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ON A PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. *We consider the Hill's equation with periodic function* $y'' + [\lambda - q(x)]y = 0$, $q(x + \pi) = q(x)$ *and investigate a periodic boundary value problem with only a finite number of simple eigenvalues.*

Key words: *ordinary differential equations, periodic boundary value problems, eigenvalues*

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1. INTRODUCTION

We consider the Hill's equation with periodic function

$$y'' + [\lambda - q(x)]y = 0,\ \ q(x+\pi) = q(x) \tag{1}$$

By y_1 and y_2 we denote the solutions of (1) which satisfy

$$y_1(0) = y_2'(0) = 1 \ \ y_1'(0) = y_2(0) = 0\ .$$

The discriminant of (1) is given by $\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$ and $\lambda_0, \lambda_1, \lambda_2, \ldots$, the zeros of $2 - \Delta(\lambda)$, are the eigenvalues of (1) subject to the boundary conditions $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$ while the zeros of $2 + \Delta(\lambda)$, $\lambda_0', \lambda_1', \lambda_2', \cdots$, are the eigenvalues of (1) subject to the boundary conditions $y(0) = -y(\pi)$ and $y'(0) = -y'(\pi)$.We shall assume that $q \in C^n$ in this article, for suitable n. We also assume, without the loss of generality $\int_0^\pi q(x)dx = 0$.For further background information see [2, 3, 5].

Since q is a π-periodic function with mean value zero

$$q(x) = \sum_{n=1}^{\infty}[a_n \cos 2nx + b_n \sin 2nx]\ .$$

We denote the even and odd harmonic parts of q by q_e and I', respectively, and let

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$$I(x)=\sum_{n=0}^{\infty}\frac{a_{2n+1}\sin 2(2n+1)x-b_{2n+1}2(2n+1)x}{2(2n+1)},$$

where $I(x+\pi/2)=-I(x)$ so that

$$q(x)=q_e(x)+I'(x). \tag{2}$$

MAIN RESULT

We use the theorems proved by Borg, Hochstadt [6] and Goldberg and Hochstadt [4]. The purpose of this paper is to generalize this above results by proving the following theorem:

Theorem Let $q\in C^n$. If all but $2n+1$ zeros of $2-\Delta(\lambda)$ (namely, $\lambda_0, \mu_1, \mu_2,...,\mu_{2n}$) are double zeros ,then

$$R_{n+1}+\sum_{k=0}^{n}C_kR_k=0\ , \tag{3}$$

where the $R_k(k=0, 1,...,n)$ satisfy $R_0=1$,

$$R'_{k+1}=-\frac{1}{4}R'''_k+\frac{1}{2}q'_eR_k+q_eR'_k-I'\int_0^x I'R_k d\tau+\frac{1}{2}r_kI', \tag{4}$$

r_k constant, and where the constants $C_0, C_1,...,C_n$ depend on $\lambda_0, \mu_1, \mu_2,...,\mu_{2n}$.

The constant r_k will be identified in proof. Theorem generalizes the previous results to any finite number of simple eigenvalues and separate the nonvanishing of instability intervals of (1) into two categories corresponding to the zeroes of $2+\Delta(\lambda)$ and zeroes of $2-\Delta(\lambda)$.

Proof: By u_1 and u_2 denote the solutions of

$$u''+[\lambda-q(x+\tau)]u=0$$

which satisfy $u_1(0,\tau)=u'_2(0,\tau)=1$ and $u'_1(0,\tau)=u_2(0,\tau)=0$. Solutions u_1 and u_2 are related to y_1 and y_2 by means of

$$u_1(x,\tau)=y'_2(\tau)y_1(x+\tau)-y'_1(\tau)y_2(x+\tau) \tag{5}$$

$$u_2(x,\tau)=y_1(\tau)y_2(x+\tau)-y_2(\tau)y_1(x+\tau) \tag{6}$$

First we proved that $u_1(x, \tau)$ and $u_2(x, \tau)$ are π-periodic functions of τ.

Since $q(x+\pi)$ is π-periodic replacement of τ by $\tau+\pi$ in the above differential equation left in invariant. Therefore if $u(x, \tau)$ is a solution so is $u(x, \tau+\pi)$. Than

$$u_1(x,\tau+\pi)=au_1(x,\tau)+bu_2(x,\tau).$$

From the initial conditions it follows that $a=1$, $b=0$ so that $u_1(x, \tau+\pi)=u_1(x, \tau)$. A similar argument applies to $u_2(x, \tau)$.

