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LAGRANGIAN EQUATIONS FOR THE MULTIBODIES HEREDITARY SYSTEMS [1]

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Abstract. *This paper is concerned with the multibodies systems in which the interaction between the bodies are described by the standard or weakly singular hereditary models. Starting from the general dynamic equation the Lagrangian equations supplemented by the generalized relaxation are constructed. A class of dynamically determinate systems is separated for which the three forms of motion equations are presented: differential Lagrangian equations of the 3rd or higher order and two integro-differential forms of equations with rheological and relaxational kernels. The typical examples are given.*

1. INTRODUCTION

The mechanics of continua with the hereditary properties obtained a wide advancement due to using of new structural materials having well-defined rheological properties in the design practice. At the present time the basics of this mechanics are widely using in the engineering calculations. Varied constructive methods using exact Galerkin-type methods for a reduction of the systems with an infinite number of freedom degrees have been developed for an investigations in the dynamics of multibodies discrete hereditary systems and for going from a complex medium to the discrete models. In the present work the methods of the analytical dynamics are developing for the discrete mechanical systems with the hereditary interactions between their particles. Lagrangian equations forms are determined for these systems.

2. PRELIMINARY FINDINGS OF THE MECHANICS OF HEREDITARY MATERIALS

In the current literature the terms “viscoelasticity” and “hereditary elasticity” are equivalent. “The hereditary theory of elasticity” is the term coined by V. Volterra. Yu. N.

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Rabotnov [1] observed that this term defines elastic after-actions very exactly. The model of a standard rheological body (Kelvin body) and a weakly singular model of hereditary body obtained the widest application for a describing of the hereditary phenomena.

3. Model of standard hereditary body

Uniaxial stress state of rheological body is described by a differential relation

$$n\dot{\sigma} + \sigma = nE\dot{\varepsilon} + \tilde{E}\varepsilon \tag{1}$$

where E and $\tilde{E}$ are an instantaneous elastic modulus and prolonged one, n is a time of relaxation.

In the mechanics of hereditary discrete systems a force of interaction between the particles of these systems by analogy with Eq. (1) is described by equation

$$n\dot{P} + P = nc\dot{y} + \tilde{c}y \tag{2}$$

where c and $\tilde{c}$ are an instantaneous rigidity and a prolonged one of an element, realized the interaction; y is a deformation of this element.
Solving the equation (2) for the force and the elongation two integral equations are obtained

$$P(t) = c\left(y - \int_0^t R(t-\tau)y(\tau)d\tau \right) \tag{3}$$

$$y(t) = \frac{1}{c}\left(P + \int_0^t K(t-\tau)P(\tau)d\tau \right) \tag{4}$$

Here

$$R(t-\tau) = \frac{c-\tilde{c}}{nc}\exp\left(-\frac{t-\tau}{n} \right) \text{ — is a kernel of relaxation,}$$

$$K(t-\tau) = \frac{c-\tilde{c}}{nc}\exp\left(-\frac{(t-\tau)\tilde{c}}{nc} \right) \text{ — is a kernel of rheology.}$$

The Eq. (3) under the fixing elongation $y=y_0$ describes a decreasing (relaxation) of a tension with a time in the hereditary element; the Eq. (4) describes the elongation of the rheological element. If at an initial moment of motion of the system the rheological and the relaxing processes caused by the preliminary history of the loading took place then the expressions in Eq. (3) and (4) are integrated over τ from ∞ to t.

Thus the standard model of hereditary interactions can be described in three ways, namely, the differential equation of state (2), the integral equation of stress relaxation (3) or the integral equation of rheology (4). In this case the kernels of the last equations are expressed by the exponential functions.

