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DYNAMIC BUCKLING OF IMPERFECTION SENSITIVE NONCONSERVATIVE DISSIPATIVE SYSTEMS UNDER FOLLOWER LOADING

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Abstract. *An analytic approach is presented for the nonlinear dynamic buckling of imperfection sensitive nonconservative discrete dissipative systems under partial follower loading in the domain of divergence. These systems under static loading lose their stability via a limit point. The analysis is confined to that region of divergence where the asymmetric stiffness matrices of perfect bifurcational systems are characterized by a full set of eigenvectors with corresponding postbuckling paths independent of each other. Thus, these systems behave dynamically like symmetric dissipative systems under conservative loading which exhibit either a point of attractor or dynamic buckling. The total potential energy criterion for dynamic buckling of conservative systems is no longer valid. Instead of this, an energy-balance equation is established that allows to determine approximate dynamic buckling loads, very good for structural design purposes, as well as lower/upper bound buckling estimates, which are readily obtained without solving the highly nonlinear initial-value problem. Comparisons of numerical results with those of other analyses obtained via numerical simulation show the reliability of the proposed approach.*

1. INTRODUCTION

The dynamic response of nonconservative (asymmetric) systems under partial follower loading according to *linear analyses* is similar to that of the associated conservative (symmetric) systems when the asymmetric stiffness matrices are symmetrizable [3]. However, this is not always true, since a limit cycle response of autonomous nonconservative (asymmetric) dissipative systems may occur in regions of existence of adjacent equilibria [4,5]. More specifically, it was shown that Ziegler's model under partial follower load exhibits in a certain region of divergence one postbuckling path passing through the 1st and 2nd branching points due to which a

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periodic attractor may occur, although its stiffness matrix is *symmetrizable* with *distinct eigenvalues* and a *complete set of eigenvectors.* A limit cycle response in a region of divergence was also found by Bolotin et al (1995) using a flat panel model. In a recent work [6] the conditions were established under which nonconservative (asymmetric) dissipative systems with symmetrizable asymmetric matrices and a full set of eigenvectors may exhibit a limit cycle response which is ruled out for symmetric systems (under conservative loading). It was also found that the above nonconservative systems behave dynamically like the associated conservative (symmetric) systems in a region of existence of adjacent equilibria where all postbuckling paths (corresponding to the full set of the stiffness matrix eigenvectors) are independent of each other. This region corresponds to a certain interval of values of the follower loading parameter which defines the degree of nonconservativeness.

This study deals with the nonlinear dynamic buckling of imperfections sensitive nonconservative dissipative systems under partial follower loading which lose their static stability via a limit point. These systems are generated from perfect bifurcational systems with trivial fundamental paths associated with asymmetric stiffness matrices with distinct eigenvalues and a full set of eigenvectors with corresponding postbuckling paths *independent* of each other. The dynamic behavior of the last systems in a certain domain of divergence is similar to that of conservative (symmetric) dissipative systems which exhibit either a point attractor or dynamic buckling (escaped motion). A major difficulty for extending the energy, geometrical and topological concepts associated with the dynamic buckling of conservative dissipative systems to the case of the above nonconservative systems is the lack of a potential loading. The *total potential energy criterion* [7] is no longer valid. However, this shortcoming is circumvented via the establishment of an energy-balance equation which, along with the equilibrium equations, permits to determine the saddle through which (or its neighborhood) dynamic buckling occurs. Then, approximate dynamic buckling loads for vanishing and non-zero damping, very accurate for structural design purposes, as well as lower/upper bound buckling estimates are readily established without integrating the systems of highly nonlinear equations of motion. Comparisons of a variety of existing results with those of the present work show the reliability of the proposed readily employed approach.

2. PROBLEM DESCRIPTION AND BASIC EQUATIONS

Consider a general n-degree-of-freedom nonlinear structural system, whose static response is governed by the set of nonlinear algebraic equations

$$V_i(q_i;\lambda;\eta) \qquad (i=1,2,\ldots,n) \tag{1}$$

where q_i are n generalized coordinates, λ a partial follower (non-potential) loading and η a loading parameter defining the degree of nonconservativeness of the system. For a certain value of η, say $\eta = \eta_c$, the external loading becomes conservative (potential loading). The validity of eqs (1) presupposes that the range of variation of η is such that adjacent equilibria exist. We postulate that $V_i(q_i;\lambda;\eta)$ are real analytic functions of q_i, η and λ, at least in the domain of interest. For a fixed value of η the sets of equilibrium states satisfying eqs (1) represent one-dimensional manifolds [in the (n+l)-dimensional load-displacement space spanned by λ and q_i], called equilibrium paths which are assumed single-valued (in the domain of interest). One can also assume that one of these paths, $q_i = q_i^F(\lambda)$, defines a fundamental equilibrium path which is initially stable; that is, all the Jacobian eigenvalues evaluated at any of its equilibrium states E have negative real parts. Then, using the implicit function theorem it is deduced that this path, $q_i = q_i^F(\lambda)$, is unique passing through point E which is stable in its neighborhood (local stability).

The components of the non-potential loading associated with eqs (1), $Q_i=Q_i(q_i;\lambda;\eta)$, are nonlinear functions of q_{i} and η, but linear functions of λ, i.e $Q_i = \lambda\overline{Q}_i(q_i;\eta)$. Denoting by $U = U(q_{\mathrm{i}})$ the positive definite function of the strain energy, eqs (1) can also be written as follows

$$V_i(q_i;\lambda;\eta)=U(q_i)\text{-}\lambda\overline{Q}_i(q_i;\eta) = 0 \qquad (i = 1,...,n) \tag{2}$$

where $U_i=\partial U/\partial q_i$. Thus, if $[V_{ij}]=[\partial Vi/\partial q_i]$ $(i,j=1,...,n)$ is the stiffness matrix, the buckling equation is given by

$$\det[V_{ij}] = 0 \tag{3}$$

from which one can obtain, at most, n distinct buckling loads. By virtue of eq. (2) one can also get

$$[V_{ij}] = [U_{ij}] - \lambda[\overline{Q}_{ij}] = 0 \quad (i, j = 1,...,n) \tag{4}$$

where $[U_{ij}] = [\partial^2 U/\partial q_i \partial q_j]$ and $[\overline{Q}_{ij}] = [\partial Q_i/\partial q_j]$ are square matrices of dimension n x n. Note that $[U_{ij}]$ is a symmetric (positive definite) matrix, while $[\overline{Q}_{ij}]$ and $[V_{ij}]$ are asymmetric matrices.

