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THE CONSTRAINT PROPAGATION ALGORITHM FOR DETERMINING THE STABILITY MARGIN OF LINEAR PARAMETER CIRCUITS AND SYSTEMS

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Lubomir Kolev, Simona Filipova-Petrakieva

Department of Theoretical Electrical Engineering Faculty of Automatica,
Technical University of Sofia 8, Kliment Ohridski Str., Sofia 1000, Bulgaria
E-mail: lkolev@tu-sofia.bg, petrakievas-te@tu-sofia.bg

Abstract. *The paper addresses the stability margin assessment for linear systems under interval parameter uncertainties. The original robust stability problem is initially transformed into an equivalent problem of estimating the eigenvalues ranges of matrices whose elements are non-linear functions of independent interval parameters.*
A new algorithm for finding the exact value of stability margin (within error bounds) is suggested. It is based on the use of the inner and outer bounds on the right ends of the eigenvalue considered in order to determine as narrow initial uncertainty region as possible. Then the constraint propagation approach is applied. It consists of two steps. First, one sweep of constraint propagation, relative to the interval components of the eigenvalue – eigenvector pair, is carried out, keeping the parameter intervals fixed. Next, the second sweep of constraint propagation, relative to the components of the interval parameters, is applied, keeping the reduced intervals of the eigenvalue – eigenvector pair fixed.
A numerical example, illustrating the applicability of the algorithm suggested, is solved at the end.

Key words: *Algoritm, stability, circuit.*

1. INTRODUCTION

It is well known that the stability analysis of linear circuits and systems under parameter uncertainties can be formulated as the problem of estimating the range of the eigenvalues of interval matrices (see e.g. [3] – [9]) because the stability margin of the curcuit studied is equal of the right end of the eigenvalue which is most to the right from the imaginary axis of the complex plane.

Let A be a real n x n matrix, $\boldsymbol{A}$ - an interval matrix containing A, and A^{-}, A^{+}, A_0 and R_A – the left end, the right end, the center and the radius of A, respectively. We consider the following "perturbed" eigenvalue problem:

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$$Ax = \lambda x,\ \ A \in \boldsymbol{A} = [A^{-}, A^{+}] = A_0 + [-R_A, R_A]\,. \tag{1}$$

Ordinary letters will denote real quantities while bold face letters will stand for their interval counterparts.

Each matrix $A \in \boldsymbol{A}$ is non-singular.

For simplicity, the elements $\boldsymbol{a}_{ij}$ of interval matrix $\boldsymbol{A}$ can be considered as independent interval, but in general, they are non-linear functions of m parameters, which take their values within prescribed intervals, i.e.

$$\begin{aligned} &a_{ij}(p) = a_{ij}(p_1, \ldots, p_m),\ \ i,j = 1,\ldots,n \\ &p_l \in \boldsymbol{p}_l,\ \ l = 1,\ldots,m \end{aligned}\ . \tag{2}$$

then the eigenvalue problem considered transforms as follows:

$$A(p)x = \lambda x,\ \ p \in \boldsymbol{p}\,. \tag{3}$$

Each matrix $A(p)$, $p \in \boldsymbol{p}$ is non-singular.

Some of the known methods for assessing the stability margin in the above formulation are based on the solution of the following sub-problems:

(i) find an inner bound on the right end of the range for the eigenvalue considered;

(ii) find an outer bound on the right end of the range for the eigenvalue considered as well as outer bounds on the right ends of the components of the associated eigenvector;

(iii) the final solution of the problem is then found by determining the exact right end of that eigenvalue range that is most to the right.

Each of these sub-problems is solved for independent interval elements of matrix $\boldsymbol{A}$ (problem (1)) in [3], [4] and [6], as well as in the case of dependent interval elements of matrix $\boldsymbol{A}$ (problem (3)) [5].

Using the above inner and outer bounds it is suggested a new algorithm for obtaining the exact stability margin as narrow initial uncertainty region as possible. This algorithm is based on an approach called constraint propagation.

