FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 45 - 64

THE TRANSVERSAL VIBRATIONS OF A NON-CONSERVATIVE DOUBLE CIRCULAR PLATE SYSTEM

UDC 531+534+517.93(045)=111

Katica (Stevanović) Hedrih, Julijana Simonović

Faculty of Mechanical Engineering University of Niš
Mathematical Institute SANU Belgrade
Yu-18 000- Niš, ul. Vojvode Tankosiċa 3/22,
Telefax: 381 18 241 663, Mobile 063 8 75 75 99
e-mail: katica@masfak.ni.ac.yu * khedrih@eunet.yu * sjupe@ptt.yu

Abstract. *The interest in the study of coupled plates as new qualitative systems has grown exponentially over the last few years because of the theoretical challenges involved in the investigation of such systems. As introduction, a review of first author's research results in area of transversal vibrations of different double plate systems is presented (see Refs. [2-7]).*
The main result of this contribution is the analytical solution of the coupled homogeneous and nonhomogeneous partial differential equations of the free and forced vibrations of the double circular plate system coupled by elastic or visco-elastic layer [1]. This solution is obtained by the use of the method of Bernoulli's particular integral as well as Lagrange's method of the constants variation. Some numerical examples are presented along with visualizations of the double plate free and forced vibrations.
The obtained analytical and numerical result is very valuable for university teaching process in the area of structural system elastodynamics as well as of hybrid deformable body system vibrations.

Key words: *Coupled subsystems, coupled dynamics, circular plate, hybrid, multi frequency, forced vibrations, Lagrange's method, analytical solution, numerical experiment*

1. INTRODUCTION

The interest in the study of coupled plates as new qualitative systems (see Refs. [2-11]) has grown exponentially over the last few years because of the theoretical challenges involved in the investigation of such systems. The recent technological innovations have caused a considerable interest in the study of components and hybrid dynamical processes of coupled rigid and deformable bodies (plates, beams and belts) denoted as hybrid systems, characterized by the interaction between subsystem dynamics, governed by coupled partial differential equations with boundary and initial conditions.

Received January 20, 2007

In the papers [7] and [9] by using the example of hybrid systems as static as well as dynamically coupled discrete subsystem of rigid bodies and continuous subsystem, the method for obtaining frequency equations of small oscillations is presented. Also, the series of theorems of small oscillations frequency equations are defined. By using examples, the analogy between frequency equations of some classes of these systems is identified. The special cases of discretization and continuation of coupled subsystems in the light of these sets of proper circular frequencies and frequency equations of small oscillations are analyzed [8].

The paper by K. Poltorak and K. Nagaya (1985) [14] was concerned with a method for solving forced vibration problems of solid sandwich plates with irregular boundaries. The exact, general solution of the equation of motion in terms of Bessel functions is found. The boundary problem is solved by using the Fourier expansion collocation method. The damping properties of an intermediate, viscoelastic layer are taken into consideration by means of a concept of a complex shear modulus. This paper by K. Poltorak and K. Nagaya (1985) [15] deals with a method for solving free vibration problems of three-layered isotropic plates of arbitrary shape with clamped edges. The direct solution of the Yan and Dowell equation of motion, in terms of Bessel functions, is found.

Composite materials are widely employed in the new lightweight structure technology for the construction of many structural members such as multilayered plates and shells. Usually, these structures have complex geometries and lay ups in order to meet specific design requirements and this leads to an anisotropic global behaviour, which is generally characterized by bending–stretching coupling. Then, the structural dynamic analysis plays a crucial role in the design and tailoring of this kind of structures in order to obtain the desired response.

The study of transversal vibrations of an elastically connected double plate system is important for both theoretical and pragmatic reasons (see Refs. [2], [3], and [13]). Many important structures may be modelled as composite structure. Like that system it is possible to use a visco-elastically connected double plate system as elements for acoustic and vibrations' isolation in a system, as a wall or ground; this is the subject of our research presented in this paper.

The obtained results have particular practical importance especially if the models refer to structures made of material with creeping features (see Ref. [12]).

2. THEORETICAL PROBLEM FORMULATION AND GOVERNING EQUATIONS OF THE BASIC PROBLEM

Let's consider two isotropic, elastic, thin circular plates, with width h_i, $i = 1,2$, modulus of elasticity E_i, Poisson's ratio μ_i and shear modulus G_i, plate mass distribution ρ_i. The plates are of constant thickness in the z-direction (see *Fig.* 1. *a*)). The contours of the plates are parallel. The plates are interconnected by a linear elastic Winkler type layer with constant surface stiffness c. This elastically connected double plate system is a composite structure type, or sandwich plate, or layered plate, and here it will be a first considered problem.

The origins of the two coordinate systems are located at the corresponding centres in the undeformed plate's middle surfaces, as shown in *Fig.* 1.*a*), and have parallel corresponding axes. The problem at hand is to determine solutions and the own vibration frequencies for such a double plate system elastically connected by an elastic spring layer distributed along plates contour surfaces.

The use of Love-Kirchhoff approximation makes the classical plate theory essentially a two dimensional model, in which the normal and transverse forces and bending and twisting moments on plate cross sections (see Ref. [16]) can be found in term of the displacement $w_i(r, \varphi, t)$, $i = 1,2$ of the middle surface points, which is assumed to be a function of two coordinates, r and φ and time t .

The plates are assumed to have the same contour forms and boundary conditions.

Let us suppose that the plate middle surfaces are plane in the undeformed state. If the plates transverse deflections $w_i(r, \varphi, t)$, $i = 1,2$ are small compared to the plates thicknesses, h_i, $i = 1,2$,(see Ref. [16]) and that plate vibrations occur only in the vertical direction.

Let us denote with $\mathrm{D}_{(i)} = \dfrac{\mathbf{E}_i h_i^3}{12(1-\mu_i^2)}$, $i = 1,2$ the bending cylindrical rigidity of plates. On the basis of previous assumptions, we suppose that plate displacements u_i (r, φ, z, t), $i = 1,2$ and v_i (r, φ, z, t), $i = 1,2$ of the generic plate point N_i (r, φ, z), $i = 1,2$ in the radial and circular direction can be expressed in function of its distance z from the corresponding plate middle surface and its transversal displacement $w_i(r, \varphi, t)$, $i = 1,2$ in direction of the axis z, and also the same displacement of the corresponding point $N_{i0}(r, \varphi, 0)$, $i = 1,2$ in the plate middle surface.

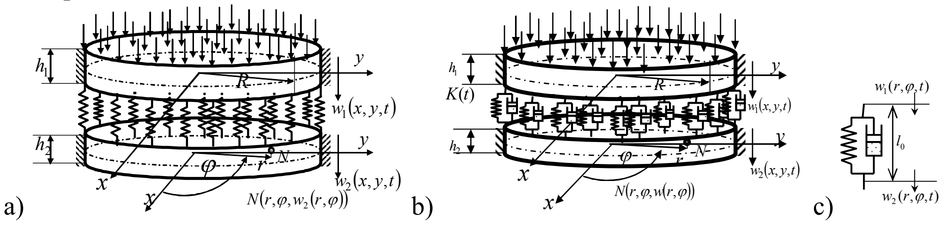

Fig. 1. a) An elastically connected double circular plate system; b) A visco-elastically connected double circular plate system; c) model of visco-elastic interconnected layer

The governing equations (see Ref. [2]) are formulated in terms of two unknowns: the transversal displacements $w_{(i)}(r, \varphi, t)$, $i = 1,2$ in direction of the axis z, of the upper plate middle surface and of the lower plate middle surface. The system of two coupled partial differential equations is derived by using d'Alembert's principle or by variational principle (see Ref. [16]). These partial differential equations of the elastically connected double plate system are in the following forms:

$$\rho_1 h_1 \frac{\partial^2 w_1(r,\varphi,t)}{\partial t^2} + \mathrm{D}_1 \Delta\Delta w_1(r,\varphi,t) - c[w_2(r,\varphi,t) - w_1(r,\varphi,t)] = 0$$

$$\rho_2 h_2 \frac{\partial^2 w_2(r,\varphi,t)}{\partial t^2} + \mathrm{D}_2 \Delta\Delta w_2(r,\varphi,t) + c[w_2(r,\varphi,t) - w_1(r,\varphi,t)] = 0 \qquad (1)$$

where is c - constant surface mechanical stiffness of elastic layer.

Let us introduce the following notations: $a_{(i)}^2 = \dfrac{c}{\rho_i h_i}$, $i = 1,2$ and $c_{(i)}^4 = \dfrac{\mathrm{D}_i}{\rho_i h_i}$, $i = 1,2$ By decoupling the equations of the previous system (1) we obtain the corresponding two partial differential equations of the decoupled plate system, which describe two partial plates founded on the elastic foundation of the Winkler type. These partial differential equations are in the following forms:

$$\frac{\partial^2 w_i(r,\varphi,t)}{\partial t^2} + c_{(i)}^4 \Delta\Delta w_i(r,\varphi,t) + a_{(i)}^2 w_i(r,\varphi,t) = 0 \,, \; i = 1,2 \qquad (2)$$

3. THE PARTICULAR SOLUTIONS OF GOVERNING BASIC DECOUPLED EQUATIONS

The solution of the previous system of partial-differential equations can be looked for by Bernoulli's method of particular integrals in the form of multiplication of two functions, of which the first $\mathbf{W}_{(i)}(r, \varphi)$, $i = 1,2$ depends only on space coordinates r and φ, and the second is a time function $T_{(i)}(t)$, $i = 1,2$ (see Refs. [2] and [3]):

$$w_i(r,\varphi,t) = W_{(i)}(r,\varphi)T_{(i)}(t)\,,\ i = 1,2 \tag{3}$$

The assumed solution is introduced in the previous system of equations (1) and (2) and after transformation we obtain the following:

$$\frac{\ddot{T}_{(i)}(t)}{T_{(i)}(t)} + c^4_{(i)}\frac{\Delta\Delta W_{(i)}(r,\varphi)}{W_{(i)}(r,\varphi)} + a^2_{(i)} = 0\,,\ i = 1,2 \tag{4}$$