The boundary between divergence (static) and flutter (dynamic) instability for geometrically perfect nonconservative systems with trivial fundamental paths ($q_i^F = 0$ for all i) corresponds to a certain value of η, $\eta = \eta_0$, which can be readily established using a linear (local) analysis [9,5]. Hence, the region of existence of adjacent equilibria (region of divergence) according to linear analyses is defined by $\eta_c \geq \eta \geq \eta_0$; thus for $\eta < \eta_0$ adjacent equilibria do not exist. However, as was shown recently, in the vicinity of η_0 and on the side of existence of adjacent equilibria a limit cycle response may occur under certain conditions [4,5,6]. In these analyses it was found, with the aid of a cantilever model under a partial follower load, that the above phenomenon of existence of limit cycles may occur for $\eta_1 \geq \eta \geq \eta_0$, where η_1=0.50, although the asymmetric stiffness matrix $[V_{ij}]$ is symmetrizable associated with distinct eigenvalues and a complete set of corresponding eigenvectors [3]. From the last analyses of Kounadis and associates it was also established that the postbuckling paths corresponding to the full set of eigenvectors are not independent of each other in the above region. However, for $\eta_c \geq \eta \geq 50$ *all post-buckling paths* are *independent* of each other; a fact due to which a limit cycle response is ruled out.

Perfect bifurcational nonconservative systems with trivial fundamental paths under partial follower loading defined by $\eta_c \geq \eta \geq \eta_1$ =0.50 may be analysed using either a static or a dynamic analysis, since both methods yield the same critical (bifurcational) load, irrespective of the mass distribution and amount of damping. These systems have a dynamic behavior similar to that of the corresponding *conservative* (symmetic) systems.

In this work attention is focused on imperfection sensitive systems which are generated from the above bifurcational nonconservative systems (defined by $\eta_c \geq \eta \geq 0.50$). These systems lose their static stability via a limit point. From Fig.1 one can see a typical nonlinear equilibrium path in the load-displacement space. A lot of numerical results of various studies [8,4] have shown that the limit point load λ_S is always higher than the critical load obtained via a dynamic analysis (which is appreciably affected by the mass distribution and the amount of damping). The numerical results of the aforementioned studies will be verified by the qualitative dynamic approach that follows.

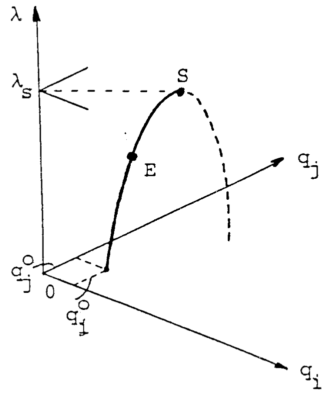

Fig.1. Typical nonlinear equilibrium path of an imperfection sensitive system under follower loading.

3.DYNAMIC ANALYSIS

The dynamic response of the above *n*-degree-of-freedom imperfection sensitive system with nonconservativeness parameter $\eta_c \geq \eta \geq 0.50$, after inclusion of small viscous damping forces, can be described by the following autonomous differential equations of motion in dimensionless form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} - Q_i = 0 \quad (i = 1,\ldots,n) \tag{5}$$

where the dots denote differentiation with respect to time, t; $K = (1/2)a_{ij}\dot{q}_i\dot{q}_j$ is the positive definite function of the total kinetic energy with diagonal elements being functions of masses [i.e. $a_{ii}=a_{ii}(m_i)$ i=1,...,n]; and non-diagonal elements that are functions of m_i and q_i [i.e. $a_{ij}=a_{ij}(m_i,q_i)$ for $i \neq j$, i,j=1,...,n]; $F = (1/2)c_{ij}\dot{q}_i\dot{q}_j$ is a (positive definite) dissipation function with coefficients c_{ij} which could be function of q_i [i.e. $c_{ij}=c_{ij}(q_i)$ with i,j=l,...,n]; $U=U(q_i)$ and $Q_i = \lambda\overline{Q}_i$ have been defined above.

The lack of potential for the above type of loading constitutes a serious difficulty for establishing a qualitative dynamic buckling analysis similar to that holding for conservative (potential) loads [7]. The total potential energy criterion for the dynamic buckling of conservative systems is no longer valid. The powerful energy criteria for the static stability (or instability) of equilibria which are associated with total potential energy functions do not exist. The asymptotically stable equilibria (attractors) on the fundamental path and the unstable equilibria (saddles) on the unstable post-buckling path are defined now on the basis of a local dynamic analysis associated with the nature of Jacobian eigenvalues. All eigenvalues of an asymptotically stable equilibrium are complex conjugate with negative real parts, while in case of a saddle at least one pair of complex-conjugate eigenvalues has positive real part. From the numerical results of the aforementioned studies it was detected that above a certain level of the loading λ (sufficiently less than λ_S) the corresponding equilibria up to the limit point S, although locally asymptotically stable, are globally unstable. Global stability (instability) is related to the boundedness (unboundedhess) of solutions of eqs (5). As was already defined [7] unboundedness of solution means dynamic buckling (escaped motion). Subsequently, we will try via a qualitative analysis [without actually solving the highly nonlinear systems of differential eqs (5)] to determine the corresponding to the lower load λ stable equilibrium $q^E(q_i^E,...,q_n^E)$ which ceases to be globally asymptotically stable. This will be facilitated with the aid of an energy-balance equation as will shown below.

Writing eqs (5) for i=1,2,...,n, multiplying all of these equations respectively by $\dot{q}_1,\dot{q}_2,...,\dot{q}_n$, integrating with respect to time t and summing up the resulting equations, we get the following energy-balance equation (including loss of energy)

$$K + 2\int_0^t F dt + U - \lambda \int_0^t (\overline{Q}_i \dot{q}_i) dt' = C \tag{6}$$

For a system initially (t=0) at rest for this type of (step) loading we may assume the following conditions

$$q_i(0) = q_i^0 \ , \ \dot{q}_i(0) = 0 \tag{7}$$

due to which

$$K(t=0) = U(t=0) = 0 \tag{8}$$

and hence C = 0.

For the above nonlinear nonconservative autonomous dissipative system it is assumed that during any interval of time, i.e. from t_1 to t_2, the energy transferred to the system by a conservative process depends only on the state of the system at t_1 and t_2, whereas the nonconservative energy transfer is dependent on the entire path between these two states. The total work supplied to the system up to time t is represented by the last integral in the LHS of eq. (6). One should also notice that velocity-dependent forces not only provide for energy dissipation (being an energy sink) but also control the magnitude of the work done by nonconservative forces by affecting the phase difference among the various degrees of freedom. Thus, the omission of damping (which is always present in actual systems), in case of systems under follower loading may lead to serious errors.

In view of eq.(8) one can write

$$K + 2\int_0^t F dt' + V = 0 \tag{9}$$

where

$$V = U - \lambda \int_0^t \left(\sum_{i=1}^n \overline{Q}_i \dot{q}_i \right) dt' \tag{10}$$

Since K and F are positive definite functions, eq. (9) implies that throughout the motion

$$V < 0 \tag{11}$$

Given that dynamic buckling is associated with an escaped solution, it is clear that dynamic buckling cannot occur when $V > O$.
Setting

$$y_i = q_i \, , \; y_{n+1} = \dot{q}_i \qquad (i=1,\ldots,n) \tag{12}$$

the above initial-value problem can be written in matrix-vector form as follows

$$\dot{y} = Y(y, \lambda; \eta) \, , \; y(t=0) = y^0 \tag{13}$$

where $y = (y_1,\ldots,y_{2n})^T$ is the state vector continuously dependent on t and λ with T denoting transpose; $Y = (Y_1,\ldots,Y_{2n})^T$ is a nonlinear smooth vector-function which is assumed to satisfy the Lipschitz condition in the domain of interest. Numerical results obtained on the basis of the qualitative analysis proposed herein are checked by solving the nonlinear initial-value problem of eqs (13) with the aid of a variant of the Runge-Kutta scheme.