4. WEAKLY-SINGULAR MODEL OF HEREDITARY BODY

A further development of the standard rheological body yields the weakly-singular model in which the weakly-singular rational-exponential functions

$$R(t-\tau)=\frac{Ae^{-\beta(t-\tau)}V(t-\tau)}{(t-\tau)^{\alpha}},\quad K(t-\tau)=\frac{A_1e^{-\beta_1(t-\tau)}}{(t-\tau)^{\alpha}}U(t-\tau)\,. \tag{5}$$

are chosen as the kernels of relaxation and rheology. Here $U(t\text{-}\tau)$ and $V(t\text{-}\tau)$ are some polynomials or the series positive power in $(t\text{-}\tau)$; β and β_1 are the coefficients of relaxation and rheology. By application of weakly-singular integral equations the deformations are more appropriately determined at the beginning period of the loading. The more complicated weakly-singular kernels are used also. Yu. N. Rabotnov [1] proposed a special function

$$\Xi(-\beta,t)=t^{-\alpha}\sum_{n=1}^{\infty}\frac{(-\beta)^n t^{n(1-\alpha)}}{\Gamma((n+1)(1-\alpha))} \tag{6}$$

where $\Gamma(z)$ is Euler function. The function (6) generates a wide class of exponential functions described the properties of the hereditary bodies.
However the weakly-singular model allows a describing of the hereditary processes only in the form of integral Eq. (3) and (4).
To take some examples of the weakly-singular kernels. A.R. Rzhanitsin [2] proposed the three-parameters kernel of rheology:

$$K(t-\tau)=\frac{a}{(t-\tau)^{\alpha}}e^{-\beta(t-\tau)},\ \ \alpha<1\,. \tag{7}$$

The four-parameters kernel was adopted by M. A. Koltunov [3]:

$$K(t-\tau)=\frac{a}{(t-\tau)^{1-\beta}}\exp(-p(t-\tau)^{\alpha}),\ \ \alpha<1,\ \ \beta<1\,. \tag{8}$$

Analogous kernel for a representation of the properties of a highly elastic rubber was proposed by G.Ya. Slonimsky [4]:

$$K(t-\tau)=\frac{a}{(t-\tau)^{\alpha}}\exp(-(t-\tau)^{1-\alpha}) \tag{9}$$

etc.

5. THE FORMS OF THE MOTION EQUATIONS OF A RHEOLOGICAL OSCILLATOR

Let us consider the scheme of the simple rheological oscillator which consistent of a mass m rested on the rheological element P and is subjected to an active force $F(t)$. A scheme of this oscillator is shown in Fig.1. Its motion equation with respect to d'Alembert's principle has the next form

$$m\ddot{y}+P=F(t) \tag{10}$$

where $P(t)$ is a reaction of the rheological element determined by Eq. (3) or (4). Eliminated reaction $P(t)$ with the use of Eq. (3) we obtain the integro-differential equation of oscillator motion with the relaxation kernel in the following form

$$m\ddot{y}+c\left(y-\int_0^t R(t-\tau)y(\tau)d\tau\right)=F(t)\ . \tag{11}$$

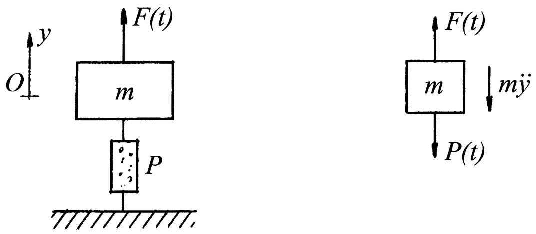

Fig.1
a) a simple oscillator with hereditary element P
b) a scheme of the forces acting on the oscillator mass.

Excluding reaction *P* from Eq. (10) with the help of Eq. (4) we obtain the integro-differential equation of motion with the kernel of rheology

$$m\ddot{y}+c\left(y+\int_0^t K(t-\tau)m\ddot{y}d\tau\right)=F(t)+c\int_0^t K(t-\tau)F(\tau)d\tau\ . \tag{12}$$

The both forms of Eq. (11) and (12) were used by A.R. Rjanitsin [7] in the research of vibration of the beams and the simple oscillators.

In the case when the properties of hereditary element are describing by the model of standard rheological body, the kernels of Eq. (3) and (4) are the exponential functions. By differentiating Eq. (11) and (12) and adding the derived expressions to the initial ones both equations have been brought to the pure differential equations without integral members:

$$nm\dddot{y}+m\ddot{y}+nc\dot{y}+\tilde{c}y=F(t)+n\dot{F}(t)\ . \tag{13}$$

This equation can be obtained by excluding reaction *P* from Eq. (10) with the help of Eq. (2).