Thus, we obtain in the space cylindrical-polar coordinates r, φ and z the following differential equations:

$$\ddot{\mathrm{T}}_{(i)}(t) + \omega^2_{(i)}\,\mathrm{T}_{(i)}(t) = 0$$

$$\Delta\Delta\,\mathrm{W}_{(i)}(r,\varphi) - k^4_{(i)}\,\mathrm{W}_{(i)}(r,\varphi) = 0\,,\ i = 1,2 \tag{5}$$

where eigen circular frequencies of the corresponding basic system of decoupled plates are:

$$\omega^2_{(i)} = k^4_{(i)}c^4_{(i)} + a^2_{(i)} = k^4_{(i)}\frac{\mathrm{D}_{(i)}}{\rho_{(i)}h_{(i)}} + \frac{c}{\rho_{(i)}h_{(i)}} = k^4_{(i)}\frac{\mathbf{E}_{(i)}h^2_{(i)}}{12\rho_{(i)}(1-\mu^2_{(i)})} + \frac{c}{\rho_{(i)}h_{(i)}}\,,\ i = 1,2 \tag{6}$$

It is easy to find the following time functions:

$$\mathrm{T}_{(i)}(t) = A_{(i)}\cos\omega_{(i)}t + B_{(i)}\sin\omega_{(i)}t\,,\ i = 1,2 \tag{7}$$

4. THE SPACE COORDINATE EIGEN AMPLITUDE FUNCTIONS

Let's consider the space coordinate amplitude functions $\mathbf{W}_{(i)}(r, \varphi)$, $i = 1,2$. For the plates in circular form, the set of the partial differential equations in the space cylindrical-polar coordinates r, φ and z is:

$$\Delta\mathbf{W}_{(i)}(r,\varphi) \pm k^2\mathbf{W}_{(i)}(r,\varphi) = 0\,,\ i = 1,2 \tag{8}$$

where Δ is the differential operator $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\varphi^2}$.

We write the solutions of previous equations in the form $\mathbf{W}_{(i)}(r, \varphi) = \Phi_{(i)}(\varphi)\,\mathbf{R}_{(i)}(r)$ and after applying this solution we obtain the following system of ordinary differential equations:

$$\Phi''_{(i)}(\varphi) \pm n^2\Phi_{(i)}(\varphi) = 0 \ \text{ and } \ \mathrm{R}''_{(i)}(r) + \frac{1}{r}\mathrm{R}'_{(i)}(r) + (\pm k^2_{(i)} \mp \frac{n^2}{r^2})\mathrm{R}_{(i)}(r) = 0\,,\ i = 1,2 \tag{9}$$

The second equation of previous system has particular solutions in the form of Neuman's and Bessel's functions, but Neuman's functions for $r = 0$ have infinite value, then particular solutions of this problem are only Bessel's function of the first kind with real argument $\mathbf{J}_n(x)$ as well as with imaginary arguments $\mathbf{I}_n(x)$, where $x = kr$. The modified Bessel's function of the first kind with imaginary arguments $\mathbf{I}_n(x)$, with order n, is in the following form:

$$\mathbf{I}_n(x) = (i)^{-n}\mathbf{J}_n(ix) = \frac{(-1)^n}{2\pi}\int_{-\pi}^{+\pi} e^{-x\cos t}\cos ntdt \tag{10}$$

If n is an integer number, than this function satisfies the following differential equation:

$$\mathbf{I}''_n(ix) + \frac{1}{(ix)}\mathbf{I}'_n(ix) - \left(1 + \frac{n^2}{(ix)^2}\right)\mathbf{I}_n(ix) = 0 \tag{11}$$

By using previous considerations and the study of equations (9), we can write their solutions in the polar coordinates as follows:

$$\Phi_{(i)n}(\varphi) = C_{(i)n}\sin(n\varphi + \varphi_{(i)0n}) \quad \text{and}$$

$$\mathrm{R}_{(i)\mathrm{nm}}(r) = \mathrm{J_n}(k_{(i)nm}r) + K_{(i)nm}\,\mathrm{I_n}(k_{(i)nm}r)\,,\ i = 1,2 \tag{12}$$

So the solutions for the space coordinate amplitude functions are in the following forms:

$$\mathbf{W}_{(i)\mathrm{nm}}(r,\varphi) = [\mathrm{J_n}(k_{(i)nm}r) + K_{(i)nm}\,\mathrm{I_n}(k_{(i)nm}r)]\sin(n\varphi + \varphi_{(i)0n})\,,\ i = 1,2 \tag{13}$$

which are the space coordinate-eigen amplitude normal functions for boundary conditions in the form constrained along the contour circular plate. The characteristic numbers are the roots of the next characteristic transcendent equation (see ref. [10])

$$\Delta_n(k_n a) = f_n(k_n a) = k_n\begin{vmatrix}\mathbf{J}_n(k_n a) & \mathbf{I}_n(k_n a)\\ \mathbf{J}'_n(k_n a) & \mathbf{I}'_n(k_n a)\end{vmatrix} = 0 \quad n = 1,2,3,4,.... \tag{13a}$$

The family (13a) of characteristic equations for each n has an infinite number of solutions (roots) and we are going to mark them with k_{nm}, $m = 1,2,3,...$ denoting a family of eigen values for each $n = 1,2,3,4,....$ The sets of equation (13a) of eigen values for each $n = 1,2,3,4,...$ can be rewritten in the form:

$$\Delta(\lambda_n) = f_n(\lambda_n) = k_n\begin{vmatrix}\mathbf{J}_n(\lambda_n) & \mathbf{I}_n(\lambda_n)\\ \mathbf{J}'_n(\lambda_n) & \mathbf{I}'_n(\lambda_n)\end{vmatrix} = 0 \tag{13b}$$

As the solutions (roots) of this equation are λ_{nm}, $n = 1,2,3,4,...$, $m = 1,2,3,...$ so we have $k_{nm} = \lambda_{nm}/a$ where $a$is the plate radius. The graphics of characteristic transcendent equations for $n = 0$, $n = 1$ and $n = 2$ are reported in figure 2 a, b and c.

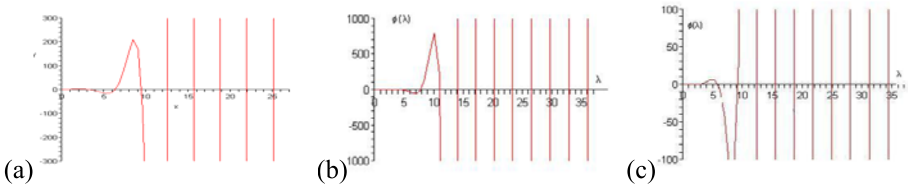

Fig. 2. The graph of characteristic transcendent equation (13b) for: (a) $n = 0$, where we can see only eight solutions (roots) λ_{0m}, $m = 1,2,...,8$; of the set with infinite number of roots, (b) $n = 1$, where we can see only eleven solutions (roots) λ_{1m}, $m = 1,2,...,11$; of the set with infinite number of roots, and (c) $n = 2$ where we can see only eleven solutions (roots) λ_{2m}, $m = 1,2,...,11$ of the set with infinite number of roots.

In Fig 2. (a) we can see from the set with infinite number of roots, corresponding to various n only a certain number of solutions (roots) denoted with λ_{nm}. For example, in Fig. 2 (a) we find the following roots $\lambda_{01} = 3.196$, $\lambda_{02} = 6.306$, $\lambda_{03} = 9.439$,..., in (b) the following roots $\lambda_{11} = 4.61$, $\lambda_{12} = 7.8$, $\lambda_{13} = 10.96$,... and in Fig. 2. (c) the following roots $\lambda_{21} = 5.9$, $\lambda_{22} = 9.2$, $\lambda_{23} = 12.4$,.... For those values of characteristic numbers the space coordinate eigen amplitude functions are represented in Fig. 3.

Last but not least, we obtain the general solutions for the transversal plates middle surface point displacement in the following forms:

$$w_{(i)}(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\mathbf{J}_n(k_{(i)_{nm}}r) + K_{(i)_{nm}}\mathbf{I}_n(k_{(i)_{nm}}r)]\sin(n\varphi + \varphi_{(i)0n})\mathrm{T}_{(i)mn}(t)\,,\ i = 1,2 \quad (14)$$

or

$$w_i(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(i)nm}(r,\varphi)\,\mathrm{T}_{(i)nm}(t) \quad (15)$$

The space coordinate eigen amplitude functions $\mathbf{W}_{(i)nm}(r, \varphi)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$ satisfy the following conditions of orthogonality:

$$\int_0^r\int_0^{2\pi}\mathrm{W}_{(i)mn}(r,\varphi)\,\mathrm{W}_{(i)sr}(r,\varphi)rdrd\varphi = \begin{cases}0 & nm \neq sr \\ v_{mnnm} & nm = sr\end{cases}, \quad (16)$$

where: $i = 1,2$, n, $m = 1,2,3,4,...\infty$, s, $r = 1,2,3,4,...\infty$, which is easily derived by using system equations (13).

