2-\(\pi\) STRUCTURES ASSOCIATED TO THE LAGRANGIAN MECHANICAL SYSTEMS

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Abstract. One defines the notion of 2-\(\pi\) structure on the phases space of a mechanical system and investigate its integrability.

Key words: 2-\(\pi\) structures, 2-\(\pi\) structures associated to the Lagrangian mechanical systems.

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INTRODUCTION

The theory of Finslerian mechanical systems has been realized by R. Miron and C. Frigioiu [7]. But, the general theory of Lagrangian mechanical systems was realized by R. Miron [4] and published also in the recent book [6] written by R. Miron and M. Bucătaru.

In the present note we study the Lagrangian case associated to the phases space of a 2-\(\pi\) structure, using our ideas which we applied in the Finslerian and Lagrangian cases [1], [2], [9].

1. LAGRANGIAN MECHANICAL SYSTEMS

Consider a Lagrange space \(L^n = (M, L(x, y))\) and a Lagrangian mechanical system \(\Sigma = (M, L(x, y), F(x, y))\), where \(F(x, y)\) are the external forces.

Following the Miron's theory we take the evolution equations of \(\Sigma\)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = F_i, \quad y^i = \dot{x}^i. \tag{1.1}
\]

These equations are equivalent with the system of differential equations of the second order:

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\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = \frac{1}{2} F^i \tag{1.2}
\]

where

\[
F^i = g^{ij} F_j \tag{1.3}
\]

and

\[
G^i = \frac{\epsilon^\alpha}{2} \left( \frac{\partial^2 L}{\partial \dot{y}^j \partial \dot{x}^i} \dot{y}^j - \frac{\partial L}{\partial \dot{x}^i} \right). \tag{1.3}'
\]

The system of differential equations (1.2) defines a dynamical system of the second order.

R.Miron characterises this system by means of a vector field on the phases space \( TM \).

So, he proves the following theorem:

**Theorem 1.1.** (Miron [4])

1\(^{st}\) The following operator

\[
1^s S = y^j \frac{\partial}{\partial x^i} - 2(G^i - \frac{1}{4} F^i) \frac{\partial}{\partial y^j}
\]

is a vector field on the manifold \( \overline{TM} = TM \setminus \{0\} \) which depend only of the Lagrangian mechanical system \( \Sigma \).

2\(^{nd}\) \( S \) is a semispray on the phases space \( \overline{TM} \).

3\(^{rd}\) The integral curves of \( S \) are the solution curves of evolution equations (1.2).

The proof of this theorem can be found in the papers [4], [8].

2. **CANONICAL NONLINEAR CONNECTIONS OF \( \Sigma \)**

The geometry of Lagrangian mechanical systems is determinated by the geometry of the pair \( (\overline{TM}, S) \).

So, the nonlinear connection \( N \) of the mechanical system \( \Sigma \) is given by the coefficients

\[
N^j_i = \frac{\partial}{\partial y^j} \left( G^i - \frac{1}{4} F^i \right) = N^j_i - \frac{1}{4} \frac{\partial F^i}{\partial y^j} \tag{2.1}
\]

where \( \tilde{N}^i_j = \frac{\partial G^i}{\partial y^j} \) are the coefficients of the canonical nonlinear connection \( \tilde{N} \) of the associated Lagrange space \( L^a = (M, L(x, y)) \) of the mechanical system \( \Sigma \).

Now, we remark that the distribution \( N \) of the canonical nonlinear connection \( N \) give rise to the direct decomposition:

\[
T_i(\overline{TM}) = N_i \oplus V_i \tag{2.2}
\]

Let \( \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \right) \) the local adapted basis to the distributions \( N \) and \( V \):
Structures Associated to the Lagrangian Mechanical Systems

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} + \frac{1}{4} \frac{\partial F^i}{\partial y^j} \frac{\partial}{\partial y^j} \tag{2.3} \]

where

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} \tag{2.3'} \]

The dual adapted basis \((dx^i, \delta y^j)\) has the 1-forms \(\delta y^j\) given by

\[ \delta y^j = \delta y^j - \frac{1}{4} \frac{\partial F^i}{\delta y^j} dx^j \tag{2.4} \]

The tensor of weak torsion of the nonlinear connection \(N\) is

\[ t^i_{jk} = \frac{\partial N^i_j}{\partial y^j} - \frac{\partial N^i_j}{\partial y^j} \tag{2.5} \]

Consequently, the nonlinear connection \(N\) of \(\Sigma\) is symmetric (because \(t^i_{jk} = 0\)).

