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INVERSE PROBLEMS OF NONLINEAR CONTROL SYSTEMS

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Abstract. *The inverse problems for nonlinear control system, described by ordinary differential equations in values of a output considered on one or more trajectories are being studied. The criterions of observability, invertibility, identifiability of nonlinear systems with using the extended output have been proved. On the basis of the modified implicit functions theorem the sufficient conditions are obtained in terms of Jacobi matrices ranks which permit the effective check-up. The set trajectories method proposed extends classes of systems for which it is possible to find unknowns of their mathematical models and is the base of new computing algorithms. It is shown that problem of determing input in the values of state known on the several trajectories always can be reduced (at least locally) to the algebraic relations and has solution for any system. The possibilities of determining the moments of inertia of a rigid body on the basis of measurements of the projection of the angular velocity onto a principal axis are studied.*

1.THE INVERSE PROBLEM FOR THE CONTROL SYSTEMS.

In the last two decades, the main developments for nonlinear control systems based on differential geometric ideas have found wide and successful using in the study of many problems of affine control systems (see [2], as example). Part of them connected with inverse problems are considered in this article. We take approach based on the known in analytical mechanics invariant relations method which has been extended in [3] to the control problems of general nonlinear systems.

Consider the systems of ordinary differential equations

$$\dot{x} = f(x,u), \; x_0 \in D \subseteq R^n , \tag{1}$$

on any solutions $x(.,x_0.u)$ of which the values of functions

$$y = h(x,u), \tag{2}$$

are given, where $x=(x^1,...,x^n)$ is a phase vector, $u=(u^1,...,u^m)$ is control or vector of

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parameters, $y=(y^1,...,y^k)$ is an output of the system (1), $t\in T=[0,t_f]$. Functions $f(x,u)$, $h(x,u)$, $u(t)$ are assumed to be continuously differentiable necessary number of times.

The problems, which we have called inverse, arise when some variables of the mathematical model (1) x, u or part of them are unknown and must be reconstructed on the basis of measurements output $y(.)$. Such the well known are observability and invertibility [1,7] problems in which accordingly the state x and control u must be determined. The presence of additional uncertainties in the model (1) is leading to modification of these basical problems. In particular, the dynamics of object in the observation problem may be described by differential including

$$\dot{x}\in F(x)=f(x,u),\ u\in U\subseteq C^N(T,R^m), \tag{3}$$

and it is natural to say about observation of uncertain system in this case. Similar situation arises when we need to determine the values of $u(t)$ considering as unknown control or varying parameters of system (1) when x_0-unknown, $x_0\in D$. This problem will be called here as identifiability problem. Our purpose is to study the solvability conditions of all such problems, when part of variables of system (1) must be determined whereas the others may be known or not. Since it is supposed that concrete realization of output is used, then solution exists and solvability question is reduced to the uniqueness of one.

From general point of view system (1), (2) generates the input-output transfer mapping with unknowns as input and y as output. For the observability problem such mapping is ($u(t)$ - for the sake of simplicity, is known function)

$$\phi_1(x_0):D\to C^N(T,R^k):\quad x_0\to x(.,x_0,u)\to y(.)=h(x,u)$$

For the invertibility problem x_0 is fixed and transfer mapping has the form

$$\phi_2(u):U\subseteq C^N(T,R^m)\to C^N(T,R^k):\quad u\to x(.,x_0,u)\to y(.)=h(x,u)$$

Since the mappings ϕ_1, ϕ_2 are single-valued then their injection is sufficient for solvability of inverse problems, a.e.

$$x_1=x_2\Rightarrow\phi_1(x_1)=\phi_1(x_2):\quad u_1=u_2\Rightarrow\phi_2(u_1)=\phi_2(u_2)$$

Consider transfer mappins for uncertain models (3). In the observe problem by means of system (3) any x_0 can be placed in correspondence with a set of solutions $X_U(x_0)=\bigcup\limits_{u(.)\in U}x(.,x_0,u)$ which is carried by functions (2) into the set of outputs $Y(x_0)=\bigcup\limits_{x(.)\in X_U(x_0)}h(x(.))$.

$$\Phi_1(x_0):\to C^N(T,R^k):\quad x_0\to X_U(x_0)\to Y_U(x_0)$$

In the similar way the many-values transfer mappings arise at the identification problem. Denote $X_D(u)=\bigcup\limits_{x_0\in D}x(.,x_0,u)$, $Y_D(u)=\bigcup\limits_{x(.)\in X_D}h(x(.))$. We have

$$\Phi_2(u):\subseteq C^N(T,R^m)\to C^N(T,R^k):\quad u\to X_D(u)\to Y_D(u)$$

The sets Y_U, Y_D generally speaking, contain all possible information regarding unknowns and it is reasonable to assume that inverse problem can be solved if, for different x or u the corresponding Y are in some way mutually distinguishable. At the same time it is naturally to suppose that not whole set Y is known, but only some (finite number at least) of its elements. Hence more strong conditions have to be implemented

$$x_1 \neq x_2 \Rightarrow Y_U(x_1) \bigcap Y_U(x_2) = 0, \quad u_1 \neq u_2 \Rightarrow Y_D(u_1) \bigcap Y_D(u_2) = 0.$$

Before introduction formal properties of system (1), (2) when the inverse problems have unique solutions define, the set of feasible functions U. Assume that U will be defined by the domain D_α of values of the functions and their derivatives up to order α: $U = U(D_\alpha)$ where $D_0 = \bigcup_{t\in T} u(t)$, $D_1 = \bigcup_{t\in T} (u(t),\dot{u}(t)),\ldots, D_\alpha = \bigcup_{t\in T} (u(t),\ldots,u(t)^{(\alpha)})$, $D_i \subseteq R^{(i+1)m}$.