The equilibrium (singular) states y^E are obtained by

$$Y(y^E, \lambda; \eta; y^0) = 0 \tag{14}$$

Note that eqs (1) due to conditions (7) become

$$V_i(q_i, \lambda; q_i^0) = 0 \tag{15}$$

which are equivalent to eq. (14).

4. DYNAMIC BUCKLING CRITERIA

Using the stability criterion of Laplace or Lagrange (boundedness of solution), dynamic buckling is defined as that state for which an escaped motion leads either to an unbounded motion (overflow) or to a large response associated with a remote stable equilibrium (point attractor). The minimum load corresponding to that state is defined as the dynamic buckling load) λ_{DD}. For loads smaller than λ_{DD}, in view of the above assumptions, the motion is captured by the (asymptotically) stable equilibria of the fundamental equilibrium path.

It is established below that dynamic buckling (escaped motion) of this non-potential system takes place via a saddle y_D^E or its neighborhood (for more than one degrees of freedom systems) of the unstable postbuckling path as this occurs in potential systems [7]. For more than one degrees of freedom systems there are also more than one trajectories leading to the saddle. This implies practically passage of the escaped trajectory rather through the neighborhood of the saddle where $V \cong 0$ than through the saddle itself. Recall also that a saddle is characterized by stable and unstable manifolds S and U tangent at yE_0 to the stable and unstable subspaces E^S and E^U of the linearised system obtained from eq. (13); namely, subspaces having Jacobian eigenvalues with negative and positive real parts, respectively. The first ones are associated with inset (stable) manifolds capturing the motion (attractors), while the second with outset (unstable) manifolds sending away the motion (repellors). This saddle due to condition (9) or (11) is also associated with the following constraint

$$V(y,\lambda;\eta;y^0)=0 \tag{16}$$

where in this case $y=(y_1,...,y_n)^T$ and $y^0=(y_1^0,...,y_n^0)^T$.

Indeed, since the total kinetic energy K of the above autonomous dissipative system is always positive (i.e. $K>0$) throughout the motion (including the case of an escaped motion), then via condition (9) we get inequality (11) or (16). Therefore, there is no motion if $V(y,\lambda;\eta;y^0)=0$. The quantity V for fixed λ becomes minimum at the equilibrium y^E [i.e.min$V=V(y^E,\lambda;\eta;y^0)=0$] of fundamental equilibrium path which by assumption is (locally) asymptotically stable. Then, due to eq. (9) K becomes maximum at y^E. The maximum value of $V(y,\lambda;\eta;y^0)=0$ for a system in motion in case of vanishing but non-zero damping (i.e. $[c_{ij}]\to 0$) corresponds to the limiting case $V(y,\lambda;\eta;y^0)=0$ which due to eq. (9) yields also $K=0$. Since K is throughout the motion a positive definite function the last case for which K becomes minimum (i.e. $K=0$) occurs when all generalized velocities (associated with K) become zero and hence $V=0$. However, as will be shown below the escaped trajectory practically passes rather through the neighborhood of the saddle (with very small negative V), than the saddle itself. Then, the phase-point is associated with

$$V(y_D^E,\lambda;\eta;y^0)\to 0^- \tag{17}$$

Clearly, both equilibria y^E and y^D correspond to the same λ.

The system for small λ, (smaller than the dynamic buckling load) undergoes nonlinear oscillations about the corresponding asymptotically stable equilibrium position y^E (of the fundamental path) with bounded amplitude

$$\|\xi(t)\|=\left(\sum_{i=1}^{n}\xi_i^2\right)^{1/2} \quad \text{with} \quad x_i=y_i\text{-}y_i^E \ \ (i=1,...,n) \tag{18}$$

which diminishes gradually with time for scleronomic definite dissipative systems [5] until the system comes to rest (at the asymptotically stable equilibrium position y^E). As the system goes away from y^E [where $V(y^E,\lambda;\eta;y^0)$=min$V<0$] then $V(y,\lambda;\eta;y^0)$ increases becoming maximum at a certain non-equilibrium point, where V is in general negative (with upper bound the limiting case $V(y^D,\lambda;\eta;y^0)=0$). At this point, for vanishing but non-zero damping, due to the constraint (9), the kinetic energy K becomes minimum, being in general positive (with lower bound $K=0$). Immediately after that instant, the motion changes sense being directed back to the state y^E to which gradually it converges. Thus, the asymptotically stable equilibrium state y^E acts as a point attractor with a basin (domain) of attraction whose boundary is formed by the union of all trajectories. As λ, increases from zero there is an overall tendency of the basin boundary to approach the unstable postbuckling equilibrium path which is uniquely defined as physical continuation of the fundamental or natural path. Note also that as λ increases the stable and unstable equilibria corresponding to the same λ approach each other (coinciding at the limit point). Consider now the case for which at a certain value of λ, there corresponds an equilibrium state y^E with a basin (of attraction) boundary which sooner or later (depending on the hypersurface of V in the V-y_i space) will touch the unstable postbuckling equilibrium path at an equilibrium point where K=0; namely, at a saddle point y_D^E where V=0. However, the escaped motion for more than one degrees of freedom practically passes rather through the neighborhood of the saddle (with $V \to 0^-$) than the saddle itself. This is due to the fact that there are more than one trajectories leading to the saddle point y_D^E. The more *narrow* and *deep* the *escape channel* with $V \leq 0$ the more *accurate* the load $\tilde{\lambda}_D$ associated with V=0. The saddle point in certain directions acts as point attractor [inset (stable) manifolds], while in other directions acts as repellor [outset (unstable) manifolds]. After a competition of opposing forces corresponding to the equilibrium points y^E and y_D^E, during which the system undergoes nonlinear oscillations, finally the motion will be captured by the attractive forces of the stable manifold of the saddle. Thereafter the motion escapes via the outset manifold of the saddle point. This sufficient criterion for dynamic buckling (escaped motion) is in agreement with Liapunov's direct method for asymptotic stability in the large [2]. According to a variant of this theorem as long as thedomain of attraction (or domain of stability) of the point y^E does not contain (inside or on its boundary) any other equilibrium point, then y^E is stable (asymptotically) in the large. When the basin boundary touches the unstable postbuckling path at a saddle (or passes through its neighborhood with very small negative V) the escaped motion is inevitable. Obviously, y_D^E and the corresponding load $\tilde{\lambda}_D$ (for vanishing damping) satisfy the following equations

$$\left.\begin{aligned} V(y_D^E, \lambda; \eta; y^0) = 0 \\ V_i(y_D^E, \lambda; \eta; y^0) = 0 \end{aligned}\right\} \quad i = 1, \ldots n \tag{19}$$

Clearly, in this case the equilibrium point y^E on the fundamental equilibrium path, although asymptotically stable locally (in the small), it is unstable globally (in the large).