The present paper discusses the problem of determination of the corresponding exact right ends of the eigenvalues of matrices whose elements are non-linear functions of independent interval parameters. The problem statement is defined in Section 2. A new constrain propagation algorithm for obtaining the exact values of the right ends of the considered intervals is suggested in Section 3. It is the generalization of the constraint propagation approach suggested in [7]. The algorithm procedure includes two steps. First, it is applied the constraint propagation technique relative to the interval components of the eigenvalue-eigenvector pair, keeping the parameter intervals fixed. Second, the same technique, relative to the components of the interval parameters, is carried out, keeping the reduced intervals of the eigenvalue-eigenvector pair fixed. A numerical example, illustrating the simplicity and applicability of the algorithm suggested, is considered in Section 4. The paper ends with conclusion remarks in Section 5.

2. PROBLEM STATEMENT

The solution of initial basic eigenvalue problem (3) will be found making the following transformation. Let the system (3) be written for the central parameters vector p^0

$$A(p^0)x = \lambda x \tag{4}$$

We will estimate only the interval of the maximum eigenvalue of matrix $A(p^0)$. In general, the new algorithm suggested later can be applied for any other real eigenvalues.

Let

$$\lambda^*(p^0) = \max(\lambda_k(p^0)), \quad k = 1,...,n \tag{5}$$

is a maximum eigenvalue while $x^* = (x_1, x_2, ..., x_n)^{\mathrm{T}}$ is the corresponding eigenvector. We make the following assumption (ensuring structural stability of the problem).

Assumption A1: Let $\lambda^*(p)$ and $x^*(p)$, corresponding to all $p \in \boldsymbol{p}$, remain real.

On account of Assumption A1, the range

$$\lambda^* = \{\lambda(p) : p \in \boldsymbol{p}\} \tag{6}$$

is a real interval.

Without any loss of generality we need the second assumption. If the pair (x^0, λ^0) is the solution of (4) then

Assumption A2: We assume that the absolute value of the *s*th component $|\, x_s^0 \,|$ of vector x^0 is the largest component of the other components, i.e.

$$|\, x_s^0 \,| \geq |\, x_i^0 \,|,\ i \neq s \tag{7}$$

Now x^0 is normalized through

$$x_s^0 = 1\,. \tag{8}$$

Further, we require that (8) be also valid for

$$x_s(p) = 1,\ p \in \boldsymbol{p}\,. \tag{8a}$$

We introduce the *n*-dimensional real vector

$$y = (y_1,\ y_2,\ \dots\ ,\ y_n)^{\mathrm{T}} \tag{9}$$

with

$$\begin{aligned} y_i &= x_i(p),\ i = 1,...,n,\ n \neq s \\ y_s &= \lambda(p) \end{aligned} \tag{10}$$

Using (10), the eigenvalue problem (3) is

$$\left| \begin{aligned} &\sum_{\substack{i=1 \\ i\neq s}}^{n} a_{ij}(p)y_j - y_s y_i + a_{is}(p) = 0,\ i = 1,...,n,\ i \neq s \\ &\sum_{\substack{i=1 \\ i\neq s}}^{n} a_{sj}(p)y_j - y_s + a_{ss}(p) = 0 \end{aligned} \right. . \tag{11}$$

where $a_{ij} = a_{ij}(p), p\in p$.

Then we obtain the exact value of the stability margin considered applying the constraint propagation approach to the non-linear system (11). (The essence of this approach is described in [1] and [7].)

In the framework (11), the constraint propagation technique consists in successive satisfaction of some constraints, given as equalities, and reduction of the initial region (box) of uncertain interval parameters. In the context of the problem considered, the constraints are given by the components of the interval eigenvalue problem under consideration.

3. A NEW CONSTRAIN-PROPAGATION ALGORITHM

This algorithm finds the exact value of the stability margin within error bounds. It is based on the fact that the stability margin is equal of the right end of the eigenvalue range that is most to the right.

Let the elements of matrix $A(p)$ be the non-linear function of independent interval parameters p_l, $l = 1,...,m$ following (2). We write them in the following linear form with respect to the elements of vector p [2]:

$$a_{ij}(p) = \alpha_{ij} p + \boldsymbol{b}_{ij},\ \ p \in \boldsymbol{p} \tag{12}$$

and substitute in system (11)

$$\left|\begin{array}{l} \sum\limits_{\substack{i=1\\ i\neq s}}^{n}[\boldsymbol{b}_{ij} + \sum\limits_{l=1}^{m}\alpha_{ijl}p_l]y_j - y_s y_i + [\boldsymbol{b}_{is} + \sum\limits_{l=1}^{m}\alpha_{isl}p_l] = 0,\ i = 1,...,n,\ i \neq s \\ \sum\limits_{\substack{i=1\\ i\neq s}}^{n}[\boldsymbol{b}_{sj} + \sum\limits_{l=1}^{m}\alpha_{sjl}p_l]y_j - y_s + [\boldsymbol{b}_{ss} + \sum\limits_{l=1}^{m}\alpha_{ssl}p_l] = 0 \end{array}\right. . \tag{13}$$