DEFINITION 1. System (1) is called observable on the output (2) in the domain $D\times D_\alpha\times T$, if for any $x_1,x_2\in D$ and admissible function $u(.)\in U(D_\alpha)$ there exists such instant $t\in T$ that $h(x(t,x_1,u))\neq h(x(t,x_2,u))$.

DEFINITION 2. Uncertain system (3) is called observable on the output (2) in the domain $D\times D_\alpha\times T$, if for any $x_1,x_2\in D$ and any two distinct admissible functions $u_1(t),u_2(t)\in D_\alpha$ there exists such instant $t\in T$ that $h(x(t,x_1,u_1))\neq h(x(t,x_2,u_2))$.

DEFINITION 3. System (1) is called invertible on the output (2) in the domain $D\times D_\alpha\times T$ if for any point $x_0\in D$ and two distinct admissible functions $u_1(t),u_2(t)\in D_\alpha$ there exists such instant $t\in T$ that $h(x(t,x_1,u_1))\neq h(x(t,x_2,u_2))$.

DEFINITION 4. System (1) is called identifiable on the output (2) in the domain $D\times D_\alpha\times T$, if for any two distinct admissible functions $u_1(t),u_2(t)\in U(D_\alpha)$ and any solutions $x_1(.)\in X_D(u_1),x_2(.)\in X_D(u_2)$ there exists such instant $t\in T$ that $h(x_1(t))\neq h(x_2(t))$.

2. EXTENDED OUTPUT

The property of solvability of the inverse problems in a sense of definitions 1-4 is connected with injection of transfer mapping with respect to a part of variables x, u. Usually the information about output is not sufficient for establishment of such property and it is desirable to use some additional relations between x, u and y. For this purpose let us introduce the family of the functions z_α, consisting of the components of y and their successive derivatives up to order α, taken by virtue of system of differential equations (1).

$$y(t) = h_0(x,y) = h(x,u)$$

$$y^{(i)}(t) = h_i(x,u,\dot{u},\ldots,u^{(i)}) = \frac{\partial h_{i-1}}{\partial x} f(x,u) + \sum_{j=0}^{i-1} \frac{\partial h_{i-1}}{\partial u^{(j)}} u^{(j+1)}, \quad i = 1,\ldots,\alpha$$

Denote $z_\alpha = (y,\dot{y},\ldots,y^{(\alpha)})$, $\upsilon_\alpha = (\dot{u},\ldots,u^{(\alpha)})$, $H_\alpha(x,u,\upsilon_\alpha) = (h_0(x,u),\ldots,h_\alpha(x,u,\upsilon_\alpha))$ and rewrite this relations in the form

$$z_\alpha(t) = H_\alpha(x,u,\upsilon_\alpha) \tag{4}$$

It is evidently that for any α the vector $z_\alpha(.)$ one-to-one corespondents to the output $y(.)$. Hence the analysis of the mapping $(x(.),u(.))\to y(.)$ defined implicitly by system (1), (2) may be replaced by the study of the algebraic relations (4), containing the same functions $x(.)$, $u(.)$, $y(.)$ and its derivatives $\upsilon_\alpha(.)$, $z_\alpha(.)$. In particular, if the transfer mapping is injection with respect to x or u then the same property has the mapping $z_\alpha=H_\alpha(x,u,\upsilon_\alpha)$. Sometimes the last ones for some α may be verified by algebraic criterions.

THEOREM 1. System (1), (2) is observable if for some α in the domain $D\times D_\alpha\times T$ for any fixed $u(.)\in U(D_\alpha)$ the mapping $x\to z_\alpha = H_\alpha(x,u,\upsilon_\alpha)$ is bijection.

The next theorem formulates the sufficient conditions of solvability of inverse problems for the system with uncertainties. For the identifiability problem we have.

THEOREM 2. System (1), (2) is identifiable if for some α in the domain $D\times D_\alpha\times T$ there exists $t\in T$ such that $pr_u H_\alpha^{-1}(z_\alpha)$ is single-valued on the set $Z_\alpha(t)=H_\alpha(x,u,\upsilon_\alpha)$. Here $pr_u H_\alpha^{-1}(z_\alpha)$ is the projection onto the space R^m of values of functions from $U(D_\alpha)$.

Proof. Assume that counter to the assertion, system is not identifiable in $D\times D_\alpha\times T$. Then there exist $u_1(t)$, $u_2(t)\in U(D_\alpha)$ and solutions $x_1(.)\in X_D(u_1)$, $x_2(.)\in X_D(u_2)$, such that, $h(x_1(t))\equiv h(x_2(t))$ for $t\in T$. Differentiating this identity necessary number of times, we obtain $H_\alpha(x_1,u_1,\upsilon_{\alpha 1})\equiv H_\alpha(x_2,u_2,\upsilon_{\alpha 2})$. Therefore mapping $pr_u H_\alpha^{-1}(z_\alpha)$ is not single-valued for any $t\in T$.