In case of non-vanishing damping eqs (19) due to condition (9) must be replaced by

$$\left.\begin{aligned} V(y_D^E, \lambda; \eta; y^0) = -2\int_0^t F dt' \\ V_i(y_D^E, \lambda; \eta; y^0) = 0 \end{aligned}\right\} \tag{20}$$

Since the integral in the RHS of eqs (20) cannot be evaluated in terms of elementary functions, a good approximation of λ_{DD} can be obtained by means of upper and/or lower bounds of the above integral with the aid of Schwarz's and Holder's inequality [see Appendix]. Then, the first of eqs (20) is replaced by

$$V(y, \lambda; \eta; y^0) < -\frac{1}{\tau} c_{ij} \bar{y}_i \bar{y}_j \tag{21}$$

where $\bar{y}_i = y_i^E - y_i^0$ and τ is the duration of time from the onset of loading until dynamic buckling (escape time). The simultaneous solution of eq. (21) and the equilibrium eqs (19) yield a lower bound (dynamic buckling estimate) $\bar{\lambda}_{DD}$ which is sufficiently accurate for structural design purposes. This is acceptable only if it corresponds to a saddle of the unstable postbuckling path. Note that $\bar{\lambda}_{DD}$ is a lower bound if τ is known. Since usually τ is unknown one can use as approximate value of τ the half period $\bar{\tau} = \pi / \omega_0$ of the linearized system, where ω_0 is its fundamental circular frequency. Note that usually $\bar{\tau} < \tau$ for small damping as the degree of freedom increases [implying more irregular energy surface $V(y,\lambda;y^0)$]. Hence, $\bar{\lambda}_{DD}$ is a good approximation of the exact dynamic buckling load λ_{DD}. From the first of eqs (20) it follows that as damping increases V decreases, while λ_{DD} increases and the corresponding saddle approaches the limit point. Obviously, $\tilde{\lambda}_D$ is a lower bound of λ_{DD}. Moreover since y_D^E (corresponding to $\tilde{\lambda}_D$) belongs to the unstable postbuckling path it is clear that the limit point λ_S is an upper bound of λ_{DD}; i.e. $\tilde{\lambda}_D < \lambda_{DD} < \lambda_S$.

From the above development it is clear that when the boundary of the basin (domain) of attraction of the point y^E includes a saddle y_D^E (by passing via it or its neighborhood with $V \to 0^-$) then an escaped motion (dynamic buckling) is initiated via the corresponding saddle (or its neighborhood) with $V \to 0^-$. This is a more general sufficient criterion for dynamic buckling than the corresponding one valid for potential systems.

For the systems under discussion as a result of inclusion of damping it is guaranteed by the theory of attractors that a wandering motion cannot be sustained indefinitely. Eventually, the motion is led to a state of stable equilibrium or to dynamic buckling (escaped motion). Moreover, it is clear that the foregoing criteria for dynamic buckling are valid provided that damping is included even in case it is negligibly small.

It is worth noticing that even in case of vanishing damping ($c_{ij} \to 0$ for all i,j) unlike conservative systems an "exact" evaluation of λ_{DD} is impossible for nonconservative systems with more than one degrees of freedom. This is due to the fact that the integrals in eq. (10), in general, cannot be analytically (in closed form) determined. However, this is possible only for nonconservative systems with one degrees of freedom. For nonconservative systems with more than one degrees of freedom one can seek lower/upper bounds or approximate evaluations of the aforementioned integrals.

5. NUMERICAL RESULTS

The proposed approach is demonstrated with the aid of two models shown in Figs. 2 and 3 of one and two degrees of freedom, respectively.

Example 1

For the first model shown in Fig.2 Lagrange (dimensionless) equation of motion and the associated initial conditions are

$$\left.\begin{array}{l}\ddot{\Theta}+(\Theta-\varepsilon)[1-\gamma(\Theta-\varepsilon)^2]+c\dot{\Theta}-\lambda\sin\eta\Theta=0\\ \Theta(0)=\varepsilon,\quad \dot{\Theta}(0)=0\end{array}\right\}\tag{22}$$

where $\gamma>0$ must be determined so that the system loses its static stability via a limit point.

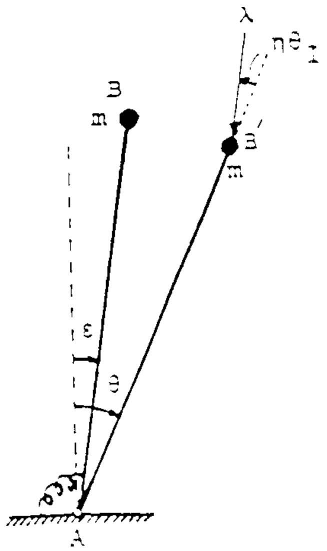

Fig.2. 0ne-degree-of-freedom
model under partial follower load.

From the static equilibrium equation

$$V_1=(\Theta-\varepsilon)[1-\gamma(\Theta-\varepsilon)^2]-\lambda\sin\eta\Theta=0\tag{23}$$

it can be easily deduced that the critical load of the perfect system is equal to $\lambda^C=1/\eta$, and moreover the corresponding critical point is an unstable symmetric branching point for $\gamma>\eta^2/6$. Hence, eqs (22) hold for $\gamma>\eta^2/6$.

Multiplying the first of eq. (1) by $\dot{\Theta}$, integrating with respect to time and taking into account the initial conditions (22) we get

$$\frac{\dot{\Theta}^2}{2}+\frac{(\Theta+\varepsilon)^2}{2}\left[1-\frac{\gamma}{2}(\Theta-\varepsilon)^2\right]+c\int_0^t\dot{\Theta}^2dt'+\frac{\lambda}{\eta}(\cos\eta\Theta-\cos\eta\varepsilon)=0\tag{24}$$

This is the energy-balance equation corresponding to eq. (9). Eqs (19) for vanishing damping are written as follows

$$\left.\begin{array}{l}\frac{(\Theta-\varepsilon)^2}{2}\left[1-\frac{\gamma}{2}(\Theta-\varepsilon)^2\right]+\frac{\lambda}{\eta}(\cos\eta\Theta-\cos\eta\varepsilon)=0\\ (\Theta-\varepsilon)[1-\gamma(\Theta-\varepsilon)^2]-\lambda\sin\eta\Theta=0\end{array}\right\}\tag{25}$$

where $\gamma>\eta^2/6$. From system (25) we obtain for given values of η, ε and γ the dynamic buckling load $\tilde{\lambda}_D\equiv\lambda_D$ and the corresponding value $\tilde{\Theta}_D\equiv\Theta_D$. Numerical values of these quantities as well as of the limit point load λ_S, and the corresponding angle Θ_S, are given in Table 1.

Table 1. Limit point and dynamic buckling load, λ_S and λ_D, with corresponding displacements Θ_S and Θ_D for ε=0.05 and various values of η and γ.