Differentiating (6) with respect to τ three times we obtain with the aid of (1), (2) and (5)

$$\frac{\partial^3}{\partial\tau^3}u_2(\pi/2,\tau) = 2q_e'(\tau)u_2(\pi/2,\tau) + 4[q_e(\tau)-\lambda]\frac{\partial u_2}{\partial\tau}(\pi/2,\tau) + \\ +2I'(\tau)[u_2'(\pi/2,\tau)+u_1(\pi/2,\tau)] \tag{7}$$

Also, it is easy to verify that

$$\frac{\partial}{\partial\tau}[u_1(\pi/2,\tau)+u_2'(\pi/2,\tau)] = -2I'(\tau)u_2(\pi/2,\tau)\ .$$

Each side of the above is a periodic function. It follows that $u_1(\pi/2,\tau)+u_2'(\pi/2,\tau) = -\int I'(\tau)u_2(\pi/2,\tau)d\tau + \frac{1}{\pi}\int_0^\pi [u_1(\pi/2,\tau)+u_2'(\pi/2,\tau)]d\tau$. Here the term $\int I'(\tau)u_2(\pi/2,\tau)d\tau$ denotes simply the term by term indefinite integral of a Fourier series. This notation will be retained in the rest of this article.

Equation (7) can now be rewritten as

$$\frac{\partial^3}{\partial\tau^3}u_2(\pi/2,\tau) = 2q_e'(\tau)u_2(\pi/2,\tau) + 4[q_e(\tau)-\lambda]\times\frac{\partial}{\partial\tau}u_2(\pi/2,\tau) - \\ -4I'(\tau)\int I'(\tau)u_2(\pi/2,\tau)d\tau + \frac{2}{\pi}I'(\tau)\int_0^\pi [u_1(\pi/2,\tau)+u_2'(\pi/2,\tau)]d\tau \tag{8}$$

By the use of standard asymptotic results [5] one finds that

$$u_2(x,\tau) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{R_k(x,\tau)}{\lambda^k} + \cos\sqrt{\lambda}x\sum_{k=1}^{\infty}\frac{T_k(x,\tau)}{\lambda^k} \tag{9}$$

$$\frac{1}{\pi}\int_0^\pi [u_1(\pi/2,\tau)+u_2'(\pi/2,\tau)]d\tau = \frac{\sin\sqrt{\lambda}\pi/2}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{r_k}{\lambda^k} + \cos\sqrt{\lambda}\,\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{t_k}{\lambda^k} \tag{10}$$

The substitution of (9) and (10) into (8) and a comparison of corresponding terms in the asymptotic expansions yields

$$R_{n+1}^{+\,'} = -\frac{1}{4}R_n^{+\,'''} + \frac{1}{2}q_e'R_n^{+\,'} + q_eR_n^{+\,'} - I'\int I'(x)R_n^{+}(x)dx + \frac{1}{2}r_n^{+}I',\, n\ge 0 \tag{11}$$

$$T_{n+1}^{+\,'} = -\frac{1}{4}T_n^{+\,'''} + \frac{1}{2}q_e'T_n^{+} + q_eT_n^{+\,'} - I'\int I'(x)T_n^{+}(x)dx + \frac{1}{2}t_n^{+}I',\, n\ge 1 \tag{12}$$

where

$$R_k^{+}(\tau) = R_k(\pi/2,\tau),\ T_k^{+}(\tau) = T_k(\pi/2,\tau),\ R_0^{+} = 1, T_1^{+} = I\ . \tag{13}$$

Similarly (8) also holds at $x = -\pi/2$ so that a substitution of (9) and (10) into (8) leads to (11) and (12) with $R_k^{+}, T_k^{+}, r_k^{+}$ and t_k^{+} replaced by $R_k^{-}, T_k^{-}, r_k^{-}$ and t_k^{-}, respectively, were

$$R_k^{-}(\tau) = R_k(-\pi/2,\tau),\ T_k^{-}(\tau) = T_k(-\pi/2,\tau),\ R_0^{-} = 1, T_1^{-} = I \tag{14}$$