It is difficult to verify the conditions of the theorems 1, 2 or another similar statements because it is connected with analysis of nonlinear mappings in some domain. In the local case, however, this mapping's injection conditions are equivalent to criterions formulated in terms of Jacobi matrices ranks, which permit the effective check-up. The next section introduces the notion of injection of nonlinear mapping with respect to a part of the variables, the main tool of investigations of nonlinear relations (4) connected with transfer mappings.

3. INJECTION OF DIFFERENTIAL MAPPING WITH RESPECT TO A PART OF THE VARIABLES

Let $x\in E\subseteq R^n$, $y\in F\subseteq R^m$, $z\in P\subseteq R^l$ are connected by the system of l independent in the domain $E\times F$ equations

$$z - H(x,y) = 0, \quad H\in C^p(E\times F, P), \quad z_0 = H(x_0, y_0) \tag{5}$$

Consider the problem of defining conditions when vector x may be uniquely determined for any z from some neighborhood of z_0. In particular, when $l=n+m$ by implicit-function theorem there exist neighborhoods S_x, S_y, S_z of the points x_0, y_0, z_0, accordingly, functions G_x, Gy such that all solutions $x\in S_x$, $y\in S_y$ of the algebraic equations (5) are given by the formulas $x=G_x(z)$, $y=G_y(z)$ for $z\in S_z$. When $l<n+m$ and $\partial h(x,y)/\partial(x,y)$ is a constant in the $E\times F$ there exist neighborhoods S_x, S_y, S_z of the points x_0, y_0, z_0, that pre-image of any $z\in S_z$ is a manifold of dimension $n+m-l$ and its coordinates are described in general by the next formulas

$$x_\alpha = G_x(z, x_\beta, y_\beta), \quad y_\alpha = G_y(z, x_\beta, y_\beta), \quad x = (x_\alpha, x_\beta), \quad y = G_x(y_\alpha, y_\beta)$$

In this case vector x may be found too if these formulas have the form

$$x = G_x(z), \quad y_\alpha = G_y(z, x, y_\beta)$$

Let us prove the conditions of such representation of manifold of solutions of transcendental equation (5).

Denote $J(x,y)=\partial H(x,y)/\partial(x,y)$, $J_x(x,y)=\partial H(x,y)/\partial x$, $J_y(x,y)=\partial H(x,y)/\partial y$ and suppose that ranks of these Jacobi matrices are constant in the considering domains.

LEMMA 1. Let in $E\times F$

$$\operatorname{rank} J(x,y) = n + \operatorname{rank} J_y(x,y)$$

Then there exist neighbourhoods S_x, S_y, S_z of points x_0, y_0, z_0 and some function $G\in C^p(S_z,S_x)$ such , that for $(x,y,z)\in E\times F\times P$ coordinates x of the *set* (x,y) of solutions of system of algebraic equations (5) are described by $x=G(z)$.

Proof. It follows from the conditions of Lemma that $\operatorname{rank}J_x=n$. Indeed, $J=(J_x,J_y)$ hence $\operatorname{rank}J\le\operatorname{rank}J_x+\operatorname{rank}J_y$ or $\operatorname{rank}J_x\ge\operatorname{rank}J-\operatorname{rank}J_y=n$. On the other hand, matrix J_x has dimension $(l\times n)$ and its rank cannot exceed n. Thus, $\operatorname{rank}J_x=n$.

Denote that $\operatorname{rank}J_y=s\le m$. Renumbering, if it is necessary, the components of vectors y, H assume that the maximum nonsingular minor J_y has the form

$$\frac{\partial(H^1,\ldots,H^s)}{\partial(y^1,\ldots,y^s)}$$

Then for any $x\in E$ in some neighborhoods S_y of the point y_0 the functions $H^1,\ldots,H^s$ are independent as functions y, while the rest depend on them.

$$z^{s+i} = g^i(x^1,\ldots,x^n,z^1,\ldots,z^s) \quad i=1,\ldots,n$$

where $\partial(g^1,\ldots,g^s)/\partial(x^1,\ldots,x^n) = n$. We will demonstrate this. Indeed, the nonsingular change of variables $(x,y)\to(x',y')$ transforms matrix $J(x,y)\to J'(x,y)= \partial z/\partial(x',y')$ without altering its rank

$$J' = \begin{pmatrix} 0_{s\times n} & E_{s\times s} & 0 \\ \partial(g^1,\ldots,g^n)/\partial(x^1,\ldots,x^n) & \partial(g^1,\ldots,g^n)/\partial(y^1,\ldots,y^n) & 0 \end{pmatrix} \tag{6}$$

We have that $\operatorname{rank} J=\operatorname{rank}J'=\operatorname{rank}\partial(g^1,\ldots,g^s)/\partial(x^1,\ldots,x^n)+s$. From (6) we may conclude that $\operatorname{rank}\partial(g^1,\ldots,g^s)/\partial(x^1,\ldots,x^n) = n$. This on the basis of the implicit function theorem is sufficient for existence of neighborhoods S_x, S_z and some function $G\in C^p(S_z,S_x)$ such that for $(x,y,z)\in E\times F\times P$, $x=G(z)$. The lemma has been proved.