η	γ	λ_S	Θ_S	$\tilde{\lambda}_D \equiv \lambda_D$	$\tilde{\Theta}_D \equiv \Theta_D$
0.20	0.04	4.611180	0.9341848	4.5153482	1.4405886
	0.06	4.548685	0.8026211	4.4392146	1.2331367
	0.08	4.501248	0.7244886	4.3816202	1.1097532
0.25	0.04	3.701544	0.9707394	3.6269415	1.4958731
	0.06	3.647977	0.8216447	3.5617775	1.2618911
	0.08	3.608227	0.7367887	3.5135602	1.1283111
0.30	0.04	3.098769	1.0243329	3.0388795	1.5761090
	0.06	3.049731	0.8475559	2.9793722	1.3008301
	0.08	3.014532	0.7530267	2.9367372	1.1527081

The above values of λ_D and Θ_D have been also computed via numerical integration using the Runge-Kutta scheme. Hence λ_D and Θ_D are exact values for a non-dissipative model.

If small damping is included one can adopt the procedure presented by Kounadis and Raftoyiannis (1990).

Example 2

Consider the nonlinear two-degree-of-freedom, partially fixed, cantilever model shown in Fig.3 for which many numerical results are available [4,5]. It consists of two weightless rigid links of equal length l and carries two concentrated masses m_1 and m_2 at B and C (where $m_1/m_2=m$). The undeformed state is specified by the initial geometric imperfections ε_1 and ε_2, while the deformed configuration by the angles Θ_1 and Θ_2 (all measured from the vertical position with respect to the corresponding link axis).

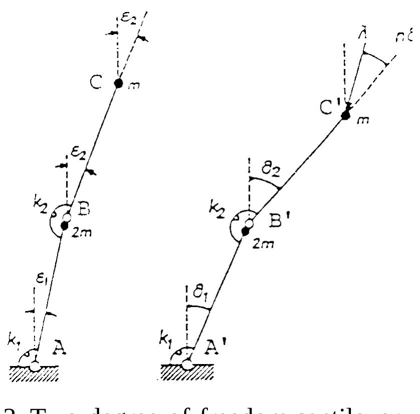

Fig.3. Two-degree-of-freedom cantilever model ABC under a partial follower compressive load at its tip.

Structural stiffness is provided by two nonlinear rotational springs (located at A and B) associated with corresponding viscous dampers. The model is subjected to a partial follower tip load λ acting at an angle $\eta\Theta$, where η defines the degree of nonconservativeness. For η=l the loading (and the model) becomes conservative. As is known the perfect model (ε_1=ε_2=0) loses its static stability via divergence for 1≥η≥4/9 [5]. It was also established that the two postbuckling equilibrium paths are independent of each other for 1≥η≥0.50, and hence a limit cycle response in this interval of values of η is ruled out. For 0.50>η≥4/9 the two postbuckling paths are not independent of each other and the model may exhibit various important phenomena (e.g. a limit cycle response, a double zero eigenvalue, flutter before divergence etc).

In this investigation attention is focused on imperfect models with 1>η≥0.50. Then the model is governed by the following Lagrange equations of motion,

$$\begin{aligned}&(1+m)\ddot{\Theta}_1+\ddot{\Theta}_2\cos(\Theta_1-\Theta_2)+\dot{\Theta}_2^2\sin(\Theta_1-\Theta_2)+(c_1^*+c_2^*)\dot{\Theta}_1-c_2^*\dot{\Theta}_2+\frac{\partial V}{\partial\Theta_1}=0\\&\ddot{\Theta}_2+\ddot{\Theta}_1\cos(\Theta_1-\Theta_2)+\dot{\Theta}_1^2\sin(\Theta_1-\Theta_2)+c_2^*\dot{\Theta}_2-c_2^*\dot{\Theta}_1+\frac{\partial V}{\partial\Theta_2}=0\end{aligned}\tag{26}$$

where

$$\begin{aligned}&\frac{\partial V}{\partial\Theta_1}=V_1=\Theta_1-\varepsilon_1+\delta_1(\Theta_1-\varepsilon_1)^2+\gamma_1(\Theta_1-\varepsilon_1)^3-(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)-\\&-\delta_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^2-\gamma_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^3-\lambda\sin[\Theta_1+(\eta-1)\Theta_2]\\&\frac{\partial V}{\partial\Theta_2}=V_2=\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1+\delta_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^2-\\&-\gamma_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^3-\lambda\sin\eta\Theta_2\end{aligned}\tag{27}$$

clearly

$$V_i=V_i(\Theta_i,\lambda;\delta_i;\eta;\gamma_i;\varepsilon_i)=U_i(\Theta_i,\lambda;\delta_i;\eta;\gamma_i;\varepsilon_i)-\lambda\overline{Q}_i(\Theta_i;\eta)=0\ \ (i=1{,}2)\tag{28}$$

are the nonlinear equilibrium equations with

$$\overline{Q}_1=\sin[\Theta_1+(\eta-1)\Theta_2],\ \ \overline{Q}_2=\sin\eta\Theta_2\tag{29}$$

while $U_i=\partial U/\partial\Theta_i$, $(i=1{,}2)$

Matrices $[\overline{Q}_{i,j}]$ and $[U_{ij}]$ in eq. (4) have the following expressions

$$[\overline{Q}_{i,j}]=\begin{bmatrix}\cos[\Theta_1+(\eta-1)\Theta_2] & (\eta-1)\cos[\Theta_1+(\eta-1)\Theta_2]\\ 0 & \eta\cos\eta\Theta_2\end{bmatrix},\ [U_{ij}]=\begin{bmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}\tag{30}$$

where

$$\begin{aligned}&U_{11}=2+2\delta_1(\Theta_1-\varepsilon_1)+3\gamma_1(\Theta_1-\varepsilon_1)^2+2\delta_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)+3\gamma_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^2\\&U_{12}=U_{21}=-1-2\delta_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)+3\gamma_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^2\\&U_{22}=1+2\delta_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)+3\gamma_2(\Theta_2-\varepsilon_2-\Theta_1+\varepsilon_1)^2\end{aligned}\tag{31}$$

Obviously, $[\overline{Q}_{i,j}]$ is nonsingular if $\Theta_1+(\eta-1)\Theta_2$ and $\eta\Theta_2$ are different than π/2. In this case the static buckling eq. (3) can also be written as

$$\det([\tilde{U}_{ij}] - \lambda I) = 0 \tag{32}$$

where $[\tilde{U}_{ij}] = [\overline{Q}_{i,j}]^{-1}[U_{ij}]$ and I is identity matrix. Since $[U_{ij}]$ is always a positive definite matrix and $[\overline{Q}_{i,j}]$ is an asymmetric nonsingular matrix $[\tilde{U}_{ij}]$ is also asymmetric and nonsingular. Then, one can obtain from eq. (32) two distinct static buckling (limit point) loads $\lambda^S_{(1)}$ and $\lambda^S_{(2)}$ for given values of η, ε_1, ε_2, γ_i and δ_i (i=1,2). Evaluating (32) at the critical state of the perfect system (ε_1=ε_2=Θ_1=Θ_2=0) we get

$$\det\begin{pmatrix} \dfrac{3\eta-1}{\eta}-\lambda & \dfrac{1-2\eta}{\eta} \\ -\dfrac{1}{\eta} & \dfrac{1}{\eta}-\lambda \end{pmatrix} = 0 \tag{33}$$