The curvature tensor \(R^i_{jk}\) of system \(\Sigma\) is as follows

\[ R^i_{jk} (x, y) = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_j}{\delta x^k} \tag{2.6} \]

Thus, the following formula holds:

\[ R^i_{jk} (x, y) = R^i_{jk} (x, y) + \frac{1}{4} \left( \frac{\partial F^m}{\partial y^j} B^i_{jk} - \frac{\partial F^m}{\partial y^j} B^i_{mj} \right) - \frac{1}{4} \frac{\delta}{\delta x^k} \left( \frac{\partial F^i}{\partial y^j} \right) + \frac{1}{4} \frac{\delta}{\delta x^j} \left( \frac{\partial F^i}{\partial y^k} \right) \tag{2.6'} \]

where

\[ B^i_{jk} = \frac{\partial N^i_j}{\partial y^k} \tag{2.7} \]

are the coefficients of the Berwald connections of mechanical system \(\Sigma\) and \(R^i_{jk}\) is the curvature tensor of the Cartan nonlinear connection of Finsler space \(F^n\).

Therefore, we have,

**Proposition 2.1.**

The nonlinear connection of the mechanical system is integrable, if and only if the d-tensor field \(R^i_{jk} (x, y)\) defined by (2.6) vanishes.
3. 2-π structures on the phases space of \( \Sigma \)

Following our methods from the papers \([1],[2]\), we define the 2-\( \pi \) structure on the manifold \( TM \), for the case of the Lagrangian mechanical systems \( \Sigma \).

**Definition 3.1.**

An almost 2-\( \pi \) structure \( \mathbf{F} \) of the mechanical system \( \Sigma \) is a tensor field \( \mathbf{F} \) of type (1.1) which has the following property:

\[
\mathbf{F}^2(X) = -\lambda^2 X, \forall X \in \chi_c(TM)
\]  

(3.1)

where \( \lambda \) is one of the numbers \( \{1, -1, i, -i\} \).

But, the canonical nonlinear connection \( \mathbf{N} \) of the mechanical system \( \Sigma \) determines by the natural way such a 2-\( \pi \) structure.

Indeed, we define \( \mathbf{F} \) on the adapted basis \( \delta \partial_i \delta \partial_j \) by

\[
\mathbf{F} \left( \delta \partial_i \right) = -\lambda \frac{\partial}{\partial y^i} \wedge \lambda \frac{\partial}{\partial y^j} \wedge \frac{\partial}{\partial x_i}.
\]

(3.2)

We remark that the definition has the geometrical meaning because this respect to change of local coordinates of the manifold \( TM \) the equations (2) are invariants.

\( \mathbf{F} \) is a tensor of type (1.1) given by

\[
\mathbf{F} = -\lambda \frac{\partial}{\partial y^i} \otimes dx^i + \lambda \frac{\partial}{\partial x_i} \otimes \delta y^i.
\]

(3.3)

Using (3.2) we can prove without difficulties (3.1) and (3.2).

Let us consider also 2-\( \pi \) structures \( \mathbf{F} \) defined by the canonical nonlinear connection \( \mathbf{N} \) of the Lagrange space \( L^n \).