4. CONDITIONS OF LOCAL SOLVABILITY OF INVERSE PROBLEMS.

The uniqueness of the local solutions of inverse problems considered is being provided by existence of the functions $x=G(z_\alpha)$, $u=G(z_\alpha)$ or $u=G(x,z_\alpha)$ being the solutions of the system (4). Sufficient conditions for the existence of such type of solutions of the system of algebraic equations system are given by lemma 1. Consider first the observability property of uncertain system (3).

THEOREM 3. Assume for some α in the domain $D\times D_\alpha$ that

$$\operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial(x,u,\upsilon_\alpha)} = n + \operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial(u,\upsilon_\alpha)}$$

Then system (3) is locally observable on the output (2) in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

Proof. Under conditions of theorem equalities (4) define in some $S_x\times S_u\times S_{\upsilon_\alpha}\subseteq D\times D_\alpha$ implicit single-valued function $x=G(z_\alpha)$, $x\in S_x$, $u\in S_u$, $\upsilon_\alpha\in S_{\upsilon_\alpha}$. Take sufficiently small interval $\tau=[0,\in)\subseteq T$, domain $d'\subseteq d\subseteq S_x$ such, that the solutions of system (3) with $x_0\in d'$ and $u(.)\in U(S_u\times S_{v_\alpha})$ stay in d for $t\in\tau$.

Assume now that system (3), (2) is not observable in the $d\times d_\alpha\times\tau$, i.e. there exist $x_1,x_2\in d'$, $u_1(.),u_2(.)\in U(d_\alpha)$ such that for $t\in\tau$ the following identity is observed: $h(x(t,x_1,u_1))\equiv h(x(t,x_2,u_2))$. Differentiating the required number of times we obtain that $z_\alpha(t)$ corresponds to two distinct states $x(t,x_1,u_1),x(t,x_2,u_2)\in d$ contrary to the fact that function $x=G(z_\alpha)$ is single-valued in considering domain.

REMARK

If the system isn't observable then we don't define the whole phase vector. But sometimes it may be done for a part of its coordinates x' where $x=(x',x'')$. Considering the $x''(t)$ as uncertaints in the system of differential equations $\dot{x}'=f(x',x'')$ (for the sake of simplicity the equations don't contain input u), we may formulate the local conditions of solving of such problem

$$\operatorname{rank}\frac{\partial H_\alpha(x)}{\partial x} = \dim x' + \operatorname{rank}\frac{\partial H_\alpha(x)}{\partial x''}$$

In particular, system (1), (2) is locally observablen when rank $\partial H_\alpha(x)/\partial x=n$.

THEOREM 4. Assume for some α in the domain $D\times D_\alpha$

$$\operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial(x,u,\upsilon_\alpha)} = m + \operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial(x,\upsilon_\alpha)}$$

Then system (1) is locally identifiable on the output (2) in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

The identifiability condition is proved analogously to observability ones. By lemma 1 there exist $S_x\times S_u\times S_{\upsilon_\alpha}\subseteq D\times D_\alpha$ and locally single-valued function $u=G(z_\alpha)$. The rest variables x,v_α may be considered as indeterminate values in the mathematical model (1).

THEOREM 5. Assume for some α in the domain $D\times D_\alpha$

$$\operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial(u,\upsilon_\alpha)} = m + \operatorname{rank}\frac{\partial H_\alpha(x,u,\upsilon_\alpha)}{\partial\upsilon_\alpha}$$

Then system (1) is locally invertible on the output (2) in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

PROOF. Fix in equations (4) the variable x. Under the conditions of the theorem and lemma 1 there exist the function G, domains S_z, S_u, S_{v_α} such that for any $x\in D$ $u=G(x,z_\alpha)$

for $z\in S_z$, $(u,\upsilon_\alpha)\in S_u\times S_{\upsilon_\alpha}$. Substitute the function $G(x,z_\alpha)$ in system of differential equations (1) instead of u, we have

$$\dot{x}=f(x,G(x,z_\alpha))=F(x,z_\alpha(t)),\quad x(0)=x_0\ . \tag{7}$$

By virtue of uniqueness of Cauchy problem's solution of system of differential equations (7) $u(t)$ may be found uniquely on the formula $u=G(x,z_\alpha)$ in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

As it turns out the invertibility conditions are wider than identifiability ones, hence the information about x_0 is essential for determing values of input $u(t)$.

EXAMPLE 1.

