FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 81 - 88

ON THE INSTANTANEOUS SCREW AXES OF TWO PARAMETER MOTIONS

UDC 621.882:530.145.6+519.233(045)=111

Murat Kemal Karacan[1], Yusuf Yayli[2]

[1]Kahramanmaras Sutcu Imam University, Faculty of Sciences and Letters,
Department of Mathematics, Avsar Campus, 46100 K.Maras, Turkey
E-mail: mkkaracan@ksu.edu.tr
[2]Ankara University, Faculty of Science, Department of Mathematics,
06100 Tandogan, Ankara, Turkey
E-mail: yayli@science.ankara.edu.tr

Abstract. *In this study, two parameter motions by using the rank of rotation matrix were analysed and some theorems were given. The locus of the instantaneous screw axes of two parameter motions for special case n=3 were investigated. Furthermore, the locus of the instantaneous screw axes is a ruled surface in the position (λ,μ) = (0,0) were shown.*

Key words: *Two parameter motion, one parameter motion, instantaneous screw axis (I.S.A), ruled surface*

2000 Mathematics Subject Classifications: 53A17

1. INTRODUCTION

In the Euclidean n-space , the two parameter motion of a rigid body is defined by

$$Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu) \tag{1.1}$$

[2], where $A \in SO(n)$ is a positive orthogonal matrix, $C \in IR_1^n$ is a column matrix, Y and X are position vectors of the same point B for the fixed and moving space with respect to orthonormal coordinate systems, respectively. The two parameter motion for (λ,μ) = (0,0) given by

$$A(0,0) = A^{-1}(0,0) = A^T(0,0) = I \text{ and } C(0,0) = 0 .$$

Thus fixed and moving spaces are coincided. If $\lambda = \lambda(t)$, $\mu = \mu(t)$, then the motion is called one parameter motions is obtained from two parameter motion. Since $A \in SO(n)$, we have

$$A^T(\lambda,\mu)A(\lambda,\mu) = A(\lambda,\mu)A^T(\lambda,\mu) = I . \tag{1.2}$$

Received February 6, 2007

For the sake of the short, we shall take as $A^T(\lambda,\mu) = A^T$ and $A(\lambda,\mu) = A$.

Definition 1.1. The differentiation of the equation (1.1) with respect to t, then it follows that

$$\dot{Y} = Y_\lambda \dot{\lambda} + Y_\mu \dot{\mu}\,,\ \dot{A} = A_\lambda \dot{\lambda} + A_\mu \dot{\mu}\,,\ \dot{C} = C_\lambda \dot{\lambda} + C_\mu \dot{\mu}\,.$$

$$\dot{Y} = \dot{A}X + \dot{C} + A\dot{X}\,,$$

where $\lambda = \lambda(t)$ and $\mu = \mu(t)$. So $\dot{Y}$, $\dot{A}X + \dot{C}$, $A\dot{X}$ are called absolutely, sliding and the relative velocities of the point B which has position vector $\vec{b}$, respectively. The solution X of system

$$\vec{V}_f = \dot{A}X + \dot{C} = 0$$

is constant on the fixed and moving spaces at the position t. The point X is called instantaneous pole points at every position t [1].

Definition 1.2. If $rank\ \dot{A} = n - 1 = r$ be an even number in the two parameter motions given by equation (1.1), then, the locus of the points having a velocity vector with stationary norm is a line at any position in the moving space. The line is called instantaneous screw axis and denoted by I.S.A. Furthermore the screw axis of the moving space is defined by

$$X = P + \sigma E\,,$$

where P is a particular solution of equation $\dot{A}X + \dot{C} = 0$ and E shows a base of solution space of homogeneous equation of $\dot{A}X = 0$ [1].

2. The rank of the rotation matrix

Theorem 2.1. Let be $A \in SO(n)$ and n an odd number, respectively. Then, the rank of A_λ and A_μ are even.

Proof. The differentiation of the equation (1.1) with respect to λ and μ, we have

$$A_\lambda A^T + A A_\lambda^T = 0 \tag{2.2}$$

and

$$A_\mu A^T + A A_\mu^T = 0\,. \tag{2.3}$$

Let be $W = A_\lambda A^T$, from equation (2.2)

$$A_\lambda A^T + (A_\lambda A^T) = 0$$

$$W + W^T = 0.$$

Then, $W = A_\lambda A^T$ is skew-symmetric matrix. Similarly, $A_\mu A^T$ is skew-symmetric matrix from (2.3). Since n is odd number, it follows that

$$\det(A_\lambda A^T) = 0$$

and

$$\det(A_\mu A^T) = 0\,,$$

we have $\det(A_\lambda) = 0$ and $\det(A_\mu) = 0$. Thus, it must be *rank* $A_\lambda = r$ (*even*) and *rank* $A_\mu = r$ (*even*).