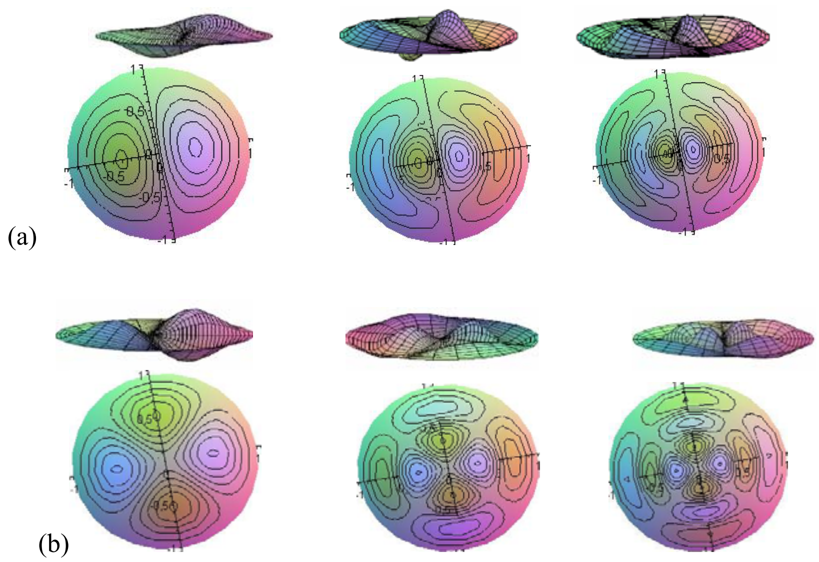

Fig. 3. The space coordinate eigen amplitude functions $W_{nm}(r,\varphi)$ for:
(a) $\lambda_{11} = 4.61$, $\lambda_{12} = 7.8$, $\lambda_{13} = 10.96$,...; (b) $\lambda_{21} = 5.9$, $\lambda_{22} = 9.2$, $\lambda_{23} = 12.4$,...
are presented above and the corresponding cross sections are presented below;

5. THE PARTICULAR SOLUTIONS OF GOVERNING SYSTEM OF COUPLED PARTIAL DIFFERENTIAL EQUATIONS FOR FREE SYSTEM OSCILLATIONS

For the solutions of the governing system of the coupled partial differential equations (1) for free double plates oscillations in the form of expansion (15) the eigen amplitude function $\mathbf{W}_{(i)nm}(r,\varphi)$, $i=1,2$, $n,m=1,2,3,4,...\infty$ are the same as in the case of decoupled plates problem and $T_{(i)nm}(t)$, $i=1,2$, $n,m=1,2,3,4,...\infty$ are unknown time functions describing their time evolution.

After introducing (15) into the following system of the coupled partial differential equations for free double plate's oscillations:

$$\frac{\partial^2 w_1(r,\varphi,t)}{\partial t^2}+c_{(1)}^4\Delta\Delta w_1(r,\varphi,t)-a_{(1)}^2[w_2(r,\varphi,t)-w_1(r,\varphi,t)]=0$$

$$\frac{\partial^2 w_2(r,\varphi,t)}{\partial t^2}+c_{(2)}^4\Delta\Delta w_2(r,\varphi,t)+a_{(2)}^2[w_2(r,\varphi,t)-w_1(r,\varphi,t)]=0 \tag{17}$$

we obtain:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi y)\ddot{\mathrm{T}}_{(1)nm}(t)+c_{(1)}^4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Delta\Delta\,\mathrm{W}_{(1)nm}(r,\varphi)\mathrm{T}_{(1)nm}(t)-$$
$$-a_{(1)}^2\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\mathrm{T}_{(2)nm}(t)-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi)\mathrm{T}_{(1)nm}(t)\right\}=0$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\ddot{\mathrm{T}}_{(2)nm}(t)+c_{(2)}^4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Delta\Delta\,\mathrm{W}_{(2)nm}(r,\varphi)\mathrm{T}_{(2)nm}(t)+$$
$$+a_{(2)}^2\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\mathrm{T}_{(2)nm}(t)-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi)\mathrm{T}_{(1)nm}(t)\right\}=0$$

By multiplying the first and second equation with $\mathrm{W}_{(i)sr}(r,\varphi)rdrd\varphi$, integrating along middle plate surface and taking into account orthogonality conditions (16) and equal boundary conditions of the plates, we obtain the mn-family of systems containing two coupled ordinary differential equations for determination of the unknown time functions $T_{(i)nm}(t)$, $i=1,2$, $n,m=1,2,3,4,...\infty$ in the following form:

$$\ddot{\mathrm{T}}_{(1)nm}(t)+[c_{(1)}^4k_{(1)nm}^4+a_{(1)}^2]\mathrm{T}_{(1)nm}(t)-a_{(1)}^2\,\mathrm{T}_{(2)nm}(t)=0$$

$$\ddot{\mathrm{T}}_{(2)nm}(t)+[c_{(2)}^4k_{(2)nm}^4+a_{(2)}^2]\mathrm{T}_{(2)nm}(t)-a_{(2)}^2\,\mathrm{T}_{(1)nm}(t)=0\,,\ n,m=1,2,3,4,...\infty,$$

or in the form:

$$\ddot{\mathrm{T}}_{(1)nm}(t)+\omega_{(1)nm}^2\,\mathrm{T}_{(1)nm}(t)-a_{(1)}^2\,\mathrm{T}_{(2)nm}(t)=0$$

$$\ddot{\mathrm{T}}_{(2)nm}(t)+\omega_{(2)nm}^2\,\mathrm{T}_{(2)nm}(t)-a_{(2)}^2\,\mathrm{T}_{(1)nm}(t)=0\quad n,m=1,2,3,4,...\infty \tag{18}$$

Eliminating the time function $T_{(2)nm}(t)$ from previous mn-family of system of the coupled second order ordinary differential equations, we obtain the mn-family of one of four order equations in the form of:

$$\ddot{\ddot{\mathrm{T}}}_{(1)nm}(t)+[\omega_{(1)nm}^2+\omega_{(2)nm}^2]\ddot{\mathrm{T}}_{(1)nm}(t)+[\omega_{(1)nm}^2\omega_{(2)nm}^2-a_{(1)}^2a_{(2)}^2]\mathrm{T}_{(2)nm}(t)=0 \tag{19}$$

with the corresponding *mn*-family frequency equation in the form of polynomial, biquadratic equation with respect to unknown own circular frequencies $\tilde{\omega}^2_{nm}$, n, m = 1,2,3,4,...∞:

$$\tilde{\omega}^4_{nm} + [\omega^2_{(1)nm} + \omega^2_{(2)nm}]\tilde{\omega}^2_{nm} + [\omega^2_{(1)nm}\omega^2_{(2)nm} - a^2_{(1)}a^2_{(2)}] = 0 \tag{20}$$

Having the two roots $\tilde{\omega}^2_{nm(s)}$, n, m = 1,2,3,4,...∞, s = 1,2:

$$\tilde{\omega}^2_{nm(1,2)} = \frac{[\omega^2_{(1)nm} + \omega^2_{(2)nm}] \mp \sqrt{[\omega^2_{(1)nm} - \omega^2_{(2)nm}]^2 + 4a^2_{(1)}a^2_{(2)}}}{2} \tag{21}$$

or in the form:

$$\tilde{\omega}^2_{nm(1,2)} = \frac{\{k^4_{(1)nm}[c^4_{(1)} + c^4_{(2)}] + a^2_{(1)} + a^2_{(2)}\} \mp \sqrt{\{k^4_{(1)nm}[c^4_{(1)} - c^4_{(2)}] + a^2_{(1)} - a^2_{(2)}\}^2 + 4a^2_{(1)}a^2_{(2)}}}{2} \tag{22}$$

Formally, we can write the system equation (18) by the following matrices of inertia A_{nm} and of quasielastic coefficients $\mathrm{C}_{nm}$of the dynamical system corresponding to the *mn*-family, with two degrees of freedom:

$$\mathbf{A}_{nm} = \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} \quad \mathbf{C}_{nm} = \begin{pmatrix} \omega^2_{(1)nm} & -a^2_{(1)} \\ -a^2_{(2)} & \omega^2_{(2)nm} \end{pmatrix} \tag{23}$$

and by using the solutions in the form of:

$$\mathrm{T}_{(i)nm}(t) = A_{(i)nm}\cos(\tilde{\omega}_{nm}t + \alpha_{nm}),\ i = 1,2 \tag{24}$$

where $\tilde{\omega}^2_{nm}$, n, m = 1,2,3,4,...∞ are unknown eigen circular frequencies, $A_{(i)nm}$ unknown amplitudes, and α_{nm} unknown phases. Then the frequency equation of the mn-family is in the form of:

$$f_{nm}(\tilde{\omega}^2_{nm}) = \left|\mathbf{C}_{nm} - \tilde{\omega}^2_{nm}\mathbf{A}_{nm}\right| = \begin{vmatrix} \omega^2_{(1)nm} - \tilde{\omega}^2_{nm} & -a^2_{(1)} \\ -a^2_{(2)} & \omega^2_{(2)nm} - \tilde{\omega}^2_{nm} \end{vmatrix} = 0 \tag{25}$$

which is equal to equations (20) with the sets of two roots $\tilde{\omega}^2_{nm(s)}$, n, m = 1,2,3,4,...∞, s = 1,2.

The relations of the amplitudes of each set are in the form:

$$\frac{\mathrm{A}^{(s)}_{(1)mn}}{a^2_{(1)}} = \frac{\mathrm{A}^{(s)}_{(2)mn}}{[\omega^2_{(1)nm} - \tilde{\omega}^2_{nm(s)}]} = C_{(s)} \quad n, m = 1,2,3,4,...\infty,\ s = 1,2 \tag{26}$$

If we take into account that it is:

$$\mathrm{A}^{(1)}_{(1)nm} = \mathrm{A}^{(2)}_{(1)nm} = 1$$

then we obtain:

$$A^{(1,2)}_{(2)nm} = \frac{\{k^4_{(1)nm}[c^4_{(1)} - c^4_{(2)}] + a^2_{(1)} - a^2_{(2)}\}}{2a^2_{(1)}} \pm \frac{1}{2}\sqrt{\left[\frac{k^4_{(1)nm}[c^4_{(1)} - c^4_{(2)}] + a^2_{(1)} - a^2_{(2)}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}} \tag{27}$$

The solutions of the *mn*-family mode time functions $T_{(i)nm}(t)$, i = 1,2, n, m = 1,2,3,4,...∞ are in the form of:

$$\mathrm{T}_{(1)nm}(t) = \mathrm{A}_{nm}\cos\tilde{\omega}_{nm(1)}t + \mathrm{B}_{nm}\sin\tilde{\omega}_{nm(1)}t + \mathrm{C}_{nm}\cos\tilde{\omega}_{nm(2)}t + \mathrm{D}_{nm}\sin\tilde{\omega}_{nm(2)}t$$

$$\begin{aligned}\mathrm{T}_{(2)nm}(t) &= \mathrm{A}^{(1)}_{(2)nm}[\mathrm{A}_{nm}\cos\tilde{\omega}_{nm(1)}t + \mathrm{B}_{nm}\sin\tilde{\omega}_{nm(1)}t] + \\ &+ \mathrm{A}^{(2)}_{(2)nm}[\mathrm{C}_{nm}\cos\tilde{\omega}_{nm(2)}t + \mathrm{D}_{nm}\sin\tilde{\omega}_{nm(2)}t]\end{aligned} \tag{28}$$

where the mn-family mode $n, m = 1,2,3,4,...\infty$ contains the set of unknown constants A_{nm}, B_{nm}, C_{nm}, D_{nm} defined by plates initial conditions.