In using the weakly-singular model of hereditary body in order to describe the motion of oscillator Eq. (11) and (12) does not recast in the pure differential form. In this case the motion is described only in two integro-differential forms.

Thus, the description of motion of the standard hereditary oscillator can be obtained by three ways: in the form of differential equation and two forms of in integro-differential equations. The description of motion of the weakly-singular oscillator can be produced in two integro-differential forms only.

6. LAGRANGIAN EQUATIONS FOR THE DISCRETE HEREDITARY SYSTEMS. THE GENERAL CASE

Ensure that in the general case for the description of the motion of hereditary systems it can be derived Lagrangian equations in the integro-differential form with the kernels of relaxation.

The general equation of the dynamics of hereditary system contained N material points is of the following form

$$\sum_{i=1}^{3N}\left(m_i\ddot{x}_i - X_i(t) + \sum_{k=1}^{K} P_k e_{ik}\right)\delta x_i = 0\ . \tag{14}$$

where $e_{ik}= e_{ik}(x_1,...,x_{3N})$ — are direction cosines of the reactions P_k, $y_k(x_1,...,x_{3N})$ — are the deformations of rheological elements. The reactions of hereditary elements are defined by the relaxation equations

$$P_k = c_k\left(y_k - \int_0^t R_k(t-\tau)y_k(\tau)d\tau\right),\quad (k=1,2,...,K) \tag{15}$$

A quantity of rheological elements in the system is arbitrary.

Let us the motion of the particles of system are bounded by s=3N-n holonomic constrains. Then the motion of system are defined by n generalized coordinates $q_1(t)$, $q_1(t)$,...,$q_n(t)$, and Cartesian coordinates are represented in the functions of the generalized coordinates $x_i= x_i(q_1, q_1,..., q_n)$. Passing to the generalized coordinates eliminating the reactions P_k from Eq. (14) with the help of Eq. (15) we obtain

$$\sum_{j=1}^{n}\sum_{i=1}^{3N}\left[m_i\ddot{x}_i - X_i(t) + \sum_{k=1}^{K} c_k e_{ik}\left(y_k - \int_0^t R_k(t-\tau)y_k(\tau)d\tau\right)\right]\frac{\partial x_i}{\partial q_j}\delta q_j = 0\ . \tag{16}$$

Using Lagrangian identities

$$\frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\ ,\qquad \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

in accordance with a procedure assumed in the analytical mechanics [6], Eq. (16) are transformed to the following form:

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial \dot{q}_j} - Q_j + \frac{\partial \Pi}{\partial \dot{q}_j} - \tilde{Q}_j\right]\delta q_j = 0\ , \tag{17}$$

where T and Q_j — are the kinetic energy and the generalized forces,

$$\Pi = \sum_{j=1}^{n}\sum_{k=1}^{3N} c_k\int_0^{x_i} e_{ik}y_k dx_i\ . \tag{18}$$

is a potential energy of hereditary elements derived with respect to instantaneous rigidities c_k. The expressions

$$\tilde{Q}_j = \sum_{k=1}^{K} b_{kj}(q_j) c_k \int_0^t R_k(t-\tau) y_k(q_j(\tau)) d\tau \, . \tag{19}$$

will be named the generalized relaxation of the reactions of rheological elements. Here b_{kj} are determined as follows

$$b_{kj}(q) = \sum_{i=1}^{3N} e_{ik} \frac{\partial x_i}{\partial q_j} \, . \tag{20}$$

On account of the independence of the variations of generalized coordinates δq_j from the general equation of dynamics (17) we obtain Lagrangian equations of the second type for hereditary discrete system in the integro-differential form

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial \Pi}{\partial q_j} + Q_j + \tilde{Q}_j \quad . \quad (j = 1,2,..,n) \, . \tag{21}$$