Then we simplify the non-linear system (13) with respect to the components of parameter vector p and get the following system

$$\sum_{l=1}^{m}\boldsymbol{g}_{il}p_l = \boldsymbol{f}_i,\ \ p_l \in \boldsymbol{p}_l,\ \ i = 1,...,n\ , \tag{14}$$

where

$$\begin{aligned} \boldsymbol{g}_{il} &= \alpha_{isl} + \sum_{\substack{j=1\\ j\neq s}}^{n}\alpha_{ijl}\boldsymbol{y}_j,\ \ i = 1,...,n \\ \boldsymbol{f}_i &= -\boldsymbol{b}_{is} - \sum_{\substack{j=1\\ j\neq s}}^{n}\boldsymbol{b}_{ij}\boldsymbol{y}_j + \boldsymbol{y}_s\boldsymbol{y}_i,\ \ i = 1,...,n,\ \ i \neq s \\ \boldsymbol{f}_s &= -\boldsymbol{b}_{ss} - \sum_{\substack{j=1\\ j\neq s}}^{n}\boldsymbol{b}_{sj}\boldsymbol{y}_j + \boldsymbol{y}_s \end{aligned} \tag{14a}$$

The system (14) is linear with respect to the components of the parameter vector p. The known methods solved the non-linear system with interval coefficients (11) with respect to the components of interval vector $\boldsymbol{y}$ [5, 7]. Now we will suggest the algorithm, based on the constraint propagation approach, to obtain the same components of $\boldsymbol{y}$ using the linear parameter system (14) yet. It consists of two iterative procedures which call each others recursively. It can be described with the following way. The main procedure includes 3 steps.

Procedure 1

Step1: We start with the initial uncertainty region – interval vector (see Steps 1 and 2 in [7])

$$\boldsymbol{y} = \boldsymbol{y}^{(0)} = \boldsymbol{y}^{out}, \tag{15}$$

where $\boldsymbol{y}^{out}$ is an interval vector whose components are the outer bounds of $\boldsymbol{y}(p)$, $p \in \boldsymbol{p}$, calculated by the method suggested in [5].

We substitute the sth component of $\boldsymbol{y}$ with interval

$$\boldsymbol{y}_s = \boldsymbol{z}_s = [(y_s^{in})^+, (y_s^{out})^+], \tag{16}$$

where $(y_s^{in})^+$ and $(y_s^{out})^+$ are the right ends on the inner and outer bounds on the exact range $\boldsymbol{y}_s^*$, calculated by the methods suggested in [5].

Step 2: We apply the constraint propagation procedure (Procedure 2 from [5]) to obtain the components of the interval vector $\boldsymbol{y}$ consisting of the eigenvalue considered and the components of the respective eigenvector, keeping the parameter intervals fixed.

The stop criteria is two serial values of the interval components of vector $\boldsymbol{y}$ to be close enough.

Step 3: We consider the components of vector $\boldsymbol{y}$ as an independent intervals and apply the constrain propagation to the linear system (14) with respect to the independent components of parameter vector $\boldsymbol{p}$. Thus we calculate the new components of the interval vector $\boldsymbol{p}$ as follows:

For i = 1 to n do

$$\left| \begin{array}{l} \boldsymbol{p}_q^{(i)} = \dfrac{\left[\boldsymbol{f}_i - \sum\limits_{\substack{l=1 \\ l \neq q}}^{m} \boldsymbol{g}_{il} \boldsymbol{p}_l \right]}{\boldsymbol{g}_{iq}}, \quad q = 1 \ldots, n \\ \boldsymbol{p}_q = \boldsymbol{p}_q \cap \boldsymbol{p}_q^{(i)} \end{array} \right. \tag{17}$$

End

The stop criteria is two serial values of the interval components of vector $\boldsymbol{p}$ to be close enough.

Go to Step 2.

In the first iteration of the procedure we start with initial vector $\boldsymbol{y}^{(0)}$, calculated by (15) and (16), but in the other iterations we use the components of interval vector $\boldsymbol{y}$ obtained as a result in the end of the Step 3.