Consider the system

$$\dot{x}_1=x_1+x_3(1+u_1)\,,\ \ \dot{x}_2=x_1-u_2\,,\ \ \dot{x}_3=x_4\,,\ \ \dot{x}_4=u_2x_3$$

$$y_1=x_1-x_2\,,\quad y_2=x_3\,.$$

To investigate the property of identifiability, we set up the derivatives of the outputs and take $z=(y_1,y_2,\dot{y}_1,\dot{y}_2,\ddot{y}_1,\ddot{y}_2)$,

$$z=(x_1-x_2\,,\ x_3\,,\ x_3(1+u_1)+u_2\,,\ x_4\,,\ x_4(1+u_1)+x_3\dot{u}_1+\ \dot{u}_2\,,\ u_2x_3)\,.$$

and obtain that for any x, u, $\dot{u}$ ($x_3\neq0$) $\operatorname{rank}\partial z/\partial(x_1,x_2,x_3,x_4,u_1,u_2,\dot{u}_1,\dot{u}_2)=6$ and $\operatorname{rank}\partial z/\partial(x_1,x_2,x_3,x_4,\dot{u}_1,\dot{u}_2)=4$. Thus, on the basis of theorem 4 the values of $u_1(t)$, $u_2(t)$ are determinated in any domain where $x_3\neq0$ by the values of components of the vector $z(t)$ only, i.e. 6 equations relating 8 unknowns $x_1,x_2,x_3,x_4,u_1,u_2,\dot{u}_1,\dot{u}_2$ may be solved in such way, that u_1,u_2 are functions of $y_1,y_2,\dot{y}_1,\dot{y}_2,\ddot{y}_1,\ddot{y}_2$. In this case we may calculate $u_1=[y_2(\dot{y}_1-y_2)-\ddot{y}_2]/y_2{}^2$, $u_2=\ddot{y}_2/y_2$.

5. USE THE SET OF TRAJECTORIES

The problems of determing the values of functions $u(.)$ allow us to use information about output obtained on the several different solutions of system of differential equations (1) under assumption that on any of ones input u is the same. The image of transfer mappings in this case consists of several functions and the set of systems for which recostruction of input is possible may be extended. Indeed, it can happen that an arbitrary function from $Y_D(u_1)$ coincides with some function from $Y_D(u_2)$, but any pair functions do not both belong simultaneously to any another set $Y_D(u_2)$. In this case it is meaningful to speak about identifiability with respect to two trajectories, etc. In that way the notion of invertibility and identifiability properties with using the set of trajectories may be introduced.

DEFINITION 5. System (1) is called λ-invertible (invertible with respect to λ trajectories) on the output (2) in the domain $D\times D_\alpha\times T$ if for any points $x_1,\dots,x_\lambda\in D$, two distinct admissible functions $u_1(t)$, $u_2(t)\in U(D_\alpha)$ there exists such instant $t\in T$ that

$$(h(x(t,x_1,u_1)),\dots,h(x(t,x_\lambda,u_1)))\neq(h(x(t,x_1,u_2)),\dots,h(x(t,x_\lambda,u_2)))$$

DEFINITION 6. System (1) is called λ-identifiable (identifiable with respect to λ trajectories) on the output (2) in the domain $D\times D_\alpha \times T$ if for any set of λ points $x_{i1},\dots,x_{i\lambda}\in D$, i=1,2, two distinct admissible functions $u_1(t)$, $u_2(t)\in U(D_\alpha)$ there exists such instant $t\in T$ that

$$(h(x(t,x_{11},u_1)),\dots,h(x(t,x_{1\lambda},u_1)))\neq(h(x(t,x_{21},u_2)),\dots,h(x(t,x_{2\lambda},u_2)))$$

Instead system (1), (2) let us consider the extended system of order λ_n with phase vector $X_\lambda=(x_1,\dots,x_\lambda)\in D^\lambda$, output $Y_\lambda=(y_1,\dots,y_\lambda)$ and initial input u

$$\dot{x}_i = f(x_i,u) \tag{8}$$

$$y_i = h(x_i) \quad i=1,\dots,\lambda \tag{9}$$

It follows directly from Definitions 3, 4 and 6, 7 that the system (1),(2) is invertible (identifiable) in domain $D\times D_\alpha\times T$ with respect to λ trajectories if and only if system (8), (9) is invertible (identifiable) in domain $D^\lambda\times D_\alpha\times T$ with respect to one trajectory. Therefore the proof of criterions of λ-invertibility and λ-identifiability involves the application of appropriate theorems of the preceding section to the system (8), (9).

THEOREM 6. Let in the domain $D^\lambda\times D_\alpha\times T$

$$\operatorname{rank}\frac{\partial(H(x_1,u,\upsilon_\alpha),\dots,H(x_\lambda,u,\upsilon_\alpha))}{\partial(u,\upsilon_\alpha)} = m+\operatorname{rank}\frac{\partial(H(x_1,u,\upsilon_\alpha),\dots,H(x_\lambda,u,\upsilon_\alpha))}{\partial\upsilon_\alpha}$$

Then system (1) is locally λ-invertible on the output (2) in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

THEOREM 7. Let in the domain $D^\lambda\times D_\alpha\times T$

$$\operatorname{rank}\frac{\partial(H(x_1,u,\upsilon_\alpha),\dots,H(x_\lambda,u,\upsilon_\alpha))}{\partial(x_1,\dots,x_\lambda,u,\upsilon_\alpha)} = m+\operatorname{rank}\frac{\partial(H(x_1,u,\upsilon_\alpha),\dots,H(x_\lambda,u,\upsilon_\alpha))}{\partial(x_1,\dots,x_\lambda,\upsilon_\alpha)}$$

Then system (1) is locally λ-identifiable on the output (2) in some domain $d\times d_\alpha\times\tau\subseteq D\times D_\alpha\times T$.