Then, the particular solutions of the governing system of coupled partial differential equations for free system oscillations corresponding to plate displacements read

$$\begin{aligned}w_1(r,\varphi,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi)[\mathrm{A}_{nm}\cos\tilde{\omega}_{nm(1)}t + \mathrm{B}_{nm}\sin\tilde{\omega}_{nm(1)}t + \\ &+ \mathrm{C}_{nm}\cos\tilde{\omega}_{nm(2)}t + \mathrm{D}_{nm}\sin\tilde{\omega}_{nm(2)}t]\end{aligned} \tag{29}$$

$$\begin{aligned}w_2(r,\varphi,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\{\mathrm{A}^{(1)}_{(2)nm}[\mathrm{A}_{nm}\cos\tilde{\omega}_{nm(1)}t + \mathrm{B}_{nm}\sin\tilde{\omega}_{nm(1)}t] + \\ &+ \mathrm{A}^{(2)}_{(2)nm}[\mathrm{C}_{nm}\cos\tilde{\omega}_{nm(2)}t + \mathrm{D}_{nm}\sin\tilde{\omega}_{nm(2)}t]\}\end{aligned}$$

The initial conditions are:

$$w_1(r,\varphi,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi)[\mathrm{A}_{nm} + \mathrm{C}_{nm}] = g_1(r,\varphi)$$

$$w_2(r,\varphi,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\{\mathrm{A}^{(1)}_{(2)nm}[\mathrm{A}_{nm}] + \mathrm{A}^{(2)}_{(2)nm}[\mathrm{C}_{nm}]\} = g_2(r,\varphi)$$

$$\left.\frac{\partial w_1(r,\varphi,t)}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(1)nm}(r,\varphi)[\tilde{\omega}_{nm(1)}\,\mathrm{B}_{nm} + \tilde{\omega}_{nm(2)}\,\mathrm{D}_{nm}] = \tilde{g}_1(r,\varphi) \tag{30}$$

$$\left.\frac{\partial w_2(r,\varphi,t)}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{W}_{(2)nm}(r,\varphi)\{\mathrm{A}^{(1)}_{(2)nm}[\tilde{\omega}_{nm(1)}\,\mathrm{B}_{nm}] + \mathrm{A}^{(2)}_{(2)nm}[\tilde{\omega}_{nm(2)}\,\mathrm{D}_{nm}]\} = \tilde{g}_2(r,\varphi)$$

where $g_i(r, \varphi)$ and $\tilde{g}_i(r,\varphi)$, $i = 1,2$ are initial condition functions for middle plate points displacement and velocity, satisfying boundary conditions. Then, by initial conditions (30) and equations (29) the unknown coefficients are defined by no homogeneous algebra equation system. By using Cramer formula the set of the unknown constants A_{nm}, B_{nm}, C_{nm}, D_{nm} for mn-family mode $n, m = 1,2,3,4,...\infty$ are defined in the following form:

$$\mathrm{A}_{\mathrm{nm}} = \frac{\iint_A [\mathrm{A}^{(2)}_{(2)nm}\, g_1(r,\varphi) - g_2(r,\varphi)]\mathbf{W}_{(1)nm}(r,\varphi) r dr d\varphi}{[\mathrm{A}^{(2)}_{(2)nm} - \mathrm{A}^{(1)}_{(2)nm}]\iint_A [\mathbf{W}_{(1)nm}(r,\varphi)]^2\, r dr d\varphi};$$

$$\mathrm{C}_{\mathrm{nm}} = \frac{\iint_A [g_2(r,\varphi) - \mathrm{A}^{(1)}_{(2)nm}\, g_1(r,\varphi)]\mathbf{W}_{(1)nm}(r,\varphi) r dr d\varphi}{[\mathrm{A}^{(2)}_{(2)nm} - \mathrm{A}^{(1)}_{(2)nm}]\iint_A [\mathbf{W}_{(1)nm}(r,\varphi)]^2\, r dr d\varphi};$$

$$\mathrm{B_{nm}} = \frac{\iint_A [\mathrm{A}^{(2)}_{(2)nm}\,\tilde{g}_1(r,\varphi) - \tilde{g}_2(r,\varphi)]\mathbf{W}_{(1)nm}(r,\varphi) r dr d\varphi}{\tilde{\omega}_{nm(1)}[\mathrm{A}^{(2)}_{(2)nm} - \mathrm{A}^{(1)}_{(2)nm}]\iint_A [\mathbf{W}_{(1)nm}(r,\varphi)]^2 r dr d\varphi};$$

$$\mathrm{D_{nm}} = \frac{\iint_A [\tilde{g}_2(r,\varphi) - \mathrm{A}^{(1)}_{(2)nm}\,\tilde{g}_1(r,\varphi)]\mathbf{W}_{(1)nm}(r,\varphi) r dr d\varphi}{\tilde{\omega}_{nm(2)}[\mathrm{A}^{(2)}_{(2)nm} - \mathrm{A}^{(1)}_{(2)nm}]\iint_A [\mathbf{W}_{(1)nm}(r,\varphi)]^2 r dr d\varphi} \tag{31}$$

The solutions (29) are the first main analytical result of our research of transversal vibrations of elastically connected double circular plates system. From analytical solutions (29), and corresponding expressions (31) of the constant we can conclude that for one *mn*-family mode n, $m = 1,2,3,4,...\infty$, to one eigen amplitude function corresponds two own circular frequencies and corresponding two-frequency time function $T_{(i)nm}(t)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$. We can conclude that the elastical Winkler type layer duplicates the number of system circular frequencies corresponding to one eigen amplitude function of the *mn*-family mode n, $m = 1,2,3,4,...\infty$.

6. Theoretical Problem Formulation and Governing Equations of Forced Oscillation of the Visco-Elastically Connected Double Plate System

Let's consider the same system of plates but connected with a visco-elastic layer (see *Fig.* 1*b*)) and external excitation force distributed along the upper and lower surface. This visco-elastically connected double plate system is a composite visco-elastic structure type.

If we present the interconnecting layer as a model of one visco-elastic element with starting element's length l_0 whose ends have displacements $w_1(r, \varphi, t)$ and $w_2(r, \varphi, t)$, and velocities $\dot{w}_1(r,\varphi,t)$ and $\dot{w}_2(r,\varphi,t)$, as shown in at *Fig.* 1*c*), using visco-elastic element constitutive relation of force, displacements and velocities in that layer (see [1]), we will formulate governing equations for this problem in terms of two unknowns: the transversal displacements $w_i(r, \varphi, t)$, $i = 1,2$ in direction of the axis z of the upper plate middle surface and of the lower plate middle surface. Then, the system of two coupled partial differential equations of the forced visco-elastically connected double plate system is in the following form [9,10]:

$$\begin{aligned}
&\frac{\partial^2 w_1(r,\varphi,t)}{\partial t^2} + c^4_{(1)}\Delta\Delta w_1(r,\varphi,t) - 2\delta_{(1)}\left[\frac{\partial w_2(r,\varphi,t)}{\partial t} - \frac{\partial w_1(r,\varphi,t)}{\partial t}\right] - \\
&-a^2_{(1)}[w_2(r,\varphi,t) - w_1(r,\varphi,t)] = \tilde{q}_{(1)}(r,\varphi,t) \\
&\frac{\partial^2 w_2(r,\varphi,t)}{\partial t^2} + c^4_{(2)}\Delta\Delta w_2(r,\varphi,t) + 2\delta_{(2)}\left[\frac{\partial w_2(r,\varphi,t)}{\partial t} - \frac{\partial w_1(r,\varphi,t)}{\partial t}\right] + \\
&+a^2_{(2)}[w_2(r,\varphi,t) - w_1(r,\varphi,t)] = \tilde{q}_2(r,\varphi,t)
\end{aligned} \tag{32}$$

where we use the same notations as in previous and define: $2\delta_{(i)} = b / \rho_i h_i$- constant surface damping coefficient of visco-elastic layer; $\tilde{q}_{(i)}(r,\varphi,t)$, $i = 1,2$ - function of continual distributed transversal forces which we use like external excitation of plates.

The solution of the previous system (32) of partial-differential equations can be looked for by Bernoulli's method of particular integrals in the form(15) of multiplication of two functions, of which the first $\mathbf{W}_{(i)}(r, \varphi)$, $i = 1,2$ depends only on space coordinates r and φ, and the second is a time function $T_{(i)}(t)$, $i = 1,2$. Here we use the same space coordinate eigen amplitude functions $\mathbf{W}_{(i)}(r, \varphi)$, $i = 1,2$ as in the case of decoupled system and for the example of mentioned boundary conditions in form (13).