By means of (2.2) and (2.3) we have

\[
\mathbf{F} = \mathbf{F} + \frac{\lambda}{4} \left[ \frac{\partial F^i}{\partial y^j} \otimes \delta y^j - \frac{\delta}{\delta x^i} \otimes \frac{\partial F^i}{\partial y^j} dx^j \right] - \frac{1}{4} \frac{\partial F^i}{\partial y^j} \otimes \frac{\partial F^i}{\partial y^j} dx^j
\]

(3.4)

Also, we have

\[
\left\{ \begin{array}{l}
\mathbf{F} \left( \frac{\delta}{\delta x^i} \right) = -\lambda \left[ \frac{\partial}{\partial y^i} + \frac{1}{4} \frac{\partial F^i}{\partial y^j} \mathbf{F} \left( \frac{\partial}{\partial y^j} \right) \right] \\
\mathbf{F} \left( \frac{\partial}{\partial y^i} \right) = \lambda \left[ \frac{\delta}{\delta x^i} + \frac{1}{4} \frac{\partial F^i}{\partial y^j} \frac{\partial}{\partial y^j} \right]
\end{array} \right.
\]

(3.5)
The condition of integrability for the $2\pi$ structure $F$ is given by
\[
N_F(X,Y) = F^2[X,Y] + [F X, FY] - F[F X, Y] - F[X, FY] = 0
\]
(3.6)
where $N_F$ is the Nijenhuis tensor. Let us calculate the integrability conditions $N_F(X,Y) = 0$, by considering the following values for the pair $(X,Y)$:
\[
\left( \frac{\delta}{\delta x_i}, \frac{\delta}{\delta x_j} \right) \left( \frac{\delta}{\delta y^a}, \frac{\delta}{\delta y^b} \right) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y^a} \right). 
\]
At first time, we have
\[
F^2 \left( \frac{\delta}{\delta x_i} \right) = -\lambda^2 \frac{\delta}{\delta x_i}, F^2 \left( \frac{\delta}{\delta y^a} \right) = -\lambda^2 \frac{\delta}{\delta y^a}
\]
(3.7)
and for \[
\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right], \left[ \frac{\delta}{\delta y^b}, \frac{\delta}{\delta y^c} \right] \quad \text{and} \quad \left[ \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^l} \right] \quad \text{one obtains}
\]
\[
\begin{align*}
\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right] &= R_{jk} \frac{\partial}{\partial y^l} \\
\left[ \frac{\delta}{\delta y^b}, \frac{\delta}{\delta y^c} \right] &= \delta N_{ij} \frac{\partial}{\partial y^l} = B_{jk} \frac{\partial}{\partial y^l} \\
\left[ \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^l} \right] &= 0
\end{align*}
\]
(3.8)
where $N_i, R_{jk}$ and $B_{jk}$ have the expression (2.1), (2.6) and (2.7).

Consequently, the following partial results are valid:
\[
F^2 \left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right] = -\lambda^2 R_{jk} \frac{\partial}{\partial y^l}, \quad \left[ F \frac{\delta}{\delta x^j}, F \frac{\delta}{\delta x^k} \right] = 0
\]
\[
F \left[ F \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right] = \lambda^2 \delta N_{ij} \frac{\delta}{\delta y^l} + F \left[ \frac{\delta}{\delta x^j}, F \frac{\delta}{\delta x^k} \right] = -\lambda^2 \delta N_{ij} \frac{\delta}{\delta x^l} \\
F^2 \left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^b} \right] = -\lambda^2 \delta N_{ij} \frac{\partial}{\partial y^l}, \quad \left[ F \frac{\delta}{\delta x^j}, F \frac{\delta}{\delta y^b} \right] = \lambda^2 \delta N_{ij} \frac{\partial}{\partial y^l}
\]
\[
F \left[ F \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^b} \right] = 0, \quad F \left[ \frac{\delta}{\delta x^j}, F \frac{\partial}{\partial y^b} \right] = -\lambda^2 R_{jk} \frac{\delta}{\delta x^l} \\
F^2 \left[ \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^b} \right] = 0, \quad \left[ F \frac{\partial}{\partial y^j}, F \frac{\partial}{\partial y^b} \right] = \lambda^2 R_{jk} \frac{\partial}{\partial y^l}
\]
Applying them, we conclude that

