k)=0$ is $-1<x_1(k)<3$.

Above obtained results can be used for proving of the existence of closed contoures and limit cycles in phase space.

If in the whole regions G where DIV**f**<0 or DIV**f**>0, then there is no oscillatory solutions of the system (1) involving limit cycles. This conclusion is a counter part of the corresponding Bendixson criterion for discrete dynamical systems.

For the existence closed phase trajectories or limit cycles in the region G, it is necessary that exists subregion (s) G_- where, DIV**f**<0and subregion(s) G_+ where DIV**f**>0. These conclusions correspond to the results presented in [6] relating to existence of the limit cycles of two dimensional discrete dynamical systems. The examples given in the paper [6] illustrate these conclusions.

4. CONCLUSION

In this paper it is first proved a discrete analogue of the Liuvill theorem on expansion of phase volume. It is shown that this theorem can be applied in analysis of stability of equilibrium point of nonlinear continuous systems [3], [4].

It could be interesting to investigate, by using the presented method, the discrete analogues of result from [3] where the conditions for equilibrium points stability of continuous systems are considered.

The same approach can be used in analysis of other dynamic characteristics of discrete nonlinear systems.

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KVALITATIVNI METOD ZA ANALIZU STABILNOSTI NELINEARNIH DISKRETNIH SISTEMA

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U radu je izložen jedan nov metod za kvalitativnu analizu stabilnosti nelinearnih diskretnih sistema baziran na diskretnoj teoriji polja. Uveden je pojam diskretne diveregencije u vidu diferencnog operatora. Dat je diskretan analog Luivilove teoreme o širenju diskretnog faznog prostora. Glavni rezultati su dati u vidu dve teoreme o potrebnim uslovima za stabilnost i dovoljnim uslovima za nestabilnost diskretnih sistema.