We now define

$$2R_k(\tau) = R_k^+(\tau) + R_k^-(\tau),\ \ 2r_k = r_k^+ + r_k^-,$$

$$2T_k(\tau) = T_k^+(\tau) + T_k^-(\tau),\ \ 2t_k = t_k^+ + t_k^-,$$

$$2R_k^*(\tau) = R_k^+(\tau) - R_k^-(\tau),\ \ 2r_k^* = r_k^+ - r_k^-,$$

$$2T_k^*(\tau) = T_k^+(\tau) - T_k^-(\tau),\ \ 2t_k^* = t_k^+ - t_k^-, \tag{15}$$

and notice that $(R_k, r_k), (R_k^*, r_k^*), (T_k, t_k)$ and (T_k^*, t_k^*) all satisfy (4)

$$R'_{k+1} = -\frac{1}{4}R'''_k + \frac{1}{2}q'_e R_k + q_e R'_k - I'\int I'(x)R_k(x)dx + \frac{1}{2}r_k I',\ k \geq 0.$$

In particular

$$R_0(x) = 1,\ \ R_1(x) = \frac{1}{2}[q_e - I^2]$$

$$R_2(x) = -\frac{1}{8}[q''_e - 3q_e^2 - 2(I')^2 - 2II'' - \frac{1}{3}I^4 - 2I\int_0^{x+\pi/2} I'(\tau)q_e(\tau)d\tau + 2q_e I^2] \tag{15a}$$

and

$$T_1(x) = I(x)$$

$$T_2(x) = -\frac{1}{4}I'' - \frac{1}{6}I^3 + \frac{1}{2}q_e I - \frac{1}{4}\int_0^{x+\pi/2} q_e(\tau)I'(\tau)d\tau. \tag{15b}$$

Let $g(x, \varepsilon)$ be Green's function satisfying

$$g'' + [\lambda - q(x+\tau)]g = \delta(x-\varepsilon),\ \ 0 \leq \varepsilon \leq 1, \tag{16}$$

and the boundary conditions $g(0, \varepsilon) = g(\pi, \varepsilon)$, $g'(0, \varepsilon) = g'(\pi, \varepsilon)$. A standard calculation [6] shows that

$$g(x) \equiv g(x,0) = (u_2(x,\tau) - u_2(x-\pi,\tau))\ /(2-\Delta(\lambda)). \tag{17}$$

It is known [2] that $\Delta(\lambda)$ is an entire function of λ of order 1/2 and has the following asymptotic representation for real λ [5]:

$$\Delta(\lambda) = 2\cos\sqrt{\lambda}\pi + 2\cos\sqrt{\lambda}\pi\sum_{k=1}^{\infty}\frac{w_k}{\lambda^k} + \frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\sum_{k=1}^{\infty}\frac{v_k}{\lambda^k} \tag{18}$$

so that

$$2 - \Delta(\lambda) = 4\sin^2\sqrt{\lambda}\pi/2 + 2(\sin^2\sqrt{\lambda}\pi/2 - \cos^2\sqrt{\lambda}\pi/2)\sum_{k=1}^{\infty}\frac{w_k}{\lambda^k} - \\ -\frac{2\sin\sqrt{\lambda}\pi/2\cos\sqrt{\lambda}\pi/2}{\sqrt{\lambda}}\sum_{k=1}^{\infty}\frac{v_k}{\lambda^k} \tag{19}$$

From the hypotheses of Theorem we also know that the only simple zeros of $2 - \Delta(\lambda)$ are $\lambda_0, \mu_1, \ldots, \mu_{2n}$, and all others must be double zeros so that

$$2-\Delta(\lambda)=(\lambda-\lambda_0)(\lambda-\mu_1)\ldots(\lambda-\mu_{2n})f^2(\lambda) \tag{20}$$

From (17), (9), (15), (20) we have

$$g(\pi/2)=(u_2(\pi/2,\tau)-u_2(-\pi/2,\tau))\ /(2-\Delta(\lambda))=$$
$$=\left[2\frac{\sin\sqrt{\lambda}\,\pi/2}{\sqrt{\lambda}}\sum_{m=0}^{\infty}\frac{R_m(\tau)}{\lambda^m}+2\cos\sqrt{\lambda}\,\pi/2\sum_{m=1}^{\infty}\frac{T_m^*(\tau)}{\lambda^m}\right]-$$
$$-(\lambda-\lambda_0)(\lambda-\mu_1)\ldots(\lambda-\mu_{2n})f^2(\lambda)$$

so that

$$(\lambda-\lambda_0)(\lambda-\mu_1)\ldots(\lambda-\mu_{2n})f(\lambda)g(\pi/2)=$$
$$=\left[2\frac{\sin\sqrt{\lambda}\,\pi/2}{\sqrt{\lambda}}\sum_{m=0}^{\infty}\frac{R_m(\tau)}{\lambda^m}+2\cos\sqrt{\lambda}\,\pi/2\sum_{m=1}^{\infty}\frac{T_m^*(\tau)}{\lambda^m}\right]\Big/f(\lambda) \tag{21}$$