Theorem 2.2. Let be $A \in SO(n)$. Then

$$rank \quad A_{\lambda\lambda} = 0 \quad \Leftrightarrow \quad rank \quad A_\lambda = 0 \text{ (} A_\lambda \text{ is a zero matrix).}$$

and

$$rank \quad A_{\mu\mu} = 0 \quad \Leftrightarrow \quad rank \quad A_\mu = 0 \text{ (} A_\mu \text{ is a zero matrix).}$$

Proof. Since

$$A(\lambda,\mu)A^T(\lambda,\mu) = I \,, \tag{2.1}$$

the differentiation of the equation (2.1) with respect to λ, it follows that

$$A_\lambda A^T + AA_\lambda^T = 0$$

$$A_{\lambda\lambda}A^T + A_\lambda A_\lambda^T + A_\lambda A_\lambda^T + AA_{\lambda\lambda}^T = 0$$

$$A_{\lambda\lambda}A^T + 2A_\lambda A_\lambda^T + AA_{\lambda\lambda}^T = 0\,. \tag{2.2}$$

Since *rank* $A_{\lambda\lambda} = 0$, we get $A_{\lambda\lambda} = 0$. We have the following from (2.2)

$$A_\lambda A_\lambda^T = 0\,.$$

For every $x \in IR_n^1$, we have

$$(A_\lambda A_\lambda^T)x^T = (0)x^T$$

$$(A_\lambda A_\lambda^T)x^T = 0$$

$$x(A_\lambda A_\lambda^T)x^T = 0$$

$$xA_\lambda A_\lambda^T x^T = 0$$

$$(xA_\lambda)(xA_\lambda)^T = 0.$$

So that

$$\langle xA_\lambda , xA_\lambda \rangle = 0$$

$$xA_\lambda = 0.$$

Since it is true for every $x \in IR_n^1$, we have

$$A_\lambda = 0 \text{ and } rank \quad A_\lambda = 0\,.$$

The differentiation of the equation (2.1) with respect to, it follows that

$$A_\mu A^T + AA_\mu^T = 0$$

$$A_{\mu\mu}A^T + A_\mu A_\mu^T + A_\mu A_\mu^T + AA_{\mu\mu}^T = 0$$

$$A_{\mu\mu}A^T + 2A_\mu A_\mu^T + AA_{\mu\mu}^T = 0\,,$$

similarly. Since *rank* $A_{\mu\mu} = 0$, we have $A_{\mu\mu} = 0$.Thus we have

$$AA = 0.$$

Since it is true for every $x \in IR$, we have the following

$$(A_\mu A_\mu^T)x^T = (0)x^T$$

$$(A_\mu A_\mu^T)x^T = 0$$

$$x(A_\mu A_\mu^T)x^T = 0$$

$$xA_\mu A_\mu^T x^T = 0$$

$$(xA_\mu)(xA_\mu)^T = 0\,.$$

Hence

$$\left\langle xA_\mu, xA_\mu \right\rangle = 0$$

$$xA_\mu = 0.$$

It is true for every $x \in IR_n^1$, we have the following

$$A_\mu = 0 \text{ and } rank \quad A_\mu = 0\,.$$

Conversely, it is obvious.

3. THE INSTANTANEOUS SCREW AXES OF TWO PARAMETER MOTIONS

Now, let's investigate a special case for $n = 3$. Let A_λ and $A_\mu$be skew-symmetric matrices and $C = 0$. We can calculate angular velocity matrix from equation $Y(\lambda,\mu) = A(\lambda,\mu)X$, it follows that

$$X = A^{-1}(\lambda,\mu)Y(\lambda,\mu)\,.$$

Differentiation of the equation $Y(\lambda,\mu) = A(\lambda,\mu)X$ with respect to t, we have

$$Y_\lambda \dot{\lambda} + Y_\mu \dot{\mu} = (A_\lambda \dot{\lambda} + A_\mu \dot{\mu})X = (A_\lambda \dot{\lambda} + A_\mu \dot{\mu})A^{-1}(\lambda,\mu)Y(\lambda,\mu).$$

$$Y_\lambda \dot{\lambda} + Y_\mu \dot{\mu} = \Omega \quad Y(\lambda,\mu).$$

Since A_λ and A_μ are skew-symmetric matrices, we can write

$$\Omega = (A_\lambda \dot{\lambda} + A_\mu \dot{\mu})A^{-1}(\lambda,\mu) = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix},$$

where Ω is an angular velocity matrix and also angular velocity matrix is a skew-symmetric. Since matrix A is orthogonal, we can write $AA^T = I$, by differentiating with respect to t, it follows that,

$$(A_\lambda \dot{\lambda} + A_\mu \dot{\mu})A^T + A(A_\lambda^T \dot{\lambda} + A_\mu^T \dot{\mu}) = 0 .$$