In general the possibilities of solving of inverse problems increase with increasing the number of solutions considered. In the [4], [5] the number of trajectories that are required to determine $u(t)$ was estimate and two numbers $\lambda_{min}=[mk^{-1}]+1$, $\lambda_{max}=m+1-k$ where $k=\operatorname{rank}\partial f(x,u)/\partial(x,u)-\operatorname{rank}\partial f(x,u)/\partial x$ with following property: the system can't be invertible with $\lambda\leq\lambda_{min}$ trajectories; if it is not invertible by $\lambda\geq\lambda_{max}$ trajrectories then it is not invertible, are found.

The examples of applications of these results show that for nonlinear systems the using of the set of trajectories expands the possibilities of the reconstruction of input signal and permits in some cases to solve the problem for output of dimension smaller then input's one, what is in principle impossible for using of one trajectory. But from the studies of linear systems with linear output by theorems 6, 7 it follows that the using of the set of trajectories doesn't give any advantages in this case.

EXAMPLE 2.
Consider the system

$$\dot{x}_1 = x_1 + x_3{}^2\ , \quad \dot{x}_2 = x_3\ , \quad \dot{x}_3 = u\ .$$
$$y = x_1\ .$$

Taking $z = (y, \dot{y}, \ddot{y}, y^{(3)}) = (x_1, x_2 + x_3{}^2,\ x_3(1+2u),\ \ 2x_3\dot{u} + u(1+2u))$ we have

$$\operatorname{rank}\frac{\partial z}{\partial(x_1, x_2, x_3, u, \dot{u})} = 4\,, \quad \operatorname{rank}\frac{\partial z}{\partial(x_1, x_2, x_3, \dot{u})} = 4$$

All the information regarding to the motion of the systems is defined by the components of vector z relating the unknowns x_1, x_2, x_3, u, $\dot{u}$ which cannot be solved in such a way that u is the function of $(y, \dot{y}, \ddot{y}, y^{(3)})$ only. Assume that output is also known on the another solution $x_1^{'}, x_2^{'}, x_3^{'}$ corresponding to the same input u. Taking $Z_2 = (y, \dot{y}, \ddot{y}, y^{(3)}, y^{'}, \dot{y}^{'}, \ddot{y}^{'}, y^{'(3)})$ we have

$$\operatorname{rank}\frac{\partial Z_2}{\partial(x_1, x_2, x_3, x_1^{'}, x_2^{'}, x_3^{'}, u, \dot{u})} = 8\,, \quad \operatorname{rank}\frac{\partial Z_2}{\partial(x_1, x_2, x_3, x_1^{'}, x_2^{'}, x_3^{'}, \dot{u})} = 7\ .$$

Hense the system considered is identifiable with respect to two solutions. Denote, that since all components of the vector Z_2 are functionaly independent, it follows that all unknowns $x_1, x_2, x_3, x_1^{'}, x_2^{'}, x_3^{'}, u, \dot{u}$ may be found and we may conclude that system is also observable with respect to two trajectories.

6. Solvability of inverse problem in the phase vector measurring.

It is obvious that the solvability properties of inverse problems of system (1), (2) depends to a considerable extent on the choice of the function (2). Assume that the information of motion of system (1) is maximal: $h(x)=x$. It turns out that this is sufficient for local identifiability of any system with respect to some number of trajectories, provided that the right sides of system (1) depend on the minimum possible number of parameters introduced. We offer the following definition.

Definition 7. The parameters of system (1) are essential if there do not exist functions F, υ such that $f(x,u) \equiv F(x, \upsilon(u))$, $(x,u) \in D \times D_0 \subseteq R^{n+m}$, the dimension of υ being less than that one of u.

Lemma 2. The parameters of the system (1) are essential in some domain $d \times d_0 \subseteq D \times D_0$ if and only if for any λ points $x_1, \dots, x_\lambda \in D$ rank $J_\lambda(x_1, \dots, x_\lambda, u) = m$. Here $\lambda = \lambda_{max} = m+1-$rank$\partial f(x,u)/\partial u$,

$$J_\lambda(x_1, \dots, x_\lambda, u) = \operatorname{colon}\left(\frac{\partial f(x_1, u)}{\partial u}, \dots, \frac{\partial f(x_\lambda, u)}{\partial u}\right)$$

Proof. Sufficiency. Let us assume that rank $J_\lambda = m$, but then, contrary to the assertion of the theorem, there exist functions F, υ such that $f(x,u) \equiv F(x, \upsilon(u))$, dim$\upsilon = s \le m$ and det$\partial \upsilon^i / \partial u^j \neq 0$ $(i,j=1,\dots,s)$. Taking the partial derivatives of $f^i(x,u)$ with respect to $u^1, \dots, u^m$ we obtain

$$\frac{\partial f^i(x,u)}{\partial u^1} = \frac{\partial F^i(x,\upsilon)}{\partial \upsilon^1}\frac{\partial \upsilon^1}{\partial u^1} + \dots + \frac{\partial F^i(x,\upsilon)}{\partial \upsilon^s}\frac{\partial \upsilon^s}{\partial u^1}$$

$$\frac{\partial f^i(x,u)}{\partial u^m}=\frac{\partial F^i(x,\upsilon)}{\partial \upsilon^1}\frac{\partial \upsilon^1}{\partial u^m}+...+\frac{\partial F^i(x,\upsilon)}{\partial \upsilon^s}\frac{\partial \upsilon^s}{\partial u^m}$$