The Procedure 1 is valid for all eigenvalues of interval matrix $A(p)$, $p \in \boldsymbol{p}$ but to simplify the presentation we described it only for the maximum one.

The right end of the eigenvalue (the sth component $\boldsymbol{y}_s$) of the interval vector $\boldsymbol{y}$, which is most to the right is equal to the stability margin of the analyzed system.

4. NUMERICAL EXAMPLE

The circuit studied is shown in Fig. 4.1. Assuming that $R_i \in \boldsymbol{R}_i$, $L \in \boldsymbol{L}$, $C \in \boldsymbol{C}$, the vector of parameters is $p = (R_1, R_2, R_3, L, C)^{\mathrm{T}}$ with $\boldsymbol{R}_1 = [97, 103]$, $\boldsymbol{R}_2 = [198, 202]$, $\boldsymbol{R}_3 = [99.999, 100.001]$, $\boldsymbol{L} = [4.999, 5.0001]m$H, $\boldsymbol{C} = [238, 267]$µF. Such systems arise in tolerance analysis of linear AC electric circuits.

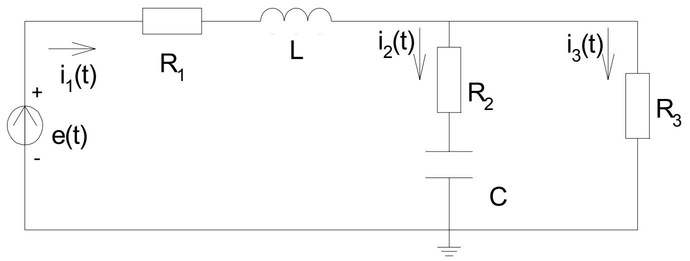

Fig. 4.1

We apply the Procedure 1 to obtain the stability margin of the circuit analyzed.

First, in Step 1 we find the initial interval vector $\boldsymbol{y}^0$ according to (15) and (16) in the following way.

It is seen from the example that the expressions (2) are non-linear functions of 5 parameters

$$A = \begin{bmatrix} -\dfrac{R_1+R_2}{L.k} & -\dfrac{1}{L.k} \\ \dfrac{1}{C.k} & -\dfrac{1}{R_3 C.k} \end{bmatrix}, \quad k = 1+\frac{R_2}{R_3} \quad M = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}. \tag{18}$$

The components of the initial centre and radii parameter vectors are:

$$p^0_{Start} = (100 \quad 200 \quad 100 \quad 5*10^{-3} \quad 250*10^{-6})^{\mathrm{T}}, \tag{19a}$$

$$Rp_{Start} = (3 \quad 2 \quad 0.001 \quad 0.01*10^{-3} \quad 12*10^{-6})^{\mathrm{T}}. \tag{19b}$$

The linearization of $a_{ij}(p)$ is made by the method suggested in [2] using (19). So we get the $a_{ij}(p)$ as the affine functions:

$$\left|\begin{aligned} a_{11} &= -0.67p_1 + 0.0001633p_2 - 1.334p_3 + 400.009p_4 - 200.087 + [-0.0225 \quad 0.0225] \\ a_{12} &= 0.0022232p_2 - 0.004446p_3 + 1.3334p_4 - 1.3338 + 10^{-4}[-6.78 \quad 6.78] \\ a_{21} &= -4.4512p_2 + 8.9024p_3 - 5.34*10^6 p_5 + 2.6703*10^3 + [-3.0127 \quad 3.0127] \\ a_{22} &= 0.0446p_2 + 0.0446p_3 + 0.535*10^6 p_5 - 40.085 + [-0.0245 \quad 0.0245] \end{aligned}\right. \tag{20}$$

The numbers of the maximum eigenvalue and the maximum component of its eigenvector are $k = 2$ and $s = 2$, respectively. Hence, we are interested in the second component of interval vector $\boldsymbol{y}$. According to (9) the vector y is:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \lambda \end{bmatrix} \tag{21}$$

resp.

$$y^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} x_2^0 \\ \lambda^0 \end{bmatrix} = \begin{bmatrix} -0.00366795 \\ -18.244878 \end{bmatrix}. \tag{22}$$

