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A REMARK ON THE STABILITY OF LINEAR GYROSCOPIC SYSTEMS

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Abstract. *This Note deals with the problem, already considered by Lord Kelvin, of the stability of linear conservative gyroscopic systems. A theorem is given which provides a necessary and sufficient condition for a class of systems. This condition is presented in the form of the positive-definiteness of a certain matrix.*

1.We consider a linear mechanical systems described by the equation

$$M\ddot{q} + B\dot{q} + Cq = 0 \tag{1}$$

where M, B and C are real n x n matrices, q is n-vector, $\cdot = \frac{d}{dt}$, and

- M is symmetric and positive definite ($M = M^T > 0$);
- B is skew/symmetric ($B^T = -B$);
- C is symmetric and negative definite ($C = C^T < 0$).

The vector q represents the generalized coordinates, M is the mass matrix, B describes the gyroscopic forces and C potential forces. As $M>0$, one can utilize the positive definite square root in a familiar way to transform equation (1) to the form

$$\ddot{x} + G\dot{x} + Kx = 0 \tag{2}$$

Where $G=M^{-1/2}BM^{-1/2}$, $K=M^{-1/2}CM^{-1/2}$ and $x=M^{1/2}q$. It is clear that $G^T=-G$ and $K^T=K$.

As equation (2) is linear, "stability of system" is determined by the stability of the equilibrium solution $x = \dot{x} = 0$. For the potential system, which can be formally obtained from (2) by setting G=0, it is well known that the system is unstable. If gyroscopic forces

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are introduced, then the system (2) may be stable or unstable. According to the Kelvin-Chetayev theorem [1] the system may be "gyroscopically stabilized" if and only if the degree of freedom is even. We assume that G is nonsingular; in particular, n is even. Gyroscopic stabilization is a delicate phenomenon and construction of necessary and/or sufficient stability conditions (in terms of the system matrices without solving the spectrum of the entire system) is very difficult. Several results have been obtained in this direction and the references on the subject can be found in [2] and [3]. We supply these results by Theorem below.

Theorem. Suppose that
a) KG^2 is symmetric matrix,
b) G has distinct eigenvalues.
Then system (2) is stable if and only if

$$G^T G > (G^2 K)^{1/2} + (GKG)^{1/2} \tag{3}$$

If $KG = GK$, then the condition (3) is equivalent to $4K - G^2 > 0$, which is in agreement with [4].

2.The proof of Theorem is based on the following two lemmas.

Lemma 1. If n=2, then (3) is necessary and sufficient for the stability of system (2).

Proof. Introduce an orthogonal 2x2 matrix U which reduces K to diagonal form, i.e.

$$U^T K U = \overline{K} = -\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

where k_1, k_2, are positive real numbers, and

$$U^T G U = \overline{G} = \gamma \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since

$$U^T (KG^2)^{1/2} U = (\overline{K}\,\overline{G}^2)^{1/2} = |\gamma| \begin{bmatrix} \sqrt{k_1} & 0 \\ 0 & \sqrt{k_2} \end{bmatrix}$$

and

$$U^T (GKG)^{1/2} U = (\overline{G}\,\overline{K}\,\overline{G})^{1/2} = |\gamma| \begin{bmatrix} \sqrt{k_2} & 0 \\ 0 & \sqrt{k_1} \end{bmatrix}$$

we have

$$U^T \left(G^T G - (KG^2)^{1/2} - (GKG)^{1/2}\right) U = -\overline{G}^2 - (\overline{K}\,\overline{G}^2)^{1/2} - (\overline{G}\overline{K}\,\overline{G})^{1/2} =$$
$$= \left(\gamma^2 - |\gamma|\left(\sqrt{k_1} + \sqrt{k_2}\right)\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consequently, condition (3) is equivalent to

$$|\gamma| > \sqrt{k_1} + \sqrt{k2} \tag{4}$$

On the other hand by setting $x = U\overline{x}$ the equation (2) is transformed into

$$\ddot{\bar{x}} + \overline{G}\dot{\bar{x}} + \overline{K}\bar{x} = 0 \tag{5}$$

The eigenvalues associated with (5) are purely imaginary and semi-simple (in other words, the system of equation (5) is stable) if and only if (4) holds.

Lemma 2. Under the assumptions (a) and (b) of Theorem there exists an orthogonal matrix T such that $T^T GT = diag(G_{(i)})$ and $T^T KT = diag(K_{(i)})$, where $G_{(i)}$, $K_{(i)}$, are skew-symmetric and symmetric 2x2 matrices respectively.

Proof. Since G is nonsingular skew-symmetric, then it can be reduced to the canonical form (see [5]).

$$T^T GT = diag(G_{(i)}) \ , \ G_{(i)} = \gamma_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where T is an orthogonal matrix and $\pm\sqrt{-1}\,\gamma_i$, i=1, ... , n/2 are the eigenvalues of G. If $T^T KT$ is partitioned into 2x2 submatrices

$$T^T KT = \left(K_{ij}\right)$$

condition (a) of Theorem is identical with the equations

$$(\gamma_i - \gamma_j)K_{ij} = 0 \ , \quad i, j = 1,..., \frac{n}{2} \tag{6}$$

Proff of the Theorem. Using the orthogonal matrix T of the previous lemma we introduce y by setting $x=Ty$ which transforms (2) into

$$\ddot{y} + T^T GT\dot{y} + T^T KTy = 0 \tag{7}$$

or

$$\ddot{y}_{(i)} + G_{(i)}\dot{y}_{(i)} + K_{(i)}y_{(i)} = 0 \ , \quad i, j = 1,..., \frac{n}{2} \tag{8}$$

where $y_{(i)} = (y_{2i-1}, y_{2i})^T$. According to lemma 1 the system (7) is stable if and only if

$$G_{(i)}^T G_{(i)} > (K_{(i)}G_{(i)}^2)^{1/2} + (G_{(i)}K_{(i)}G_{(i)})^{1/2} \ , \ i = 1,..., \frac{n}{2} \tag{9}$$

or, in terms of the original matrices,

$$G^T G > (KG^2)^{1/2} + (GKG)^{1/2}$$

This completes the proof of the Theorem.

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NAPOMENA O STABILNOSTI LINERANIH SISTEMA GIROSKOPSKIH SISTEMA

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Ovaj rad bavi se problemom stabilnosti linearnih konzervativnih giroskopskih sistema koji je već razmatrao Lord Kelvin. Data je teorema koja obezbedjuje potrebne i dovoljne uslove za jednu klasu sistema. Ovaj uslov je predstavljen u obliku pozitivne definitnosti odredjene matrice.