We determine

$$\frac{\partial F^i(x,\upsilon)}{\partial \upsilon^j}=\frac{\partial \upsilon^s}{\partial u^m}=\sum_{i=1}^{s}\eta_{ij}(u)\frac{\partial f^i}{\partial u^j}, \quad (i=1,...,n;\, j=1,...s)$$

where η_{ij} are independent of x and are the same for i=1,...,n. From this we have

$$\frac{\partial f^i(x,u)}{\partial u^{s+1}}=\sum_{p=1}^{s}\kappa_p(u)\frac{\partial f^i(x,u)}{\partial u^p}, \quad (i=1,...,n)$$

Thus, the (s+1)-th column of matrix $\partial f(x,u)/\partial u$ is a linear combination of the first s columns. Then, since the coefficients $\kappa_p(u)$ are independent of x, by construction of matrix J_λ its (s+1)-th column is the sum of the first s columns with the same coefficients. However, this contradicts the condition rank$J_\lambda=m$.

Necessity. Let rank$J_\lambda=s\le m$ for any $x_1,...,x_\lambda\in D$. We will show that in this case $f(x,u)$ can be represented by smaller number of parameters. We introduce s points $x_1,...,x_\lambda\in D$. such that the rows

$$\frac{\partial f^{l_1}(x_1,u)}{\partial u},...,\frac{\partial f^{l_1}(x_s,u)}{\partial u}$$

of matrix J_λ are linearly independent and $\det(\partial f^{l_i}/\partial u^j)\neq 0$ (i,j=1,...,s). Then for any l=1,...,n

$$\frac{\partial f^l(x,u)}{\partial u^1}=\sum_{i=1}^{s}\eta_i(x,u)\frac{\partial f^{l_i}}{\partial u^1},...,\frac{\partial f^l(x,u)}{\partial u^m}=\sum_{i=1}^{s}\eta_i(x,u)\frac{\partial f^{l_i}}{\partial u^m} \tag{10}$$

Solving the first s equations for $\eta_i(x,u)$ we find that $\eta_i(x,u)=\sum_{j=1}^{s}\eta_{ij}\partial f^i/\partial u^j$, where $\eta_{ij}(x_1,...,x_\lambda,u)$ and are the same for any functions $f^i(x,u)$ l=1,...,n. Substituting $\eta_i(x,u)$ into the (s+1)-th equation of system (10), we obtain

$$\frac{\partial f^i(x,u)}{\partial u^{s+1}}=\sum_{j=1}^{s}\kappa_j(u)\frac{\partial f^i(x,u)}{\partial u^j}, \tag{11}$$

where $\kappa_j(u)=\sum_{j=1}^{s}\eta_{ij}(x_1,...,x_s,u)\partial f^{l_i}(x_i,u)/\partial u^{s+1}$ are independent of the number l=1,...,n.

Considering (10) as a partial differential equation with respect to the functions $f^l(x,u)$ for any fixed x, we conclude that all solutions f_l depend functionally on the integrals $\alpha_1(u)$,..., $\alpha_s(u)$ of the system of characteristics of equation (10): $f^l(x,u)\equiv F^l(x,\alpha_1(u),...,\alpha_s(u))$ l=1,...,n . Thus, contrary to our assumption, function $f(x,u)$ can be represented by a smaller number of parameters and this proves the lemma.

The following theorem is a corollary of Lemma 2 and Theorem 4.

THEOREM 8. Assume that in the domain $D\times D_0$ parameters of the right hand side of system (1) are essential. Then system of differential equations (1) is identifiable with respect to λ trajectories in some domain $d\times d_0\times\tau\subseteq D\times D_0\times T$ upon measuring the phase vector.

Proof. Let us analyze the identifiability of the system under consideration, taken α=1, $z=(x,f(x,u))$. Then matrices J, J_x, J_u have the form

$$J=\begin{pmatrix} E_n & 0 \\ \partial f/\partial x & \partial f/\partial u \end{pmatrix},\ J_x=\begin{pmatrix} E_n \\ \partial f/\partial x \end{pmatrix},\ J_u=\begin{pmatrix} E_n \\ \partial f/\partial u \end{pmatrix}$$

Considering system (1) on λ solutions, we can conclude that for extended matrices J_λ, $J_{\lambda x}$, $J_{\lambda u}$ we have rankJ_λ=rank$J_{\lambda x}$+rank$J_{\lambda u}$. Hence sufficient conditions of identifiability are rank$J_{\lambda u}=m$. Evidently rank of this matrix coincides with rank of matrix in the conditions of Lemma 2, and this proves the theorem.

EXAMPLE 3.
Consider a one-dimensional system containing k+1 unknown control functions

$$\dot{x}=u_0+u_1x+u_2x^2+...+u_kx^k,\quad y=x$$

Matrix $J_u(x)$ has the form $J_u(x)=(1,x,x^2,..,x^k)$; the maximum number of solutions required $\lambda=k+2$-rank$J_u=k+1$. We set up matrix $J_{k+1\ u}=(x_1,x_2,..,x_{k+1},u)$ and determine its rank:

$$J_{k+1u}(x_1,x_2,...,x_{k+1},u)=\begin{pmatrix} 1 & x_1 & x_1{}^2 & \cdots & x_1{}^k \\ 1 & x_2 & x_2{}^2 & \cdots & x_1{}^k \\ . & . & . & \cdots & . \\ 1 & x_{k+1} & x_{k+1}{}^2 & \cdots & x_{k+1}{}^2 \end{pmatrix}$$

Matrix $J_{k+1\ u}$ is a square matrix and since its determinant is a Vandermonde determinant, its rank on k+1 different solutions x_1, x_2,.., x_{k+1} is equal to k+1. Thus, the system under consideration cannot be represented by a smaller number of control functions, consequently it is locally identifiable on k+1 trajectories.