7. The Analytical Solutions of the Time Functions of the Forced Transversal Vibrations of a Double Circular Plate System with Visco-Elastic Layer

Our next defined task is to derive analytical solution of the governing system of coupled partial differential equations for forced system oscillations, equations (32). We consider the eigen amplitude functions $\mathbf{W}_{(i)nm}(x, y)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$ expansion with unknown time functions $T_{(i)nm}(t)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$ describing their time evolution [2] as mentioned above in the case of decoupled plates problem. Then after introducing (15) into (32), we obtain the following system of no homogeneous second order ordinary differential equations with respect to the unknown time functions $T_{(i)nm}(t)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$ for the mn-family mode:

$$\begin{aligned}
&\ddot{\mathrm{T}}_{(1)nm}(t) + 2\delta_{(1)}\dot{\mathrm{T}}_{(1)nm} + \omega^2_{(1)nm}\mathrm{T}_{(1)nm}(t) - a^2_{(1)}\mathrm{T}_{(2)nm}(t) - 2\delta_{(1)}\dot{\mathrm{T}}_{(2)nm}(t) = f_{(1)nm}(t)\\
&\ddot{\mathrm{T}}_{(2)nm}(t) + 2\delta_{(2)}\dot{\mathrm{T}}_{(2)nm} + \omega^2_{(2)nm}\mathrm{T}_{(2)nm}(t) - a^2_{(2)}\mathrm{T}_{(1)nm}(t) - 2\delta_{(2)}\dot{\mathrm{T}}_{(1)nm} = f_{(2)nm}(t)
\end{aligned} \tag{33}$$

where known time functions $f_{(1)nm}(t)$ and $f_{(2)nm}(t)$ are defined by the following expressions:

$$f_{(1)nm}(t) = \frac{\int_0^r\int_0^{2\pi} \tilde{q}_{(1)}(r,\varphi,t)\,\mathrm{W}_{(1)nm}(r,\varphi)\,r\,dr\,d\varphi}{\int_0^r\int_0^{2\pi}[\mathrm{W}_{(1)nm}(r,\varphi)]^2\,r\,dr\,d\varphi}$$

and

$$f_{(2)nm}(t) = \frac{\int_0^r\int_0^{2\pi} \tilde{q}_{(2)}(r,\varphi,t)\,\mathrm{W}_{(1)nm}(r,\varphi)\,r\,dr\,d\varphi}{\int_0^r\int_0^{2\pi}[\mathrm{W}_{(1)nm}(r,\varphi)]^2\,r\,dr\,d\varphi} \tag{34}$$

We can obtain the basic linear unperturbed equations of the coupled system of differential equations (33) neglecting the external excitations. Also, for the linear system, we can formally define the following matrices: mass inertia moment matrix $\mathbf{A}$, damping coefficient matrix $\mathbf{B}$, and quasielastic coefficients matrix $\mathbf{C}$ (see Ref. [14]):

$$\mathbf{A}_{nm} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},\quad \mathbf{B} = \begin{pmatrix} 2\delta_{(1)} & -2\delta_{(1)} \\ -2\delta_{(2)} & 2\delta_{(2)} \end{pmatrix},\quad \mathbf{C}_{nm} = \begin{pmatrix} \omega^2_{(1)nm} & -a^2_{(1)} \\ -a^2_{(2)} & \omega^2_{(2)nm} \end{pmatrix} \tag{35}$$

and the characteristic equation of the linearized coupled system is in the following form:

$$\left|\lambda^2\mathbf{A}_{nm} + \lambda\mathbf{B} + \mathbf{C}_{nm}\right| = \begin{vmatrix} \lambda^2 + 2\delta_{(1)}\lambda + \omega^2_{(1)nm} & -a^2_{(1)} - 2\delta_{(1)}\lambda \\ -a^2_{(2)} - 2\delta_{(2)}\lambda & \lambda^2 + 2\delta_{(2)}\lambda + \omega^2_{(2)nm} \end{vmatrix} = 0 \tag{36}$$

with four roots for every eigen amplitude function mode *nm*:

$$\lambda_{1,2nm} = -\hat{\delta}_{1nm} \mp i\hat{p}_{1nm} \text{ and } \lambda_{3,4nm} = -\hat{\delta}_{2nm} \mp i\hat{p}_{2nm} \tag{37}$$

We obtain own amplitude numbers from (see Refs. [2] and [3]):

$$\frac{A_{1nm}^{(s)}}{a_{(1)}^2 + 2\delta_{(1)}\lambda_{snm}} = \frac{A_{2mn}^{(s)}}{\lambda_{snm}^2 + 2\delta_{(1)}\lambda_{snm} + \omega_{(1)nm}^2} = \tilde{C}_s \text{ or } \frac{A_{1nm}^{(s)}}{K_{21nm}^{(s)}} = \frac{A_{2nm}^{(s)}}{K_{22nm}^{(s)}} = C_{snm}$$

and we rewrite the solution of linear coupled system in the form:

$$T_{(1)nm}(t) = K_{21nm}^{(1)} e^{-\hat{\delta}_{1nm}t} R_{01}\cos(\hat{p}_{1nm}t + \alpha_{01}) + K_{21nm}^{(2)} e^{-\hat{\delta}_{2nm}t} R_{02}\cos(\hat{p}_{2nm}t + \alpha_{02})$$

$$T_{(2)nm}(t) = K_{22nm}^{(1)} e^{-\hat{\delta}_{1nm}t} R_{01}\cos(\hat{p}_{1nm}t + \alpha_{01}) + K_{22nm}^{(2)} e^{-\hat{\delta}_{2nm}t} R_{02}\cos(\hat{p}_{2nm}t + \alpha_{02}) \tag{38}$$

where amplitudes and phases R_{0i} and α_{0i} are constants, defined by the initial conditions.

To obtain an approximation of the solution of the coupled equations (32) for the forced vibrations by using the Lagrange's method of constant variations, we propose solutions in the following forms:

$$T_{(1)nm}(t) = K_{21nm}^{(1)} e^{-\hat{\delta}_{1nm}t} R_{1nm}(t)\cos\Phi_{1nm}(t) + K_{21nm}^{(2)} e^{-\hat{\delta}_{2nm}t} R_{2nm}(t)\cos\Phi_{2nm}(t)$$

$$T_{(2)nm}(t) = K_{22nm}^{(1)} e^{-\hat{\delta}_{1nm}t} R_{1nm}(t)\cos\Phi_{1nm}(t) + K_{22nm}^{(2)} e^{-\hat{\delta}_{2nm}t} R_{2nm}(t)\cos\Phi_{2nm}(t) \tag{39}$$

where two amplitudes $R_{inm}(t)$ and two phases $\Phi_{inm}(t) = \hat{p}_{inm}t + \phi_i(t)$, $i = 1{,}2$ are unknown functions. By introducing the condition that the first derivatives of the time functions $\dot{T}_{(i)nm}(t)$:

$$\begin{aligned}\dot{T}_{(1)nm}(t) = &-\hat{\delta}_1 K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\cos\Phi_1(t) - K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\hat{p}_1 \sin\Phi_1(t) - \\ &-\hat{\delta}_2 K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\cos\Phi_2(t) - K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\hat{p}_2 \sin\Phi_2(t) + \\ &+K_{21}^{(1)} e^{-\hat{\delta}_1 t} \dot{R}_1(t)\cos\Phi_1(t) - K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t) + \\ &+K_{21}^{(2)} e^{-\hat{\delta}_2 t} \dot{R}_2(t)\cos\Phi_2(t) - K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)\end{aligned}$$

$$\begin{aligned}\dot{T}_{(2)nm}(t) = &-\hat{\delta}_1 K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\cos\Phi_1(t) - K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\hat{p}_1 \sin\Phi_1(t) - \\ &-\hat{\delta}_2 K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\cos\Phi_2(t) - K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\hat{p}_2 \sin\Phi_2(t) + \\ &+K_{22}^{(1)} e^{-\hat{\delta}_1 t} \dot{R}_1(t)\cos\Phi_1(t) - K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t) + \\ &+K_{22}^{(2)} e^{-\hat{\delta}_2 t} \dot{R}_2(t)\cos\Phi_2(t) - K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)\end{aligned}$$

have the same forms as in the case where amplitudes $R_{inm}(t)$ and difference of phases $\phi_{inm}(t)$ are constants:

$$\begin{aligned}\dot{T}_{(1)nm}(t) = &-\hat{\delta}_1 K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\cos\Phi_1(t) - K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\hat{p}_1 \sin\Phi_1(t) - \\ &-\hat{\delta}_2 K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\cos\Phi_2(t) - K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\hat{p}_2 \sin\Phi_2(t)\end{aligned}$$

$$\begin{aligned}\dot{T}_{(2)nm}(t) = &-\hat{\delta}_1 K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\cos\Phi_1(t) - K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t)\hat{p}_1 \sin\Phi_1(t) - \\ &-\hat{\delta}_2 K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\cos\Phi_2(t) - K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t)\hat{p}_2 \sin\Phi_2(t)\end{aligned} \tag{39*}$$

we obtaine first two conditions for the derivatives of the unknown functions $\dot{R}_i(t)$ and $\dot{\Phi}_i(t)$.

$$\begin{aligned}
&K_{21}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\cos\Phi_1(t)-K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t)+\\
&+K_{21}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\cos\Phi_2(t)-K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)=0\\
&K_{22}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\cos\Phi_1(t)-K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t)+\\
&+K_{22}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\cos\Phi_2(t)-K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)=0
\end{aligned}\tag{40}$$

After multiplying first equation (40) with cofactor $K_{22}^{(1)}$ or $K_{22}^{(2)}$ and second with $-K_{21}^{(1)}$ or $-K_{21}^{(2)}$ and summing these two equation the system of equations follows :

$$\begin{aligned}
&\dot{R}_1(t)\cos\Phi_1(t)-R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t)=0\\
&\dot{R}_2(t)\cos\Phi_2(t)-R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)=0
\end{aligned}\tag{41}$$

The second derivatives $\ddot{T}_{(i)nm}(t)$ are in the forms:

$$\begin{aligned}
&\ddot{T}_{(1)nm}(t)=\hat{\delta}_1^2K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\cos\Phi_1(t)-\hat{\delta}_1K_{21}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\cos\Phi_1(t)+\\
&+2\hat{\delta}_1K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\sin\Phi_1(t)+\hat{\delta}_1K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t)-\\
&-K_{21}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\hat{p}_1\sin\Phi_1(t)-K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1^2\sin\Phi_1(t)-\\
&-K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\dot{\phi}_1(t)\cos\Phi_1(t)+\hat{\delta}_2^2K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\cos\Phi_2(t)-\\
&-\hat{\delta}_2K_{21}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\cos\Phi_2(t)+2\hat{\delta}_2K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\sin\Phi_2(t)+\\
&+\hat{\delta}_2K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)-K_{21}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\hat{p}_2\sin\Phi_2(t)-\\
&-K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2^2\sin\Phi_2(t)-K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\dot{\phi}_2(t)\cos\Phi_2(t)\\[1ex]
&\ddot{T}_{(2)nm}(t)=\hat{\delta}_1^2K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\cos\Phi_1(t)-\hat{\delta}_1K_{22}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\cos\Phi_1(t)+\\
&+2\hat{\delta}_1K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\sin\Phi_1(t)+\hat{\delta}_1K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t)-\\
&-K_{22}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\hat{p}_1\sin\Phi_1(t)-K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1^2\sin\Phi_1(t)-\\
&-K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\dot{\phi}_1(t)\cos\Phi_1(t)+\hat{\delta}_2^2K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\cos\Phi_2(t)-\\
&-\hat{\delta}_2K_{22}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\cos\Phi_2(t)+2\hat{\delta}_2K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\sin\Phi_2(t)+\\
&+\hat{\delta}_2K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\dot{\phi}_2(t)\sin\Phi_2(t)-K_{22}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\hat{p}_2\sin\Phi_2(t)-\\
&-K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2^2\sin\Phi_2(t)-K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\dot{\phi}_2(t)\cos\Phi_2(t)
\end{aligned}\tag{42}$$