If the forces X_i are potential then it is built up the general expression of the potential energy including potential energy of hereditary elements and potential energy of the forces X_i. By analogy with Eq. (18) it is appropriate to introduce a function of the reactions relaxation of the rheological elements $\tilde{\Pi}$ by the formula

$$\tilde{\Pi} = \sum_{i=1}^{3N} \int_0^{x_i} \sum_{k=1}^{K} c_k \int_0^t R_k(t-\tau) y_k(\tau) d\tau \, e_k(x_i) dx_i \, . \tag{22}$$

In this case the generalized relaxation of reactions (19) are calculated with the formulae

$$\tilde{Q}_j = \frac{\partial \tilde{\Pi}}{\partial q_j} \quad (j = 1,2,...,n) \, .$$

The generalized forces and relaxation can be defined with the help of the expression of a virtual work of the active forces and relaxation also (see below Expample 3).

Integro-differential equations in relaxation (21) are universal.

It is easy to make sure that in general case the construction of Lagrangian equations for the hereditary systems in the rheological form by analogy with Eq. (12) presents no possibilities. Excluding the reactions P_k from the general equation of dynamics (14) with the help of state equations in the rheological form

$$y_k = \frac{1}{c_k} \left(P_k + \int_0^t K_k(t-\tau) P_k(\tau) d\tau \right), \tag{23}$$

we obtained the next equation

$$\sum_{i=1}^{3N} \left[m_i \ddot{x}_i - X_i(t) + \sum_{k=1}^{K} c_k e_{ik} \left(y_k - \int_0^t K_k(t-\tau) P_k(\tau) d\tau \right) \right] \delta x_i = 0 \, . \tag{24}$$

The further excluding of P_k in the general case is impossible.

In the same manner we can ensure that in the general case it is impossible to construct Lagrangian equations for the standard rheological systems in the pure differential form using rheological relations as (2).

7. DYNAMICALLY DETERMINATE HEREDITARY DISCRETE SYSTEMS.

Out of all the set of hereditary systems let us set off the systems allowing the construction of Lagrangian equations in the pure differential form and in two integral forms. This systems will be named *dynamically determinate*. This class of systems is important in solving the applied problems.

Writing down the general equation of dynamics in the form

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+\sum_{k=1}^{K}b_{kj}P_k\right]\delta q_j=0\ , \tag{25}$$

where the coefficients $b_{kj}(q)$ are determined by formula (20).

By independence of the variations δq_j out of Eq. (25) we obtain n equations of Lagrange contained the reactions of rheological interactions

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+\sum_{k=1}^{K}b_{kj}P_k\delta q_j=0\,. \quad (j=1,2,..,n), \tag{26}$$

Definition: Hereditary discrete system is termed dynamically determinate when the quantity of the rheological elements k involved in this system is equal or less then the number of its degrees of freedom m ($k \le n$) and for which it is possible to separate system of equations of order $k \le n$ from Lagrangian Eq. (26)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+\sum_{k=1}^{K}b_{kj}P_k\delta q_j=0\,. \quad (j=1,2,..,k)\,, \tag{27}$$

that the determinant forming by the elements b_{kj} is not equal zero:

$$\left|b_{kj}(q)\right|=\left|\sum_{i=1}^{3N}e(q)\frac{\partial x_i}{\partial q_j}\right|\neq 0\,. \tag{28}$$

In view of Eq.(28) the subsystem (27) is solvable for the reactions P_k (k=1,2,...,K) for which the next relations are obtained

$$P_k=-\sum_{\nu}^{K}a_{k\nu}(q)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right). \tag{29}$$

The reactions of interactions in the standard hereditary system are determined by the following equations

$$n_k\dot{P}_k+P_k=n_kc_k\dot{y}_k(q)+\tilde{c}_ky_k \tag{30}$$

Excluding the reactions P_k from Eq. (27) and (30) we obtain k differential equations of the third order

$$\left(1+n_k\frac{d}{dt}\right)\sum_{\nu=1}^{K}a_{k\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right)+nc_k\dot{y}_k(q)+\tilde{c}_ky_k(q)=0\,. \quad (k=1,2,...,K)\,. \tag{31}$$