Applying the method suggested in [5] we get the following outer bounds of the eigenvector considered components

$$\boldsymbol{y}^{out} = \begin{bmatrix} [-0.0037587 & -0.0035775] \\ [-19.4124 & -17.08791] \end{bmatrix}, \tag{23}$$

the right end of the inner bound on the eigenvalue considered

$$(\lambda^{in})^{+} = (y_s^{in})^{+} = (y_2^{in})^{+} = -17.1669 \tag{24}$$

and the right end of the exact range

$$(\lambda^{exact})^{+} = (y_s^{exact})^{+} = -17.1669 \tag{25}$$

Then based on (15) and (16) and using (23) and (24) we get the initial interval vector

$$\boldsymbol{y}^{(0)} = \begin{bmatrix} [-0.0037587 & -0.0035775] \\ [-17.1669 & -17.08791] \end{bmatrix}. \tag{26}$$

Secondly, we repeat the Steps 2 and 3 from the Procedure 1 until we obtain two serial values of the interval components of vector $\boldsymbol{p}$ to be close enough with the accuracy $\varepsilon = 10^{-9}$. The final components of the centre and radii parameter vectors are reached on the 6th iteration and they are the following:

$$p^{0}_{Final} = (100 \quad 200 \quad 100 \quad 5*10^{-3} \quad 250*10^{-6})^{\mathrm{T}}, \tag{27a}$$

$$Rp_{Final} = (0.01327 \quad 0.0066612 \quad 0.001 \quad 0.01*10^{-3} \quad 0.00577889*10^{-6})^{\mathrm{T}}. \tag{27b}$$

Thus the right end of the sth component of the vector $\boldsymbol{y}$ is

$$(y_s)^{+} = -17.1669 \tag{28}$$

But

$$(y_s)^{+} = (y_s^{out})^{+} = (\lambda^{out})^{+} \tag{29}$$

then

$$(y_s^{out})^{+} = (\lambda^{out})^{+} = -17.1668 \tag{30}$$

It is obvious from (25) and (30) that the right ends of outer bound and of the exact range considered, calculated by the new algorithm and the method from [5], are equal. Therefore, the new method suggested determines the right end of the stability margin of the circuit studied.

The calculation times for this example are shown in Table 1.

Table 1

Step 1 in Procedure 1 (methods from [5])	Whole the Procedure 1
0.17 s	0.22 s

5. Conclusion

The new algorithm suggested obtains the exact value of the stability margin within error bounds. Some of the known methods for assessing it are based on determining the exact ranges and the respective inner and outer bounds of the considered eigenvalue-eigenvector pair solving the non-linear system (11) with respect to the components of the interval vector

y. Unlike this approach, a new constraint propagation algorithm which applies to the linear parameter system (14) is proposed with respect to the components of the parameter vector ***p***. The algorithm suggested obtains directly the right end of the exact range of the eigenvalue analyzed and thus determines the exact value of the stability margin.

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OGRANIČENJA PROPAGACIJA ALGORITAMA ZA ODREDJIVANJE GRANICA STABILNOSTI LINEARNIH PARAMETARA KOLA I SISTEMA

Lubomir Kolev, Simona Filipova-Petrakieva

Rad se bavi procenom granica stabilnosti za linearne sisteme pri neodredjenim parametrima intervala. Originalni problem stabilnosti je inicijalno transformisan u ekvivalentni problem procene sopstvenih vrednosti matrica čiji su elementi nelinearne funkcije nezavisnih intervalnih parametara.

Novi algoritam za odredjivanje tacčne vrednosti granica stabilnosti (sa opsegom greške) je predložen. Zasniva se na unutrašnjim i spoljašnjim krivama tačnih granica sopstvenih vrednosti da bi se odredila uska inicijalna nepouzdanost oblasti. Onda je primenjen pristup ograničenja propagacije. On se sastoji iz dve faze. Prvo, pomeranje ograničenja propagacija, relevantnog za intervalne komponentne sopstvene vrednosti – par sopstvenoj vrednosti je ostvaren, tako što je parametar intervala fiksiran. Sledeće, drugo pomeranje ograničenja širenja, relativno za komponente intervalnih parametara, je primenjeno, tako što se skraćeni intrvali sopstvenih vrednosti – par sopstvenih vrednosti- fiksiraju.

Dat je i numerički primer koji pokazuje primenjljivost predloženog algoritma.

Ključne reči: *algoritam, stabilnost, kolo*