Let be $W = (A_\lambda \dot{\lambda} + A_\mu \dot{\mu})A^T$, then

$$W + W^T = 0 .$$

The Sliding velocity are given by

$$\vec{V}_f = (A_\lambda \dot{\lambda} + A_\mu \dot{\mu})X + (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) = 0 . \quad (3.1)$$

The solution X of the equation (3.1) is pole points. Thus, it must be

$$rank(A_\lambda \dot{\lambda} + A_\mu \dot{\mu}) = rank\Omega = 2 .$$

Then

$$Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu)$$

$$A(\lambda,\mu)X = Y(\lambda,\mu) - C(\lambda,\mu)$$

$$A^{-1}(\lambda,\mu)A(\lambda,\mu)X = A^{-1}(\lambda,\mu)(Y(\lambda,\mu) - C(\lambda,\mu))$$

$$X = A^{-1}(\lambda,\mu)(Y(\lambda,\mu) - C(\lambda,\mu)).$$

If we write this value X in the equation (3.1), we have

$$(A_\lambda \dot{\lambda} + A_\mu \dot{\mu})[A^{-1}(\lambda,\mu)\left(Y(\lambda,\mu) - C(\lambda,\mu)\right)] + (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) = 0 . \quad (3.2)$$

If we say

$$Y(\lambda,\mu) - C(\lambda,\mu) = Y^*(\lambda,\mu) ,$$

then the equation (3.2), we have

$$\Omega \wedge Y^*(\lambda,\mu) + (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) = 0 . \quad (3.3)$$

The solution of equation (3.3) gives fixed pole points of the motion. For the solution of equation (3.3), it must be verified the condition below

$$\left\langle \Omega,\ C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle = 0 .$$

In general, this condition can't verify. But we can separate the vector $C_\lambda \dot{\lambda} + C_\mu \dot{\mu}$ in two composites which are perpendicular and parallel to the $\vec{\Omega}$, it follows that

$$U = (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) \ - \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega,\Omega \right\rangle}\Omega$$

and

$$V = \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega,\Omega \right\rangle}\Omega .$$

U is perpendicular to $\bar{\Omega}$ and V is parallel to $\bar{\Omega}$, in which

$$C_\lambda \dot{\lambda} + C_\mu \dot{\mu} = (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) - \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega, \Omega \right\rangle} \Omega + \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega, \Omega \right\rangle} \Omega .$$

That is

$$\begin{aligned} \left\langle \Omega, U \right\rangle &= \left\langle \Omega, (C_\lambda \dot{\lambda} + C_\mu \dot{\mu}) - \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega, \Omega \right\rangle} \Omega \right\rangle \\ &= \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle - \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega, \Omega \right\rangle} \left\langle \Omega, \Omega \right\rangle \\ &= \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle - \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle \\ &= 0 \end{aligned}$$

and

$$V = \frac{\left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle}{\left\langle \Omega, \Omega \right\rangle} \Omega$$

$$V = \sigma \bar{\Omega}.$$

If we write the vector U replacing by $C_\lambda \dot{\lambda} + C_\mu \dot{\mu}$ in equation (3.3), we have

$$\Omega \wedge Y^*(\lambda, \mu) + U = 0 .$$

Since the condition $\langle \Omega, U \rangle = 0$ is verified and $rank\, \Omega = 2$, then the system $\Omega \wedge Y^*(\lambda,\mu) + U = 0$ can solve. Now let's solve the equation $\Omega \wedge Y^*(\lambda,\mu) + U = 0$, so we have

$$\Omega \wedge Y^*(\lambda, \mu) + U = 0$$

$$\Omega \wedge Y^*(\lambda, \mu) + U = 0$$

$$\Omega \wedge (\Omega \wedge Y^*(\lambda, \mu) + U) = 0$$

$$\Omega \wedge (\Omega \wedge Y^*(\lambda, \mu)) + \Omega \wedge U = 0$$

$$\left\langle \Omega, Y^*(\lambda, \mu) \right\rangle \Omega - \left\langle \Omega, \Omega \right\rangle Y^*(\lambda, \mu) + \Omega \wedge U = 0$$