Now the Green's function $g(x)$, as a function of λ must be a meromorphic function. But the boundary value problem (1) is selfadjoint from which it follows that $g(x)$ can have only simply poles. We conclude therefore that each of (21) must be an entire function. The denominator $f(\lambda)$ in (21) can also be written as (see (20))

$$f(\lambda)=\left[\frac{2-\Delta(\lambda)}{(\lambda-\lambda_0)(\lambda-\mu_1)\ldots(\lambda-\mu_{2n})}\right]^{1/2} \tag{22}$$

and by use of (19) we see that

$$f(\lambda)\approx 2\sin(\sqrt{\lambda}\,\pi/2)\big/\lambda^{n+1/2}\quad \lambda\to\infty \tag{23}$$

It follows that the right side of (21) is $O(\lambda^n)$, so that by Liouville's theorem it must be a polynomial in λ, of degree n, say $P_n(\lambda)$.Than, using (21),(19) and (20)

$$\left[2\frac{\sin(\sqrt{\lambda}\,\pi/2)}{\sqrt{\lambda}}\sum_{m=0}^{\infty}\frac{R_m(\tau)}{\lambda^m}+2\cos\sqrt{\lambda}\,\pi/2\sum_{m=1}^{\infty}\frac{T_m^*(\tau)}{\lambda^m}\right]\Big/\Big[\Big(4\sin^2\sqrt{\lambda}\,\pi/2+$$
$$+2(\sin^2\sqrt{\lambda}\,\pi/2-\cos^2\sqrt{\lambda}\,\pi/2)\sum_{k=1}^{\infty}\frac{w_k}{\lambda^k}-\frac{2\sin\sqrt{\lambda}\,\pi/2\cos\sqrt{\lambda}\,\pi/2}{\sqrt{\lambda}}\sum_{k=1}^{\infty}\frac{v_k}{\lambda^k}\Big)\Big/ \tag{24}$$
$$(\lambda-\lambda_0)(\lambda-\mu_1)\ldots(\lambda-\mu_{2n})\Big]^{1/2}=P_n(\lambda)\equiv\sum_{k=0}^{n}A_{n-k}(\tau)\lambda^k$$

By squaring the above and comparing corresponding terms in the asymptotic series we obtain

$$\sum_{m=0}^{\infty}\frac{R_m(\tau)}{\lambda^m}=\sum_{k=0}^{n}\frac{A_k(\tau)}{\lambda^k}\times\left[\left(1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{w_k}{\lambda^k}\right)\Big/\left(1-\frac{\lambda_0}{\lambda}\right)\left(1-\frac{\mu_1}{\lambda}\right)\ldots\left(1-\frac{\mu_{2n}}{\lambda}\right)\right]^{1/2} \tag{25}$$

We note that all the terms under the radical are independent of τ and depend only on λ_0, $\mu_1,\ldots,\mu_{2n}$. One can now rewrite (25) as

$$\sum_{m=0}^{\infty}\frac{G_m}{\lambda^m}\sum_{m=0}^{\infty}\frac{R_m(\tau)}{\lambda^m}=\sum_{k=0}^{n}\frac{A_k(\tau)}{\lambda^k} \tag{26}$$

From (26) we see incidentally that $G_0 = 1$, $A_0(\tau) = 1$.From the coefficient of λ^{-n-1} we see that

$$\sum_{k=0}^{n+1} G_{n+1-k}R_k = 0$$

or equivalently

$$R_{n+1}+\sum_{k=0}^{n} C_k R_k = 0$$

if we let $C_k = G_{n+1-k}$, thereby establishing Theorem.

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O PERIODIČNOM GRANIČNOM PROBLEMU

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Razmatramo Hilovu jednačinu sa perodičnom funkcijom $y'' + [\lambda - q(x)]y = 0$, $q(x + \pi) = q(x)$ *i ispitujemo periodični granični problem samo sa konačnim brojem sopstvenih vrednosti.*

Ključne reči: *obične diferencijalne jednačine, periodični granični problem, sopstvene vrednosti*