7. DETERMINATION OF MOMENTS OF INERTIA

Rotational motion of a free rigid body is described by the Euler equations:

$$A_1\dot{\omega}_1=(A_2-A_3)\omega_2\omega_3,\quad A_2\dot{\omega}_2=(A_3-A_1)\omega_3\omega_1,\ A_3\dot{\omega}_3=(A_1-A_2)\omega_1\omega_2 \tag{12}$$

Instead of moments of inertia A_1, A_2, A_3 we take the dimensionless parameters

$$a_1=\frac{A_2-A_3}{A_1},\quad a_2=\frac{A_3-A_1}{A_2},\quad a_3=\frac{A_1-A_2}{A_3}$$

and rewrite (12) as follows:

$$\dot{\omega}_1=a_1\omega_2\omega_3,\quad \dot{\omega}_2=a_2\omega_3\omega_1,\ \dot{\omega}_3=a_3\omega_1\omega_2 \tag{13}$$

Let us consider the problem of determing the constant parameters a_1, a_2, a_3 on the basis of measurements of the one projection of the angular velocity onto one of a principal axis $y=\omega_1$. We write the extended output (4) (allowing for the fact that $\dot{a}_i=0$, i=1,2,3)

$$y=\omega_1,\quad \dot{y}=a_1\omega_1\omega_3,\quad \ddot{y}=a_1\omega_1(a_2\omega_3{}^2+a_3\omega_2{}^2),$$

$$y^{(3)} = a_1{}^2(a_2\omega_3{}^2 + a_3\omega_2{}^2)\omega_2\omega_3 4a_1a_2a_3\omega_1{}^2\omega_2\omega_3 .$$

The last derivative can be represented as $y^{(3)} = \dot{y}\ddot{y}/y + 4a_2a_3y^2\dot{y}$, and we have

$$y^{(i)} = h_i(y,\dot{y},\ddot{y},a_2a_3) \quad i = 3,4 \tag{14}$$

Therefore, for any $z_\alpha = (y,\dot{y},...,y^\alpha)$, $\alpha \geq 3 \operatorname{rank} \partial z_\alpha / \partial(\omega,a) = 4$, $\operatorname{rank} \partial z_\alpha / \partial\omega = 3$. The conditions of the theorem 4 are not valid. All the information regarding the motion of the systems is defined by the components of vector z_α relating the unknowns ω, a. This equations cannot be solved in such way that a_1, a_2, a_3 are the functions of z_α, and we may hot assert that given output one-to-one associate with some a.

At the same time the system considered is frequently encountered as an object of validation of particular methods of identification of nonlinear systems. We apply theorem 5 to our system. As vector z we take the $z_\alpha = (y,\dot{y},y^{(3)})$ and calculate $\operatorname{rank} \partial(\dot{y},\ddot{y},y^{(3)})/\partial(a_1,a_2,a_3) = 3$ since

$$\det \partial(\dot{y},\ddot{y},y^{(3)})/\partial a_1a_2a_3) = 4a_1{}^2\omega_1{}^3\omega_2{}^2\omega_3{}^2(a_2\omega_3{}^2 - a_3\omega_2{}^2) \tag{15}$$

On the basis of theorem 6, the system considered is invertible in the neighborhood of points where expression (15) is nonzero, and correspondence between parameters a_1, a_2, a_3 and output ω_1 is one-to-one in this case. We may conclude that for determination of inertia moments on the basis of measurements of the projection of the angular velocity onto one of a principal axis the method, utilizing informationon about initial state is required.

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INVERZNI PROBLEM NELINEARNIH UPRAVLJAČKIH SISTEMA

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U radu se proučavaju inverzni problemi za nelinearne upravljačke sisteme, opisane običnim diferencijalnim jednačinama po vrednostima izlaza razmatranim na jednoj ili više trajektorija. Dokazani su kriterijumi opservabilnosti, invertibilnosti i mogućnosti identifikacije sa korišćenjem proširenog izlaza. Na osnovu modifikovane teoreme implicitnih funkcija dobijeni su dovoljni uslovi u obliku ranga Jakobijevih matrica što dozvoljava efikasnu proveru. Predloženi metod skupa trajektorija proširuje klase sistema za koje je moguće naći nepoznate njihovih matematičkih modela i predstavlja osnovu novih računskih algoritama. Pokazano je da se problem odredjivanja ulaza po poznatim veličinama stanja na nekoliko trajektorija, uvek može redukovati (barem lokalno) na algebarske relacije i ima rešenje za svaki sistem. Proučene su i mogućnosti odredjivanja momenata inercije krutog tela na osnovu merenja projekcije ugaone brzine na glavnu osu.