\[
\begin{align*}
F \left[ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k} \right] &= -\lambda^2 \frac{\partial N_i^j}{\partial y^k} \frac{\partial}{\partial x^l} - \lambda^2 \frac{\partial N_i^j}{\partial y^k} \frac{\partial}{\partial x^l} \\
F \left[ \frac{\partial}{\partial y^i}, F \frac{\partial}{\partial y^k} \right] &= \lambda^2 \frac{\partial N_i^j}{\partial y^k} \frac{\partial}{\partial x^l} - \lambda^2 \frac{\partial N_i^j}{\partial y^k} \frac{\partial}{\partial x^l}
\end{align*}
\]

Taking into account the expression of the torsion tensor \( t_{ih} = \frac{\partial N_i^j}{\partial y^k} - \frac{\partial N_i^j}{\partial y^k} \)
the following result follows:

**Theorem 3.1.**
An almost \( 2\pi \)-structure \( F \) of the mechanical system \( \Sigma \) is integrable if and only if the curvature tensor \( R_{ik} \) is given by:

\[
R_{ik}(x, y) = R_{ik}(x, y) + \frac{1}{4} \left( \frac{\partial F^m}{\partial y^j} B_{jm}^i - \frac{\partial F^m}{\partial y^j} B_{jm}^i \right) - \frac{1}{4} \frac{\partial}{\partial x^l} \left( \frac{\partial F^i}{\partial y^j} \right) + \frac{1}{4} \frac{\partial}{\partial x^l} \left( \frac{\partial F^i}{\partial y^j} \right)
\]

is equal to zero.

**§4. The Metrical 2-\( \pi \) Structure Associated to Mechanical System \( \Sigma \)**

Let us consider the N-lift \( G \) of the fundamental tensor \( g_{ij} \) of \( L^m \)

\[
G = g_{ij} dx^i \otimes dx^j + g_{ij} dy^i \otimes dy^j
\]

We remark that \( G \) is a Riemannian structure on \( \tilde{T}M \) which depend only of the mechanical system \( \Sigma \).

By means of (2.4) \( G \) can be written in the form:

\[
G = g_{ij} dx^i \otimes dx^j + g_{ij} \left( \delta y^i - \frac{1}{4} \frac{\partial F^i}{\partial y^k} dx^k \right) \otimes \left( \delta y^j - \frac{1}{4} \frac{\partial F^j}{\partial y^k} dx^k \right).
\]

We have

**Theorem 4.2.** The pair \( (F, G) \) is an almost metric 2- \( \pi \) structure of the mechanical system \( \Sigma \).
Proof. It is sufficient to prove the formula \( G(FX, FY) = \lambda^2 G(X, Y) \) using the adapted basis \( \left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) \).

We get
\[
G\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}, \quad G\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) = 0, \quad G\left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g_{ij},
\]
\[
G\left( F\left( \frac{\delta}{\delta x^i} \right), F\left( \frac{\delta}{\delta y^j} \right) \right) = \lambda^2 G\left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = \lambda^2 g_{ij} = \lambda^2 G\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right)
\]
\[
G\left( F\left( \frac{\partial}{\partial y^i} \right), F\left( \frac{\partial}{\partial y^j} \right) \right) = 0
\]
\[
G\left( F\left( \frac{\delta}{\delta x^i} \right), F\left( \frac{\partial}{\partial y^j} \right) \right) = \lambda^2 G\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \right) = \lambda^2 g_{ij} = \lambda^2 G\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right)
\]
q.e.d

REFERENCES

7. Miron, R., Frigioiu C. Finslerian Mechanical Systems. Algebras, Groups and Geometries, Nr.4, 2005

2-π STRUKTURA PRIDRUŽENA LAGRANE-OVOM MEHANIČKOM SISTEMU

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Definisano je kretanje dinamičkog sistema 2-π structure u faznom prostoru mehaničkog sistema i izučavana je njegova integrabilnost.

Ključne reči: 2-π structure, 2-π structure pridružene Lagrange-ovom mehaničkom sistemu.