The other *n-k* equations of the system (26) are present with the help of Eq. (29) in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right)-\frac{\partial T}{\partial q_j}-Q_j-\sum_{k=k+1}^{n} b_{kj}\sum_{\nu=1}^{K} a_{k\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right)=0\,.\ \ (j=k+1,\dots,n)\,. \tag{32}$$

As standard both weakly singular hereditary dynamically determinate systems allows the construction of two integro-differential forms of the motion equations, namely: Excluding the reactions P_k from Eq. (26) we obtain integro-differential equations contained the kernels of relaxation

$$\sum_{\nu=1}^{K} a_{k\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right)+c_k\left(y_k(q)-\int_0^t P_k(t-\tau)y_k(q(\tau))d\tau\right)=0\,.\ \ (k=1,2,\dots,K)\,. \tag{33}$$

The remaining equations of this system have the form (32).

Excluding the reactions P_k from Eq. (23) with the help of (29) we obtain the integro-differential equations with the kernels of relaxation in the form

$$\begin{aligned}&\sum_{\nu=1}^{K} a_{k\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right)+\\&+c_k\left(y_k(q)-\int_0^t K_k(t-\tau)\sum_{\nu=1}^{K} a_{k\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\nu}-\frac{\partial T}{\partial q_\nu}-Q_\nu\right)d\tau\right)=0\,.\ \ (k=1,2,\dots,K)\end{aligned}\,. \tag{34}$$

The other *n-K* equations of system are written in the form (32).

So that by analogy with (11), (12) and (13) the motion of the dynamically determinate systems can be described by three forms of the modified Lagrangian equations (31), (33) and (33).

8. The simple dynamically determinate systems

When the structure of these systems allows to select the deformations of hereditary elements as the generalized coordinates

$$q_j=y_j(x_1,\dots,x_{3N},t)\,,\ \ (j=1,2,\dots,n) \tag{35}$$

the forms of Lagrangian equations are simplified. In this case the rheological Eq. (2), (3), (4) for these systems are represented as

$$\begin{aligned}&n_j\dot P_j+P_j=n_jc_j\dot q_j+\tilde c_j q_j\ \ ,\\&P_j=c_j\left(q_j-\int_0^t R_j(t-\tau)q_j(\tau)d\tau\right),\\&q_j=\frac{1}{c_j}\left(P_j+\int_0^t K_j(t-\tau)P_j(\tau)d\tau\right).\end{aligned} \tag{36}$$

The general equation of dynamics in the generalized coordinates are writing in the form

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+P_j\right]\delta q_j=0\,. \tag{37}$$

Respective Lagrangian equations, contained the reactions P_j have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+P_j=0\,. \quad (j=1,2,...,n) \tag{38}$$

Excluding the reactions of hereditary elements from (38) and the first equation (36) we obtain the system of differential Lagrangian equations in the form

$$\left(n_j\frac{d}{dt}+1\right)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j\right)+n_j c_j \dot{q}_j+\tilde{c}_j q_j=0\,. \quad (j=1,2,...,n) \tag{39}$$

This is a system of the 3rd order. If a number of the hereditary elements is less then the number of the degrees of freedom ($k < n$), just then some equations of the system (39) will have a classical form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j\,, \quad (j=K+1,...,n) \tag{40}$$

Excluding the reactions P_j from Eq. (38) by the relaxation equations (36) we get integro-differential Lagrangian equations with the kernels of relaxation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-c_j\int_0^t R_j(t-\tau)q_j(\tau)d\tau+c_j q_j=Q_j \quad (j=1,2,...,n) \tag{41}$$

And, finally, eliminating the reactions of constraints from Eq. (38) and rheological Eq. (36) we get the system of integro-differential Lagrangian equations in rheological form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j+\int_0^t K_j(t-\tau)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j\right)d\tau+c_j q_j=Q_j\,. \quad (j=1,2,...,n) \tag{42}$$

Thus the dynamically determinate hereditary systems allows the construction of Lagrangian equations in the pure differential form for standard systems and in two form of integro-differential equations as for weakly-singular both for standard hereditary systems.