Eq. (33) yields

$$\lambda^2 - 3\lambda + \frac{1}{\eta} = 0 \tag{34}$$

from which we obtain the critical loads

$$\lambda^C_{(1,2)} = \frac{1}{2}\left(3 \pm \sqrt{9 - \frac{4}{\eta}}\right) \tag{35}$$

with double critical point

$$\lambda^C_0 = \frac{3}{2},\ \eta_0 = \frac{4}{9} \tag{36}$$

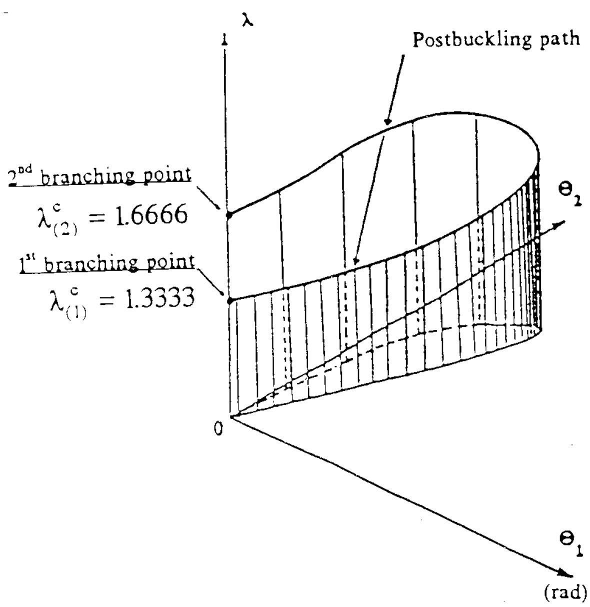

Fig. 4. One postbuckling equilibrium path (passing through the 1st and 2nd branching point) for η=0.45.

It has been established that the model for 0.50>η≥4/9 exhibits one postbuckling path passing through the first and second branching point [5]. A typical example corresponding to η=0.45 is shown in Fig. 4. In this region of existence of adjacent equilibria (region of divergence) the model may exhibit a limit cycle response. A pertinent example for η=0.45, $c_1^* = 0.04$, $c_2^* = 0.10$ and $\lambda_{(1)}^c < \lambda = 1.50 < \lambda_{(2)}^c$ is shown in Fig. 5. In fig. 6 one can see the nonlinear equilibrium paths (natural and complementary), λ vs Θ_i, i=1,2 for the first and second buckling modes for η=0.45. Moreover, for 1≥η≥0.50 the postbuckling paths are independent of each other and the model experiences a dynamic behavior similar to symmetric conservative systems [6]. Therefore, the subsequent analysis holds for 1≥η≥0.50.

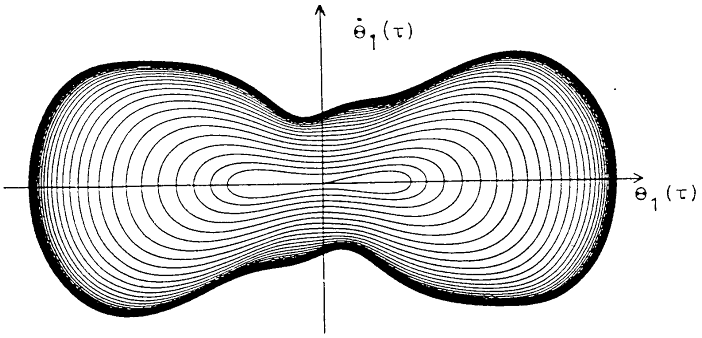

Fig.5. Phase-plane portrait associated with stable limit cycles for a model with $\eta = 0.45 < \eta^* = 0.50$, $c_1^* = 0.04$, $c_2^* = 0.10$, and $\lambda_{(1)}^C < \lambda = 1.50 < \lambda_{(2)}^C$ where $\lambda_{(1)}^C = 1.3333$ and $\lambda_{(2)}^C = 1.6666$.

Multiplying the 1st and 2nd of eqs (26) by $\dot{\Theta}_1$ and $\dot{\Theta}_2$ respectively, integrating with respect to time and summing up the resulting equations we get the following energy-balance equation [see eq. (6)]

$$K + U + 2\int_0^\tau F d\tau' - \lambda\left\{\int_0^\tau \sin[\Theta_1 + (\eta - 1)\Theta_2]\dot{\Theta}_1 d\tau' + \int_0^\tau \sin\eta\Theta_2 \bullet \dot{\Theta}_2 d\tau'\right\} = C \tag{37}$$

where K, U and F are given by

$$\begin{aligned}
K &= \frac{1}{2}(1+m)\dot{\Theta}_1^2 + \frac{1}{2}[\dot{\Theta}_2^2 + 2\dot{\Theta}_1\dot{\Theta}_2\cos(\Theta_1 - \Theta_2)] \\
U &= \frac{1}{2}(\Theta_1 - \varepsilon_1)^2 + \frac{1}{3}\delta_1(\Theta_1 - \varepsilon_1)^3 + \frac{1}{4}\gamma_1(\Theta_1 - \varepsilon_1)^4 + \\
&\quad + \frac{1}{2}(\Theta_2 - \varepsilon_2 - \Theta_1 + \varepsilon_1)^2 + \frac{1}{3}\delta_2(\Theta_2 - \varepsilon_2 - \Theta_1 + \varepsilon_1)^3 + \frac{1}{4}\gamma_2(\Theta_2 - \varepsilon_2 - \Theta_1 + \varepsilon_1)^4 \\
F &= \frac{1}{2}c_1^*\dot{\Theta}_1^2 + \frac{1}{2}c_2^*(\dot{\Theta}_2 - \dot{\Theta}_1)^2
\end{aligned} \tag{38}$$