After introducing the first $\dot{T}_{(i)nm}(t)$,eqs. (39*) and second $\ddot{T}_{(i)nm}(t)$, eqs (42) derivatives of the proposed solutions (39) in the system of nonhomogeneous equations (33) we obtain two more equations in the derivatives of the unknown functions $\dot{R}_{inm}(t)$ and $\dot{\phi}_{inm}(t)$:

$$-K_{21}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\hat{p}_1\sin\Phi_1(t)-K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\dot{\phi}_1(t)\cos\Phi_1(t)-$$
$$-K_{21}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\hat{p}_2\sin\Phi_2(t)-K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\dot{\phi}_2(t)\cos\Phi_2(t)=f_{(1)nm}$$

$$-K_{22}^{(1)}e^{-\hat{\delta}_1 t}\dot{R}_1(t)\hat{p}_1\sin\Phi_1(t)-K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\hat{p}_1\dot{\phi}_1(t)\cos\Phi_1(t)-$$
$$-K_{22}^{(2)}e^{-\hat{\delta}_2 t}\dot{R}_2(t)\hat{p}_2\sin\Phi_2(t)-K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_2(t)\hat{p}_2\dot{\phi}_2(t)\cos\Phi_2(t)=f_{(2)nm}$$

After multiplying the first equation with cofactor $K_{22}^{(1)}$ or $K_{22}^{(2)}$ and second with $-K_{21}^{(1)}$ or $-K_{21}^{(2)}$ and summing these two equations the system of equations follows:

$$\dot{R}_1(t)\sin\Phi_1(t)+R_1(t)\dot{\phi}_1(t)\cos\Phi_1(t)=\frac{K_{22}^{(2)}f_{(1)nm}-K_{21}^{(2)}f_{(2)nm}}{e^{-\hat{\delta}_1 t}\hat{p}_1(K_{22}^{(1)}K_{21}^{(2)}-K_{22}^{(2)}K_{21}^{(1)})}$$

$$\dot{R}_2(t)\sin\Phi_2(t)+R_2(t)\dot{\phi}_2(t)\cos\Phi_2(t)=\frac{K_{22}^{(1)}f_{(1)nm}-K_{21}^{(1)}f_{(2)nm}}{e^{-\hat{\delta}_2 t}\hat{p}_2(K_{22}^{(1)}K_{21}^{(2)}-K_{22}^{(2)}K_{21}^{(1)})} \quad (43)$$

Solving the obtained subsystems of four nonhomogeneous algebraic equations (41) and (43) with respect to the derivatives $\dot{R}_{inm}(t)$ and $\dot{\phi}_{inm}(t)$, we can write the system of the first-order differential equations as follows:

$$\dot{R}_{1nm}(t)=-\frac{f_{(1)nm}(t)K_{22nm}^{(2)}-f_{(2)nm}(t)K_{21nm}^{(2)}}{\hat{p}_{1nm}(K_{21nm}^{(1)}K_{22nm}^{(2)}-K_{22nm}^{(1)}K_{21nm}^{(2)})}e^{\hat{\delta}_{1nm}t}\sin\Phi_{1nm}(t)\,;$$

$$\dot{\phi}_{1nm}(t)=-\frac{f_{(1)nm}(t)K_{22nm}^{(2)}-f_{(2)nm}(t)K_{21nm}^{(2)}}{R_{1nm}(t)\hat{p}_{1nm}(t)(K_{21nm}^{(1)}K_{22nm}^{(2)}-K_{22nm}^{(1)}K_{21nm}^{(2)})}e^{\hat{\delta}_{1nm}t}\cos\Phi_{1nm}(t)$$

$$\dot{R}_{2nm}(t)=-\frac{K_{22nm}^{(1)}f_{(1)nm}(t)-K_{21nm}^{(1)}f_{(2)nm}(t)}{\hat{p}_{2nm}(K_{22nm}^{(1)}K_{21nm}^{(2)}-K_{22nm}^{(2)}K_{21nm}^{(1)})}e^{\hat{\delta}_{2nm}t}\sin\Phi_{2nm}(t)\,;$$

$$\dot{\phi}_{2nm}(t)=-\frac{K_{22nm}^{(1)}f_{(1)nm}(t)-K_{21nm}^{(1)}f_{(2)nm}(t)}{R_{2nm}(t)\hat{p}_{2nm}(K_{21nm}^{(1)}K_{22nm}^{(2)}-K_{22nm}^{(1)}K_{21nm}^{(2)})}e^{\hat{\delta}_{2nm}t}\cos\Phi_{2nm}(t) \quad (44)$$

where we denoted $\Phi_{inm}(t)=\hat{p}_{inm}t+\phi_{inm}(t)$.

If we use trigonometrical transformation of mentioned solutions (39) and define four more variables like:

$$A_{(i)}(t)=R_{(i)}(t)\cos\phi_i(t)\,;\ B_{(i)}(t)=-R_{(i)}(t)\sin\phi_i(t)\,,\ i=1,2 \quad (45)$$

and integrate the system of equations (44), using the obtained solutions we can rewrite the solutions in the following final forms:

$$T_{(1)nm}(t) = K_{21nm}^{(1)} e^{-\hat{\delta}_{1nm}t}[A_{(01)nm}\cos\hat{p}_{1mn}t + B_{(01)nm}\sin\hat{p}_{1nm}t] +$$
$$+K_{21nm}^{(2)} e^{-\hat{\delta}_{2nm}t}[A_{(02)nm}\cos\hat{p}_{2nm}t + B_{(02)nm}\sin\hat{p}_{2nm}t] +$$
$$+K_{21nm}^{(1)}\int_0^t \left[\frac{f_{(1)nm}(\tau)K_{22nm}^{(2)} - f_{(2)nm}(\tau)K_{21nm}^{(2)}}{\hat{p}_{1nm}(K_{21nm}^{(1)}K_{22nm}^{(2)} - K_{22nm}^{(1)}K_{21nm}^{(2)})} e^{\hat{\delta}_{1nm}(\tau-t)}\sin\hat{p}_{1nm}(\tau-t)\right]d\tau +$$
$$+K_{21nm}^{(2)}\int_0^t \left[\frac{K_{22nm}^{(1)}f_{(1)nm}(\tau) - K_{21nm}^{(1)}f_{(2)nm}(\tau)}{\hat{p}_{2nm}(K_{22nm}^{(1)}K_{21nm}^{(2)} - K_{22nm}^{(2)}K_{21nm}^{(1)})} e^{\hat{\delta}_{2nm}(\tau-t)}\sin\hat{p}_{2nm}(\tau-t)\right]d\tau$$

$$T_{(2)nm}(t) = K_{22nm}^{(1)} e^{-\hat{\delta}_{1nm}t}[A_{(01)nm}\cos\hat{p}_{1mn}t + B_{(01)nm}\sin\hat{p}_{1nm}t] +$$
$$+K_{22nm}^{(2)} e^{-\hat{\delta}_{2nm}t}[A_{(02)nm}\cos\hat{p}_{2nm}t + B_{(02)nm}\sin\hat{p}_{2nm}t] +$$
$$+K_{22nm}^{(1)}\int_0^t \left[\frac{f_{(1)nm}(\tau)K_{22nm}^{(2)} - f_{(2)nm}(\tau)K_{21nm}^{(2)}}{\hat{p}_{1nm}\left(K_{21nm}^{(1)}K_{22nm}^{(2)} - K_{22nm}^{(1)}K_{21nm}^{(2)}\right)} e^{-\hat{\delta}_{1nm}(\tau-t)}\sin\hat{p}_{1nm}(\tau-t)\right]d\tau + \quad (46)$$
$$+K_{22nm}^{(2)}\int_0^t \left[\frac{f_{(1)nm}(\tau)K_{22nm}^{(1)} - f_{(2)nm}(\tau)K_{21nm}^{(1)}}{\hat{p}_{2nm}\left(K_{22nm}^{(1)}K_{21nm}^{(2)} - K_{22nm}^{(2)}K_{21nm}^{(1)}\right)} e^{\hat{\delta}_{2nm}(\tau-t)}\sin\hat{p}_{2nm}(\tau-t)\right]d\tau$$

The solutions (46) are the main analytical result for time functions of forced transversal vibrations of visco-elastically connected double circular plates system, so the solutions for middle surface points displacements in functions of r, φ and t are in forms (15) where the space coordinate eigen amplitude functions $\mathbf{W}_{(i)}(r, \varphi)$, $i = 1,2$ are in forms (13). From the analytical solutions (46), we can conclude that for one mn -family mode $n, m = 1,2,3,4,...\infty$, to one eigen amplitude function corresponds two circular damped frequencies and corresponding two-frequency time functions $T_{(i)nm}(t)$, $i = 1,2$, $n, m = 1,2,3,4,...\infty$, in the case of free oscillations of the system, and that for forced oscillations in those functions contain terms corresponding to different combinations (sums and differences) between frequencies of forced external excitations and eigen circular damped frequencies.