9. THE EXAMPLES OF THE CONSTRUCTION OF MOTION EQUATIONS FOR HEREDITARY SYSTEMS

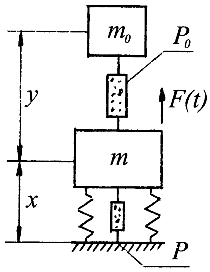

Fig.2. Vibroprotectional system with dynamic damper.

Example 1. Vibroprotectional system with dynamic damper. The vibroprotection of an object m is realized by a damper with a rigidity c and the rheological energyabsorber P (See Fig.2). The dynamic damper is represented by mass m_0 with the rheological element P_0. This is the dynamically determinate rheological system. The absolute displacement x of mass m and the relative displacement y of mass m_0 can be chosen as the generalized coordinates. The deformations of rheological elements P and P_0 are determined by these displacements.

The hereditary elements of this system are standard and their state is described by the equations

$$n\dot{P} + P = nc\dot{x} + \tilde{c}x \ , \qquad n_0\dot{P}_0 + P = n_0c_0\dot{y} + \tilde{c}_0 y \tag{43}$$

The kinetic and potential energies are of the following form

$$T = \frac{1}{2}(m\dot{x}^2 + m_0(\dot{x}+\dot{y})^2), \quad \Pi = \frac{1}{2}Cx^2 + mgx + m_0g(x+y). \tag{44}$$

The generalized active forces are

$$Q_x = F(t), \quad Q_y = 0. \tag{45}$$

The equations of motion according to the formulae (39) have the next form:

$$\left(1+n\frac{d}{dt}\right)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} - Q_x + \frac{\partial \Pi}{\partial x}\right) + nc\dot{x} + \tilde{c}x = 0 \ , \qquad \left(1+n_0\frac{d}{dt}\right)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} - Q_y + \frac{\partial \Pi}{\partial y}\right) + n_0c_0\dot{y} + \tilde{c}_0 y = 0 \ , \tag{46}$$

Thus the motion of the rheological vibroprotectional system with dynamic damper is described by two equations of the third order.

If the weakly-singular model is used for a description of the motion of system shown in Fig. 2, then the reactions of the rheological elements are determined by equations

$$P = c\left(x - \int_0^t R(t-\tau)x(\tau)d\tau \right),$$
$$P_0 = c_0\left(y - \int_0^t R_0(t-\tau)y(\tau)d\tau \right), \tag{47}$$

where $R(t\text{-}\tau)$ and $R_0(t\text{-}\tau)$ are weakly-singular kernels of relaxation and in respective with Eq. (41) equations of motion are represented as

$$m\ddot{x} + m_0(\ddot{x}+\ddot{y}) - F(t) + (C+c)x - c\int_0^t R(t-\tau)x(\tau)d\tau = \text{-}(m_0+m)g\,,$$
$$m_0(\ddot{x}+\ddot{y}) - c_0\int_0^t R_0(t-\tau)y(\tau)d\tau = \text{-}\,m_0 g\,. \tag{48}$$

If the state of hereditary elements are described by the rheological equations with weakly-singular or standard kernels in the form

$$x = \frac{1}{c}\left(P + \int_0^t K(t-\tau)P(\tau)d\tau \right),$$
$$y = \frac{1}{c_0}\left(\int_0^t K_0(t-\tau)P(\tau)d\tau \right), \tag{49}$$

then depending on the formulae (42) the integro-differential equations in rheological form are written as follows

$$m\ddot{x} + m_0(\ddot{x}+\ddot{y}) + (C+c)x +$$
$$+\int_0^t K(t-\tau)\left[m\ddot{x} + m_0(\ddot{x}+\ddot{y}) + (m+m_0)g - F(\tau)\right]d\tau = F(t) - (m+m_0)g\,, \tag{50}$$
$$m_0(\ddot{x}+\ddot{y}) + c_0 y + \int_0^t K_0(t-\tau)[m_0(\ddot{x}+\ddot{y}) + m_0 g]d\tau = -m_0 g\,.$$

A choosing of one in three presented above forms of Eq. (46), (48), (50) for investigation of the dynamic properties of vibroprotection is controlled by the type of initial information about the properties of hereditary elements.