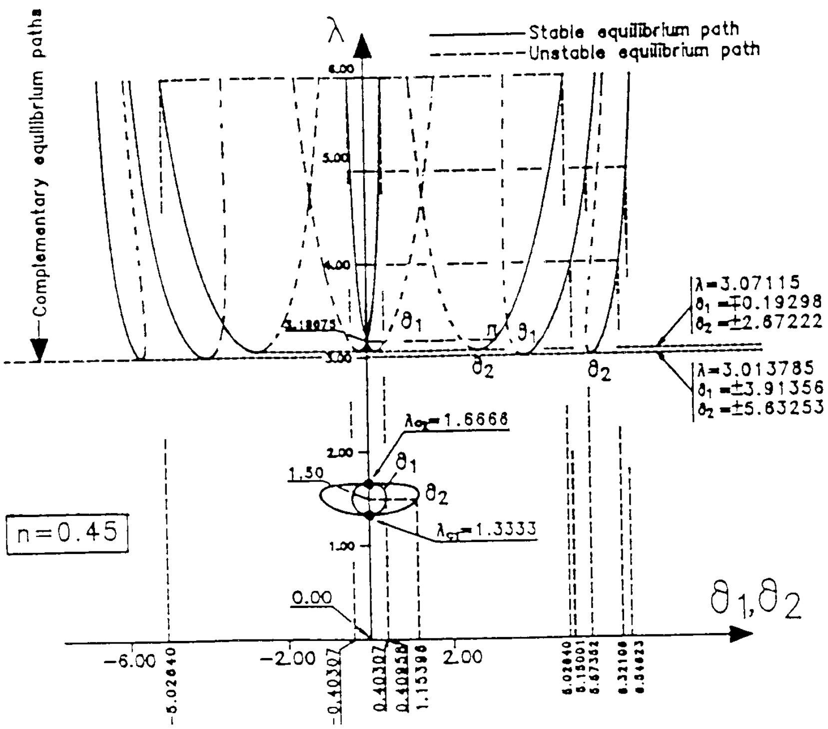

Fig.6. Natural and complementary nonlinear equilibrium paths of the 1st and 2nd buckling modes, λ vs Θ_i, i=1,2), for η=0.45.

Assuming that the model is initially (τ=0) at rest for this (step) type of loading we have

$$\Theta_i(0) = \varepsilon_i \,, \dot{\Theta}_i(0) = 0 \,, \ (i = 1{,}2) \tag{39}$$

which imply $K(\tau=0)=U(\tau=0)=0$ and hence C=0 (40)

The energy-balance eq. (37), due to eq. (40), is given by eq. (9), where

$$V = U - \lambda \left\{ \int_0^\tau \sin[\Theta_1 + (\eta - 1)\Theta_2]\dot{\Theta}_1 d\tau' + \int_0^\tau \sin \eta\Theta_2 \bullet \dot{\Theta}_2 d\tau' \right\} \tag{41}$$

Obviously, the integrand of the first of these two integrals is not an integrable function. However, it can be approximated in various ways. For instance, after application of the first mean-value theorem for integrals one can adopt the following approximation valid at the instant of dynamic buckling (escaped motion).

$$\int_0^{\tau} \sin[\Theta_1 + (\eta - 1)\Theta_2]\dot{\Theta}_1 d\tau' \cong (\Theta_1^D - \varepsilon_1)\sin[\Theta_1^{st} + (\eta - 1)\Theta_2^{st}] \tag{42}$$

where Θ_1^D corresponds to the saddle point (on the unstable postbuckling path) through which (or its neighborhood) dynamic buckling occurs; Θ_1^{st} and Θ_2^{st} correspond to the stable equilibrium point on the fundamental path associated with the same λ. Note that for a load λ slightly smaller than the dynamic buckling load, damped oscillations take place around this stable equilibrium state.

Another approximation of the above integral which could also be adopted is the following

$$\int_0^{\tau} \sin[\Theta_1 + (\eta - 1)\Theta_2]\dot{\Theta}_1 d\tau' \cong \cos[\varepsilon_1 + (\eta - 1)\varepsilon_2] - \cos[\Theta_1^D + (\eta - 1)\Theta_2^D] + \\ + (1-\eta)(\Theta_2^D - \varepsilon_2)\sin[\Theta_1^{st} + (\eta - 1)\Theta_2^{st}] \tag{43}$$

By virtue of eqs (41) and (42) on one hand and of eqs (41) and (43) on the other, we get the following approximations of V

$$V \cong U - \lambda\left\{(\Theta_1^D - \varepsilon_1)\sin[\Theta_1^{st} + (\eta - 1)\Theta_2^{st}] + \frac{1}{\eta}(\cos\eta\varepsilon_2 - \cos\eta\Theta_2^D)\right\} \tag{44}$$

and

$$V \cong U - \lambda\left\{\cos[\varepsilon_1 + (\eta - 1)\varepsilon_2] - \cos[\Theta_1^D + (\eta - 1)\Theta_2^D] + \\ + (1-\eta)(\Theta_2^D - \varepsilon_2)\sin[\Theta_1^{st} + (\eta - 1)\Theta_2^{st}] + \frac{1}{\eta}(\cos\eta\varepsilon_2 - \cos\eta\Theta_2^D)\right\} \tag{45}$$

For the case of vanishing damping ($c_{ij}\rightarrow 0$ for all i,j=1,2) eqs (19) with the aid of equilibrium eqs (27) (V_1=V_2=0) and the above approximations of V (eqs [44] and [45]) lead to approximate values of $\tilde{\lambda}_D$, $\tilde{\Theta}_1^D$ and $\tilde{\Theta}_2^D$ without solving the nonlinear initial-value problem defined by eq. (26) and $\Theta_i(0)=\varepsilon_i$, (i=1,2), $\dot{\Theta}_i(0) = 0$. Numerical values of these quantities on the basis of this analysis and thereafter by using the Runge-Kutta scheme are given in Tables 2 and 3 for various values of the parameters. Corresponding values of limit point loads are also included in these Tables.

Clearly the results of both aproximations are very good, while those which are based on eq.(44) are very close to the "exact" results which were also reported by Kounadis et al [4].

Similar results can be seen in Table 3 in case of a system with a higher degree of nonconservativeness (i.e. for η=0.60).

In case of non-zero damping one can use the approximate formula (21) along with equilibrium eqs (27) (V_1=V_2=0). Table 4 shows how the numerical results for the system of Table 3 are changed if small damping (with $c_1^* = 0.01$ and $c_2^* = 0.05$) is included. In this case the RHS of eq. (21) is given by

$$\frac{1}{\tau}c_{ij}\bar{y}_i\bar{y}_j = \frac{c_1^*}{\tau}(\Theta_1^D - \varepsilon_1)^2 + \frac{c_2^*}{\tau}(\Theta_2^D - \varepsilon_2 - \Theta_1^D + \varepsilon_1)^2 \tag{46}$$

Table 2. Static, dynamic (obtained numerically) and dynamic (obtained analytically) loads for a system with η=0.90, γ_1=γ_2=0, δ_1=-2.50, δ_2=-0.75, ε_1=0.05, m=2, $c_1^* = c_2^* \to 0$ and various values of ε_2.

ε_2	Static buckling load	Dynamic buckling load numerically obtained	Dynamic buckling load eq.(44)		Dynamic buckling load eq.(45)	
			Value	Error(%)	Value	Error(%)
-0.030	0.35359	0.34361	0.346954	0.97	0.34104	0.75
-0.031	0.35843	0.34756	0.350503	0.96	0.34424	0.96
-0.0314	0.35993	0.34922	0.351988	0.79	0.34556	1.05
-0.0315	0.36031	0.34965	0.352366	0.78	0.34589	1.08
-0.032	0.36226	0.35181	0.354294	0.71	0.34758	1.20
-0.033	0.36643	0.35644	0.358377	0.54	0.35107	1.51
-0.034	0.37103	0.36153	0.362817	0.36	0.35474	1.88
-0.035	0.37622	0.36723	0.367710	0.13	0.35861	2.35

Table 3. Static, dynamic (obtained numerically) and dynamic (obtained analytically) loads for a system with η=0.60, γ_1=γ_2=0, δ_1=-2.50, δ_2=-0.75, ε_1=0.05, m=2, $c_1^* = c_2^* \to 0$ and various values of ε_2.