Choosing for external excitation periodic forces, we can rewrite the functions $f_{(i)nm}(t) = h(_{0i})\cos\Omega_i t$, $i = 1,2$ in the following forms:

$$f_{(1)nm}(t) = \frac{\int_0^r\int_0^{2\pi} \tilde{F}_{(01)}\tilde{\tilde{F}}_{(1)}(r,\varphi)\,\mathrm{W}_{(1)nm}(r,\varphi) r dr d\varphi}{\int_0^r\int_0^{2\pi}[\mathrm{W}_{(1)nm}(r,\varphi)]^2 r dr d\varphi}\cos\Omega_1 t = h_{(01)}\cos\Omega_1 t$$

and

$$f_{(2)nm}(t) = \frac{\int_0^r\int_0^{2\pi} \tilde{F}_{(02)}\tilde{\tilde{F}}_{(2)}(r,\varphi)\,\mathrm{W}_{(1)nm}(r,\varphi) r dr d\varphi}{\int_0^r\int_0^{2\pi}[\mathrm{W}_{(1)nm}(r,\varphi)]^2 r dr d\varphi}\cos\Omega_2 t = h_{(02)}\cos\Omega_2 t\,.$$

where $\tilde{q}_i(r,\varphi,t)=\tilde{F}_{(0i)}\tilde{\tilde{F}}_{(i)}(r,\varphi)\cos\Omega_i t$ are known specific area distributed external transversal excitations along the upper plate upper contour surface as well as the lower plate lower contour surface.

In the special observed cases of homogeneous double plate system with equal plate mass distributions and thicknesses, and considering external excitation only in the upper plate we obtained the following solutions:

$$\begin{aligned}T_{(1)nm}(t)&=[A_{(01)nm}\cos(\sqrt{\omega_{nm}^2-a^2})t+B_{(01)nm}\sin(\sqrt{\omega_{nm}^2-a^2})t]+\\&+e^{-2\delta t}[A_{(02)nm}\cos(\sqrt{a^2+\omega_{nm}^2-4\delta^2})t+B_{(02)nm}\sin(\sqrt{a^2+\omega_{nm}^2-4\delta^2})t]-\\&-\frac{h_{01}}{2\sqrt{\omega_{nm}^2-a^2}}\int_0^t[\cos\Omega_1 t\sin[(\sqrt{\omega_{nm}^2-a^2})\cdot(\tau-t)]]d\tau+\\&+\frac{h_{01}}{2\sqrt{a^2+\omega_{nm}^2-4\delta^2}}\int_0^t[e^{2\delta(\tau-t)}\cos(\Omega_1 t)\sin[(\sqrt{a^2+\omega_{nm}^2-4\delta^2})\cdot(\tau-t)]]d\tau\end{aligned}$$

$$\begin{aligned}T_{(2)nm}(t)&=[A_{(01)nm}\cos(\sqrt{\omega_{nm}^2-a^2})t+B_{(01)nm}\sin(\sqrt{\omega_{nm}^2-a^2})t]-\\&-e^{-2\delta t}[A_{(02)nm}\cos(\sqrt{a^2+\omega_{nm}^2-4\delta^2})t+B_{(02)nm}\sin(\sqrt{a^2+\omega_{nm}^2-4\delta^2})t]-\\&-\frac{h_{01}}{2\sqrt{\omega_{nm}^2-a^2}}\int_0^t[\cos(\Omega_1 t)\sin[(\sqrt{\omega_{nm}^2-a^2})\cdot(\tau-t)]]d\tau-\\&-\frac{h_{01}}{2\sqrt{a^2+\omega_{nm}^2-4\delta^2}}\int_0^t[e^{2\delta(\tau-t)}\cos(\Omega_1 t)\sin[(\sqrt{a^2+\omega_{nm}^2-4\delta^2})\cdot(\tau-t)]]d\tau\end{aligned}\tag{47}$$

8. Numerical Results

For numerical experiment and analysis, we take into consideration a homogeneous double plate system containing two equal circular plates with radius $a=1[m]$ and graded from steel material. By using Maple and the possibilities of visualizing these numerical results, we present them as space surfaces of the plate middle surface during the time, and also as time-history diagrams of the plate middle surface points displacements. On the basis of numerical results, series of characteristic middle surface forms of coupled plates during the time are presented in the Figs. 4, 5 and 6.

In Fig. 4. characteristic transversal displacements of the middle surface points of lower and upper plates are presented in function of r, φ and t, in three different time moments for:

a) one eigen amplitude function form of oscillations ($n=1$, $m=0$);

b) two eigen amplitude function forms of oscillations ($n=0$, $m=1$ summed with forms for $n=1$, $m=1$) and

c) three eigen amplitude function forms of oscillations ($n=0$, $m=1$ summed with forms for $n=1$, $m=1$ and $n=2$, $m=1$)

In Fig. 5. the characteristic transversal displacements of the middle surface points on characteristic diameters for lower and upper plates in function of r for $\varphi_j = const$ at characteristic forms along time t, three eigen amplitude function forms of oscillations ($n=0, m=1$ summed with forms for $n=1, m=1$ and $n=2, m=1$) are presented for: three different values of the external excitation frequency, when external distributed force is applied to upper plate a) $\Omega \approx \tilde{\omega}_{11} = \sqrt{\omega_{11}^2 - a^2}$; b) $\Omega \approx \tilde{\omega}_{21} = \sqrt{\omega_{21}^2 - a^2}$ and c) $\Omega \approx \tilde{\omega}_{31} = \sqrt{\omega_{31}^2 - a^2}$

In Fig. 6. characteristic transversal displacements of the middle surface points are presented:

6.1: on the series of characteristics diameters and cycles in function of r, φ for $\varphi_j = const$ and $r = const$, at characteristic forms along time t /

6.2: on the series of characteristics diameters in function of r, for $\varphi_j = const$ at characteristic forms along time t, for lower and upper plates three eigen amplitude function forms of oscillations ($n=0, m=1$ summed with forms for $n=1, m=1$ and $n=2, m=1$) for: three different values of the external excitation frequency, when external distributed force is applied to upper plate a) $\Omega \approx \tilde{\omega}_{11} = \sqrt{\omega_{11}^2 - a^2}$; b) $\Omega \approx \tilde{\omega}_{21} = \sqrt{\omega_{21}^2 - a^2}$ and c) $\Omega \approx \tilde{\omega}_{31} = \sqrt{\omega_{31}^2 - a^2}$.

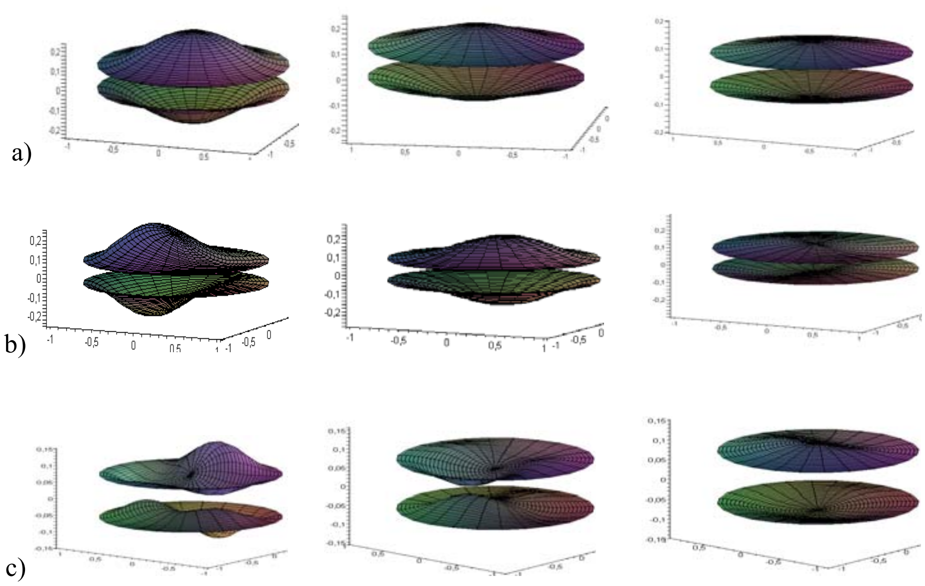

Fig. 4. Characteristic transversal displacements of the middle surface points of lower and upper plates in function of r, φ and t, in three different time moments, for: a) one eigen amplitude function form of oscillations ($n = 1$, $m = 0$); b) two eigen amplitude function forms of oscillations ($n = 0$, $m = 1$ summed with forms for $n = 1$, $m = 1$) and c) three eigen amplitude function forms of oscillations ($n = 0$, $m = 1$ summed with forms for $n = 1$, $m = 1$ and $n = 2$, $m = 1$)

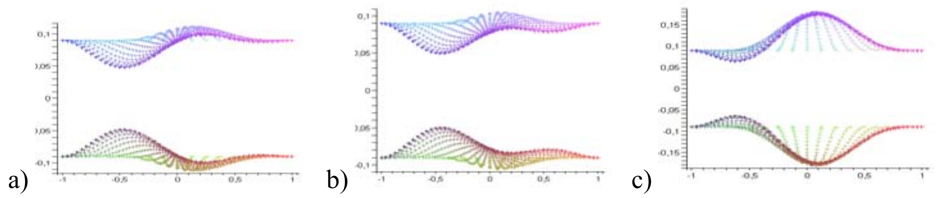

Fig. 5. Characteristic transversal displacements of the middle surface points on characteristic diameters for lower and upper plates in function of r for $\varphi_j = const$ at characteristic forms along time t, three eigen amplitude function forms of oscillations ($n = 0$, $m = 1$ summed with forms for $n = 1$, $m = 1$ and $n = 2$, $m = 1$) for: three different values of the external excitation frequency, when external distributed force is applied to upper plate

a) $\Omega \approx \tilde{\omega}_{11} = \sqrt{\omega_{11}^2 - a^2}$; b) $\Omega \approx \tilde{\omega}_{21} = \sqrt{\omega_{21}^2 - a^2}$ and c) $\Omega \approx \tilde{\omega}_{31} = \sqrt{\omega_{31}^2 - a^2}$

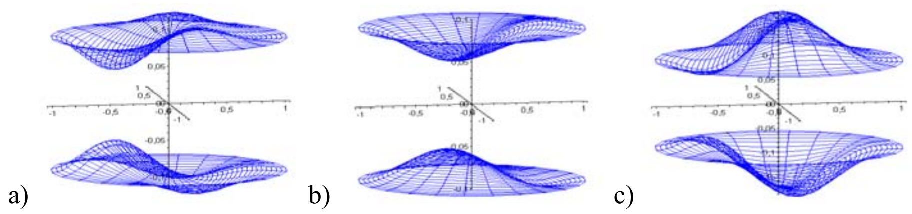

Fig. 6.1.