Example 2. The construction of the equations of motion of a rheological pendulum. (Fig. 3).

Standard rheological model. The hereditary properties of pendulum thread are defined by equation

$$n\dot{P} + P = nc\dot{\rho} + \tilde{c}\rho\,. \tag{51}$$

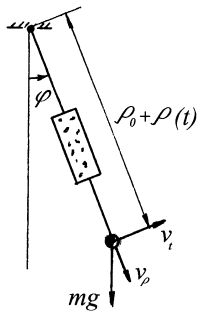

Fig.3. The rheological pendulum.

When an angle of displacement $\varphi(t)$ and elongation $\rho(t)$ are chosen as the generalized coordinates then this system is simple dynamically determinate by definition. Kinetic and potential energies of pendulum are written as

$$T = \frac{1}{2} m(\upsilon_t^2 + \upsilon_\rho^2) = \frac{m}{2}((\rho_0 + \rho)^2 \dot{\varphi}^2 + \dot{\rho}^2), \quad Q_\rho = mg\cos\varphi, \; Q_\varphi = -mg(\rho_0 + \rho)\sin\varphi. \tag{52}$$

In accordance with (39) the equations of motion of rheological pendulum have the form

$$m\left(1 + n\frac{d}{dt}\right)[\ddot{\rho} - (\rho_0 + \rho)\dot{\varphi}^2 + mg\cos\varphi] + nc\dot{\rho} + \tilde{c}\rho = 0, \quad m[(\rho_0 + \rho)^2\ddot{\varphi} + 2(\rho + \rho_0)\dot{\rho}\dot{\varphi}] + mg(\rho_0 + \rho)\sin\varphi = 0. \tag{53}$$

Weakly-singular model. The properties of hereditary pendulum thread by equations

$$P = c\left(\rho - \int_0^t R(t-\tau)\rho(\tau)d\tau\right), \quad \rho = \frac{1}{c}\left(P + \int_0^t K(t-\tau)P(\tau)d\tau\right). \tag{54}$$

Corresponding equations in the terms of relaxation (41) have the next form

$$m[\ddot{\rho} - (\rho_0 + \rho)\dot{\varphi}^2] + c\rho = mg\cos\varphi + c\left(\int_0^t R(t-\tau)\rho(\tau)d\tau\right), \quad m[(\rho_0 + \rho)^2\ddot{\varphi} + 2(\rho + \rho_0)\dot{\rho}\dot{\varphi}] + mg(\rho_0 + \rho)\sin\varphi = 0, \tag{55}$$

The rheological form is obtained if the first equation (55) is replaced by the following one

$$m[\ddot{\rho} - (\rho_0 + \rho)\dot{\varphi}^2] + c\rho + \int_0^t K(t-\tau)\{m[\ddot{\rho} - (\rho_0 + \rho)\dot{\varphi}^2] - mg\cos\varphi\}d\tau = mg\cos\varphi. \tag{56}$$

Example 3. Dynamically indeterminate hereditary system. This system is pictured in Fig. 4. The quantity of its hereditary elements is more then the number of the degrees of freedom. The properties of these elements are defined by the equations in the terms of relaxation

$$P_1 = c_1\left(y_1 - \int_0^t R_1(t-\tau)y_1(\tau)d\tau \right)$$
$$P_2 = c_2\left(y_2 - \int_0^t R_2(t-\tau)y_2(\tau)d\tau \right) \tag{57}$$
$$P_3 = c_3\left(y_1 + y_2 - \int_0^t R_3(t-\tau)(y_1+y_2)d\tau \right)$$

Kinetic and potential energies with respect to the formulae (18)-(22) are described by the expressions

$$T = \frac{1}{2}(m_1\dot{y}_1^2 + m_2(\dot{y}_1+\dot{y}_2)^2)$$
$$\Pi = \frac{1}{2}(c_1y_1^2 + c_2y_2^2 + c_3(y_1+y_2)^2 + c(y_1+y_2)^2)\,. \tag{58}$$