ε_2	Static buckling load	Dynamic buckling load numerically obtained	Dynamic buckling load eq.(44)		Dynamic buckling load eq.(45)	
			Value	Error(%)	Value	Error(%)
-0.030	0.446565	0.41463	0.424115	2.29	0.421875	1.75
-0.031	0.447357	0.41552	0.424877	2.25	0.422674	1.72
-0.032	0.448155	0.41641	0.425642	2.22	0.423475	1.70
-0.033	0.448958	0.41731	0.426409	2.18	0.424279	1.67
-0.034	0.449766	0.41822	0.427180	2.14	0.425086	1.64
-0.035	0.450580	0.41913	0.427953	2.11	0.425895	1.61
-0.040	0.454734	0.42378	0.431860	1.91	0.429974	1.46
-0.045	0.459039	0.42860	0.435838	1.69	0.434112	1.29
-0.050	0.463504	0.43359	0.439886	1.45	0.438305	1.09
-0.060	0.472972	0.444149	0.448199	0.91	0.446845	0.61

Table 4. Static, dynamic (obtained numerically) and dynamic (obtained analytically) loads for a system with η=0.60, γ_1=γ_2=0, δ_1=-2.50, δ_2=-0.75, ε_1=0.05, m=2, $c_1^* = 0.01$, $c_2^* = 0.05$, and various values of ε_2.

ε_2	Static buckling load	Dynamic buckling load numerically obtained	Dynamic buckling load eq.(44)		Dynamic buckling load eq.(45)	
			Value	Error(%)	Value	Error(%)
-0.030	0.446565	0.415513	0.424577	2.18	0.422573	1.70
-0.031	0.447357	0.416411	0.425337	2.14	0.423370	1.67
-0.032	0.448155	0.417315	0.426099	2.10	0.424168	1.64
-0.033	0.448958	0.418224	0.426865	2.07	0.424969	1.61
-0.034	0.449766	0.419140	0.427633	2.03	0.425772	1.58
-0.035	0.450580	0.420062	0.428403	1.99	0.426577	1.55
-0.040	0.454734	0.424771	0.432299	1.77	0.430641	1.38
-0.045	0.459039	0.429652	0.436266	1.54	0.434763	1.19
-0.050	0.463504	0.434717	0.440304	1.29	0.438942	0.97
-0.060	0.472972	0.445464	0.448598	0.70	0.447454	0.45

Taking τ as the half fundamental period of the linearized undamped eqs (26) we find $\tau=\pi/\omega_0$. Then, one can easily get

$$\omega_0 = \left[0.5 \left(a_2 - \sqrt{a_2^2 - 4a_4} \right) \right]^{1/2} \tag{47}$$

where $a_2 = 3.5 - 1.60\tilde{\lambda}_D$, $a_4 = 0.30(\tilde{\lambda}_D^2 - 3\tilde{\lambda}_D + \frac{1}{0.60})$

In all cases considered in Table 4 approximations based on eqs (44) and (45) as well as on eq. (46) give an error less than 2.18%.

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APPENDIX

For a function $\dot{X}(t)$, integrable in the interval [τ_0, τ], one can write the following Schwarz's and Holder's inequality

$$\left| X(\tau) - X(\tau_0) \right|^2 = \left| \int_{\tau_9}^{\tau} \dot{X}(t) dt \right|^2 \leq (\tau - \tau_0) \int_{\tau_9}^{\tau} \left| \dot{X}(t) \right|^2 dt \tag{a}$$

In case that X(t)=(x_1(t),...,x_n(t))^T inequality (a) becomes

$$\overline{X}^T \overline{X} = \sum_{i=1}^{n} \left| \int_{\tau_9}^{\tau} \dot{x}_i dt \right|^2 \leq (\tau - \tau_0) \int_{\tau_9}^{\tau} \left| \dot{X}^T \dot{X} \right| dt \tag{b}$$

where $\overline{X} = X(\tau) - X(\tau_0)$.

If the elements of the non-negative damping matrix [c_{ij}] are scalar quantities using relation (b) one can write

$$\frac{c_{ij}}{\tau}\bar{q}_i\bar{q}_j \le \int_o^{\tau} c_{ij}\dot{q}_i\dot{q}_j dt \quad (i,j=1,..,n) \tag{c}$$

where $\bar{q}_i = q_i(\tau) - q_i^0$ with $q_i^0 = q_i(0)$

With the aid of eqs (12), eq.(a) and eq.(c), eqs $V(y,\lambda;y^0) = V_i(y,\lambda;y^0) = 0$ become

$$\left.\begin{array}{l} V(y^E;\lambda;y^0) \le -\frac{1}{\tau}c_{ij}\bar{y}_i\bar{y}_j \\ V_i(y^E;\lambda;y^0) = 0 \end{array}\right\} \tag{d}$$

where $\bar{y}_i = y_i^E - y_i^0$ with y_i^E on the unstable path (saddle), while the summation indices i and j range from 1 to n.

DINAMIČKO IZVIJANJE NEDOVOLJNO OSETLJIVOG NEKONZERVATIVNOG DISIPATIVNOG SISTEMA POD DEJSTVOM VODJENOG OPTEREĆENJA

Anthony N. Kounadis

Prikazan je analitički pristup nelinearnom dinamičkom izvijanju nedovoljno osetljivih nekonzervativnih diskretnih disipativnih sistema izloženih dejstvu delimično vodjenog opterećenja u oblasti divergencije. Ovi sistemi pod dejstvom statičkog opterećenja gube stabilnost kroz graničnu tačku. Analiza je ograničena na oblast divergencije gde su matrice asimetrične krutosti idealnih bifurkacionih sistema karakterisane potpunim skupom sopstvenih vektora sa odgovarajućim medjusobno nezavisnim putanjama izvijanja. Tako, ovi sistemi se ponašaju dinamički kao simetrični disipativni sistemi pod dejstvom konzervativnog opterećenja koje pokazuje bilo tačku atraktora ili dinamičkog izvijanja. Kriterijum totalne potencijalne energije za dinamičko izvijanje konzervativnih sistema više ne odgovara. Umesto njega, uspostavljena je jednačina energijskog balansa koja dopušta da se odredi aproksimativno opterećenje dinamičkog izvijanja, veoma dobra za namene projektovanja konstrukcija, kao i za predvidjanje donje/gornje granice izvijanja, koja se lako dobija bez rešavanja problema sa izrazito nelinearnim početnim uslovima. Poredjenje numeričkih rezultata sa dobijenim u drugim analizama numeričke simulacije pokazuje valjanost izloženog pristupa.