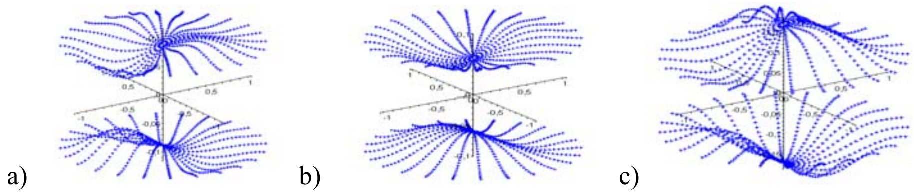

Fig. 6.2.

Fig. 6. Characteristic transversal displacements of the middle surface points: 6.1: on the series of characteristic diameters and cycles in function of r, φ for $\varphi_j = const$ and $r = const$, at characteristic forms along time t; 6.2: on the series of characteristics diameters in function of r, for $\varphi_j = const$ at characteristic forms along time t, for lower and upper plates three eigen amplitude function forms of oscillations ($n = 0$, $m = 1$ summed with forms for $n = 1, m = 1$ and $n = 2, m = 1$) for: three different values of the external excitation frequency, when external distributed force is applied to upper plate a) $\Omega \approx \tilde{\omega}_{11} = \sqrt{\omega_{11}^2 - a^2}$; b) $\Omega \approx \tilde{\omega}_{21} = \sqrt{\omega_{21}^2 - a^2}$ and c) $\Omega \approx \tilde{\omega}_{31} = \sqrt{\omega_{31}^2 - a^2}$

9. CONCLUDING REMARKS

The analytical solutions of system coupled partial differential equations in every of *nm*- family of corresponding dynamical free (unperturbed) processes are obtained by using method of Bernoulli's particular integral and Lagrange's method of constants variations for solution of forced transversal oscillations.

From the obtained ordinary differential equations and corresponding analytical solutions for time functions corresponding to one eigen amplitude function mode we can conclude that they are uncoupled from other eigen amplitude time functions.

From the analytical solutions for the case of pure elastic layer between plates, we can conclude that for one *mn*-family mode n, $m = 1,2,3,4,...\infty$, to one eigen amplitude function correspond two circular frequencies and corresponding two-frequency time functions $T_{(i)nm}(t)$, $i = 1,2$, n, $m = 1,2,3,4,...\infty$, in the case of free oscillations of the system, and that for forced oscillations these functions contain terms corresponding to different combinations (sums and differences) between frequencies of forced external excitations and eigen circular frequencies.

We can see that integral part, i.e. particular analytical solutions of coupled partial differential equations, of derived solutions correspond to the coupled forced and free middle surface vibrations regimes, and describe multi-frequency vibrations with frequencies which are different combinations (sums and differences) between frequencies of forced external excitations and eigen circular damped frequencies (see solutions (46)). These analytical solutions can be used for analyses of possible regimes of resonances or phenomena of dynamical absorption. By using Maple program, the visualizations of the characteristic forms of the plate middle surfaces through time are presented.

The obtained analytical and numerical result is very valuable for university teaching process in the area of structural system elastodynamics as well as of hybrid deformable body system vibrations.

Acknowledgement: *Parts of this research were supported by the Ministry of Sciences and Environmental Protection of Republic of Serbia through Mathematical Institute SANU Belgrade Grant ON144002 "Theoretical and Applied Mechanics of Rigid and Solid Body. Mechanics of Materials" and Faculty of Mechanical Engineering University of Niš. Parts of this research results are submitted for presentation at SSM-JDM Congress of Theoretical and Applied Mechanics.*

REFERENCES

1. Goroško Oleg Aleksandrovič and Katica (Stevanović) Hedrih, (2001), *Analitička dinamika (mehanika) diskretnih naslednih sistema, (Analytical Dynamics (Mechanics) of Discrete Hereditary Systems)*, University of Niš, 1999-2000-2001, p. 426.
2. Hedrih (Stevanović) K., (2006), *Transversal Vibrations of Double-Plate Systems*, *Acta Mechanica Sinica, Springer*, (2006) 22, pp. 487-501 (hard cower and on line).
3. Hedrih (Stevanović), K., (2007), *Double plate system with discontinuity in the elastic bounding layer*, *Acta Mechanica Sinica* DOI 10.1007/s10409-007-0061-x (corrected proof, in press)
4. Hedrih (Stevanović) K., (2005), Integrity of Dynamical Systems, *Journal Nonlinear Analysis*, 63 (2005) 854 – 871.
5. Hedrih (Stevanović) K., (2005), *Partial Fractional Order Differential Equations of Transversal Vibrations of Creep-connected Double Plate Systems*, in Monograph - *Fractional Differentiation and its Applications,* Edited by Alain Le Mahaute, J. A. Tenreiro Machado, Jean Claude Trigeassou and Jocelyn Sabatier, U-Book, pp. 289-302.
6. Hedrih (Stevanović), K., (2007), *Transversal vibrations of the axially moving sandwich belts*, *Archive of Applied Mechanics*, Springer, DOI 10.1007/s00419-006-0105-x
7. Katica (Stevanović) Hedrih, (2006), *The frequency equation theorems of small oscillations of a hybrid system containing coupled discrete and continuous subsystems*, F*acta Universitatis, Series Mechanics, Automatic Control and Robotics*, Vol.5, No 1, 2006 pp. 25 - 41 http://facta.junis.ni.ac.yu/facta/

8. Hedrih (Stevanović) K., (2006), *Modes of the Homogeneous Chain Dynamics*, *Signal Processing*, *Elsevier,* 86(2006), 2678-2702.. ISSN: 0165-1684 www.sciencedirect.com/science/journal/01651684
9. Hedrih (Stevanović) K., (2003), *Frequency equations of small oscillations mixed systems of the coupled discrete and continuous subsystems*, Mehanika tverdogo tela (Rigid Body Mechanics), Donetsk, UDC 531.1:534.012:534.013:, ISSN 0321-1975, vip. 33, pp. 174-189. Ukraine.
10. Hedrih (Stevanović) K., (2005), Mathematical Analogy and Phenomenological Mapping: Vibrations of Multi Plate and Multi Beam Homogeneous Systems, *Scientific Buletletin of the "Politehnica" University of Timisoara, Romania, Transaction on Mechanics, Editura Politehnica, Tom 50 (64), Special Issue, 2005, pp. 11-18. ISSN 1224-6077*.
11. Hedrih (Stevanović) K., (2004), *Discrete Continuum Method,* COMPUTATIONAL MECHANICS, WCCM VI in conjunction with APCOM'04, *Sept. 5-10, 2004, Beijing, China,* © 2004 Tsinghua University Press & Springer-Verlag, pp. 1-11, CD.
12. Hedrih (Stevanović) K., (2005), Transveral vibrations of creep connected multi plate homogeneous systems, CD Proceedings, Fifth EUROMECH Nonlinear Dynamics Conference, Eindhoven University of Technology, Editors: D.H. van Campen, M.D. Lazurko, W.P.J.M. van den Over, ID 11-428, pp. 1445-1454.
13. Hedrih (Stevanović) K. and Simonović J., (2006), *Characteristic Eigen Numbers and Frequencies of the Transversal Vibrations of Sandwich System*, SEECCM 06 –First South-East European Conference on Computational Mechanics, Proceedings, Editors – M. Kojic and M Papadrakakis, Kragujevac, pp. 90-94. + CD. hppt://www.seeccn.kg.ac.yu. ISBN 86-81037-13-7, COBISS.SR-ID 131758092, udc 531/533(082)
14. Krzysztof POLTORAK, Kosuke NAGAYA,A Method for Solving Free Vibration Problems of Three-Layered Plates with Arbitrary Shape, Journal of Acoustical Society of America,78-6(1985),pp.2042-2048.
15. Krzysztof POLTORAK,Kosuke NAGAYA,A Method for Solving Forced Vibration Problems of Three-Layered Plates with Arbitrary Shape, Journal of Acoustical Society of America,80-5(1986), pp.1408-1413.
16. Rašković, D., (1965), *Teorija oscilacija, (Theory of Oscillations),* Naučna knjiga, 1965, 503.(in Serbian).

TRANSVERZALNE OSCILACIJE NEKONZERVATIVNOG SISTEMA DVE KRUŽNE PLOČE

Katica (Stevanović) Hedrih, Julijana Simonović

Interes proučavanja iscilacija spregnutih ploča , kao kvalitativno novih sistema, poslednjih godina eksponencijalno raste naročito radi izazova teorijskog proučavanja takvih sistema. Kao uvod dat je pregled rezultata proučavanja prvog autora iz oblasti transverzalnih oscilacija različitih sistema dveju ploča. (pogledati ref-ce. [2-7]).

Osnovnovni rezultat našeg doprinosa je analitičko rešenje sistema spregnutih homogenih i nehomopgenih parcijalnih diferencijalnih jednačina koje opisuju slobodne i prinudne oscilacije sistema dve kružne ploče povezane elastičnim ili visko-elastičnim slojem [1]. Ova rešenja su dobijena Bernoulli-jevom metodom partikularnih integrala kao i Lagrange-ovom metodom varijacije konstanata. Prikazanano je nekoliko slika numeričkog eksperimenta sa slobodnim i prinudnim oscilacijam sistema kružnih ploča.

Dobijeni analitički i numerički rezultati su veoma korisni u procesu univerzitetske nastave iz oblasti elastodinamike strukturnih sistema kao i oscilacija hibridnih sistema deformabilnih tela.

Ključne reči: *spregnuti podsistemi, spregnute dinamike, kružne ploče, hibridni, višefrekventni, prinudne oscilacije, Lagrange-ova metoda, analitičko rešenje, numerički eksperiment*