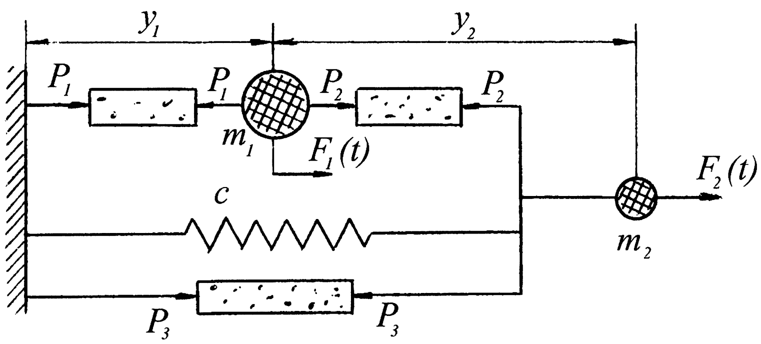

Fig.4. Dynamically indeterminate hereditary system.

For definition of the generalized forces we invoke the expression of virtual work of active forces and relaxation

$$\delta A = F_1\delta y_1 + F_2(\delta y_1 + \delta y_2) + \left(c_1\int_0^t R_1y_1d\tau - c_2\int_0^t R_2y_2d\tau \right)\delta y_1 -$$
$$-\left(c_2\int_0^t R_2y_2d\tau - c_3\int_0^t R_3(y_1+y_2)d\tau \right)(\delta y_1 + \delta y_2)\ . \tag{59}$$

From this expression follows that

$$Q_1+\tilde{Q}_1=F_1+F_2+c_1\int_0^t R_1(t-\tau)y_1(\tau)d\tau+c_3\int_0^t R_3(t-\tau)(y_1+y_2)d\tau$$
$$Q_2+\tilde{Q}_2=F_2+c_2\int_0^t R_2(t-\tau)y_2(\tau)d\tau+c_3\int_0^t R_3(t-\tau)(y_1+y_2)d\tau\,. \quad (60)$$

Then the equations of motion will look like

$$m_1\ddot{y}_1+m_2(\ddot{y}_1+\ddot{y}_2)+c_1y_1+(c+c_3)(y_1+y_2)=Q_1+\tilde{Q}_1$$
$$m_2(\ddot{y}_1+\ddot{y}_2)+c_2y_2+(c+c_3)(y_1+y_2)=Q_2+\tilde{Q}_2\,. \quad (61)$$

It is a pity that the limited size of this article does not allow to illustrate completely the technique of application of all variety of motion equations forms of discrete hereditary systems.

REFERENCES

1. Rabotnov Yu.N., (1977), Elements of Hereditary Mechanics of Solids, Moscow, Nauka. (in Russian).
2. Rzhanitsin A.R., (1949), Some Questions of the Mechanics of Deforming in Time Systems, Moscow, GTTI (in Russian)
3. Koltunov M.A., (1968), To the question of the choosing of kernels in solution of problems accounting creep and relaxation, Mechanics of Polymers, No 4 (in Russian).
4. Slonimsky G.L., (1961), On the law of deforming of high elastic polymer bodies, Rep. Acad. Sci. USSR, Vol. 140, No 2 (in Russian).
5. Savin G.N., Ruschitsky Ya. Ya., (1976), Elements of Hereditary Media Mechanics, Kyiv, Vyscha shkola (in Ukrainian).
6. Pars L.A., (1964), A Treatise on Analitical Dynamics, Heinemann, London.

LAGRANGE-OVE JEDNAČINE ZA NASLEDNE SISTEME SA VIŠE TELA

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Ovaj rad se bavi sistemima više tala kod kojih je uzajamno dejstvo između tela opisano običnim (standardnim) ili slabim singularnim naslednim modelima. Počevši od opšte dinamičke jednačine sastavljene su Lagrange-ove jednačine dopunjene generalizovanom (uopštenom) relaksacijom. Izdvojena je klasa dinamički određenih sistema za koje su prikazane jednačine tri oblika kretanja: diferencijalne Lagrange-ove jednačine trećeg ili višeg reda i dve integro-diferencijalne forme jednačina sa reološkim i relaksacionim jezgrima. Dati su tipični (odgovarajući) primeri.