$$Y^*(\lambda, \mu) = \frac{\left\langle \Omega, Y^*(\lambda, \mu) \right\rangle \Omega + \Omega \wedge U}{\left\langle \Omega, \Omega \right\rangle}$$

$$Y^*(\lambda,\mu) = \frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + \frac{\langle \Omega, Y^*(\lambda,\mu) \rangle \Omega}{\langle \Omega,\Omega \rangle}$$

$$Y^*(\lambda,\mu) = \frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + \sigma\Omega \,, \tag{3.4}$$

where $a \wedge (b \wedge c) = \langle a, c\rangle\, b - \langle a, b\rangle\, c$. If we rewrite the equation (3.4) in the equation $Y^*(\lambda,\mu) = Y(\lambda,\mu) - C(\lambda,\mu)$, we have

$$Y(\lambda,\mu) = Y^*(\lambda,\mu) + C(\lambda,\mu) \;=\; \frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + C(\lambda,\mu) + \sigma\Omega..$$

Hence we get

$$Y(\lambda,\mu) = Q + \sigma\Omega \,,\ \sigma \in IR \,. \tag{3.5}$$

This means that, it is a line which passes through the point Q with directrix Ω. The line is called a fixed pole axis in the fixed space, the expression of the fixed pole axis in the moving space can be found by writing instead of the value $Y(\lambda,\mu)$ in the equation (1.1).

$$Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu)$$

$$\frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + C(\lambda,\mu) + \sigma\Omega = A(\lambda,\mu)X + C(\lambda,\mu)$$

$$\frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + \sigma\Omega = A(\lambda,\mu)X$$

$$X = A^{-1}(\lambda,\mu)\frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + A^{-1}(\lambda,\mu)\sigma\Omega$$

$$X = A^{-1}(\lambda,\mu)\frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + \sigma \quad A^{-1}(\lambda,\mu)\Omega \,.$$

In the position $(\lambda,\mu) = (0,0)$, it follows that

$$\begin{aligned} X &= \frac{\Omega \wedge U}{\langle \Omega,\Omega \rangle} + \sigma\Omega \\ X &= P + \sigma\Omega \end{aligned} \,. \tag{3.6}$$

Since $C(0,0) = 0$ in the position $(\lambda,\mu) = (0,0)$, then $P = Q$ and the pole axis of the fixed and moving spaces are a coincidence. Thus X and Y are the pole axes in fixed and moving spaces which the lines pass through the points Q and P with directrix $\vec{\Omega}$. The locus of the lines which have a velocity vector with stationary norm in the position $(\lambda,\mu) = (0,0)$ of the motion are called instantaneous screw axes. The equations

$$Y = Q + \sigma \quad \Omega \,,\ \sigma \in IR$$

$$X = P + \sigma \quad \Omega , \ \sigma \in IR .$$

depend on parameters $\dot{\lambda}$ and $\dot{\mu}$. Since the parameters $\dot{\lambda}$ and $\dot{\mu}$ depend on *t*, there are ∞^2 one parameter motions and ∞ screw axes, respectively [2]. The locus of the instantaneous screw axes is a ruled surface. Indeed the following equations determine a ruled surface.

$$Y(t,\sigma) = Q(t) + \sigma \quad \Omega(t) ,$$

$$X(t,\sigma) = P(t) + \sigma\Omega(t) .$$

REFERENCES

1. Hacısalihoğlu , H.H., "On The Geometry of Motion In The Euclidean n-Space". Communications, Ankara Universitesi, Seri 23 A, p: 95-108, 1974.
2. Bottema, O., Roth, B., "Theotical Kinematics", North Holland Publ. Com. 1979.
3. Bukcu,B., " Cayley Formula and its Applications in E_1^3, Ankara University, Graduate School and The Natural Science, Ph. D. Thesis, 2003.

O SKUPU TRENUTNIH OSA DVOPARAMETARSKOG KRETANJA

Murat Kemal Karacan, Yusuf Yayli

U radu su izučavana dvoparametarska kretanja korišćenjem ranga matrice rotacije, i definisano je nekoliko teorema. Položaj skupa trenutnih osa dvoparametarskog kretanja za specijalan slučaj n=3 *je izučen. Pokazano je da je položaj skupa trenutnih osa površ rulete u položaju (*λ,μ*)* = (0,0).

Ključne reči: *dvoprametarsko kretanje, jednoparametarsko kretanje, skup trenutnih osa (I.S.A), površi ruleta.*